Fundamental bounds on Quantum Metrology in the presence of Decoherence

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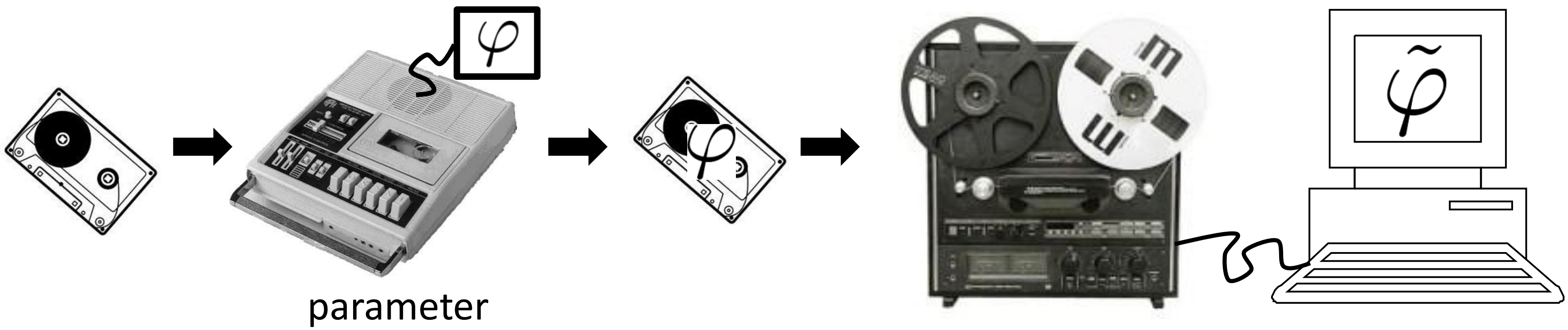
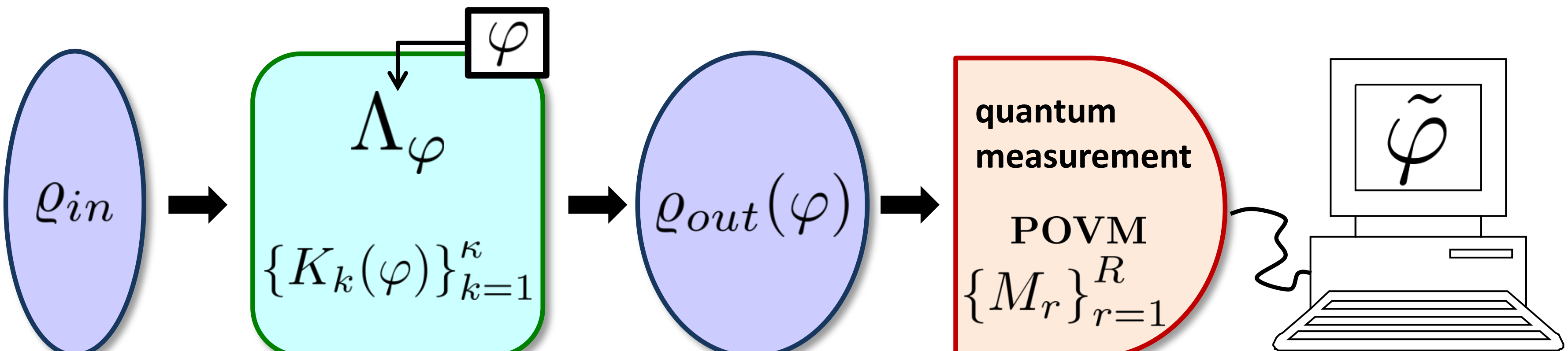
QUANTUM METROLOGY: PARAMETER ESTIMATION THEORY
VS
GENERAL METROLOGY: CLASSICAL PARAMETER ESTIMATION GAME



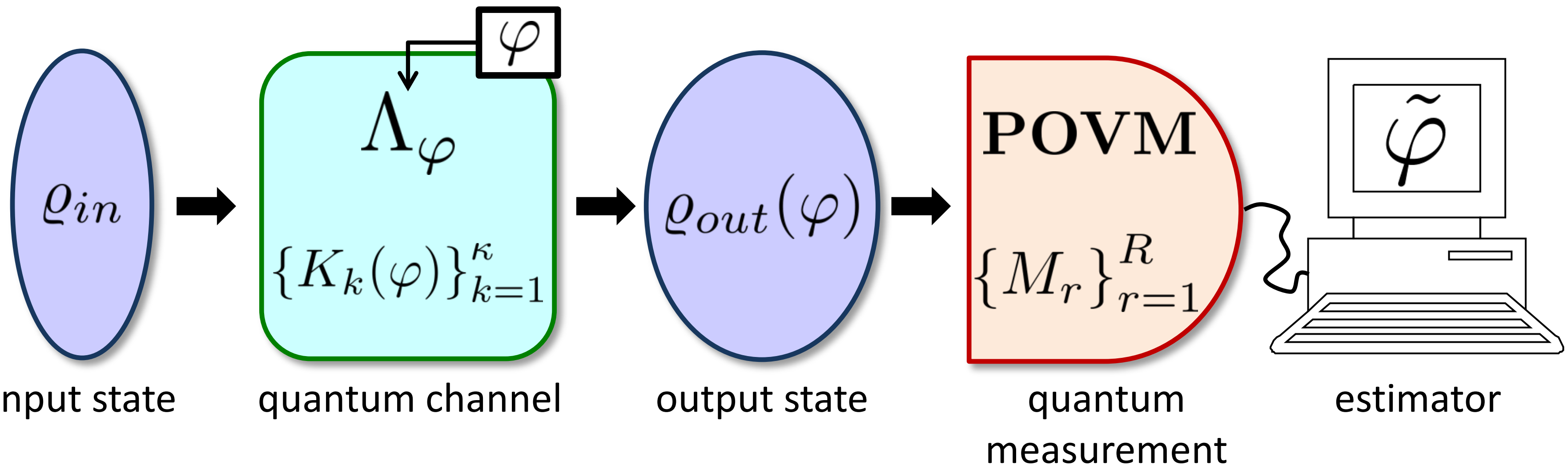
Parallelism: classical parameter estimation — quantum channel estimation



quantum state	quantum channel parameterised by φ	output state $\sum_{k=1}^{K} K_k(\varphi)\varrho_{in}K_k^\dagger(\varphi)$	general set of positive op^s $\sum_{r=1}^{R} M_r = \mathbf{1}$	estimate from probabilities $p_r = \mathrm{Tr}\{M_r\,\varrho_{out}(\varphi)\}$

QUANTUM ~~PARAMETER~~ CHANNEL ESTIMATION THEORY

Setup for the optimal estimation strategy:



AIM:

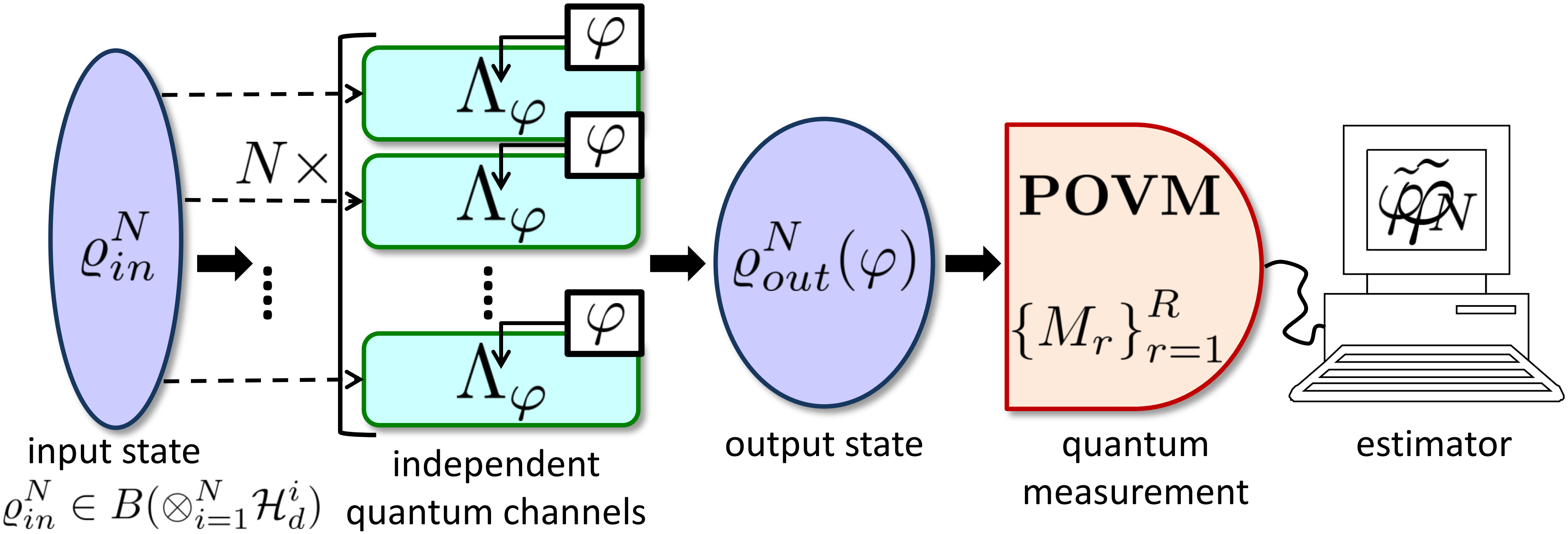
- Find the optimal method of establishing $\tilde{\varphi}$ as close to φ, for all $\varphi \in \mathcal{S}_\varphi$.
- Minimise the average error: $$\Delta\tilde{\varphi} = \sqrt{\left\langle (\tilde{\varphi} - \varphi)^2 \right\rangle}$$
- **Very hard!**

  We are given the set $\{\Lambda_\varphi : \varphi \in \mathcal{S}_\varphi\}$, for which we need to optimise over:

$\boxed{\textbf{the input state + the set of all POVMs + the estimator.}}$

Can we ask any general questions?

N independent **realisations of the estimated channel:**



What is the scaling of the average error, $\Delta\tilde{\varphi}_N$, with the number of realisations *N*?

- **Classically**, as the realisations are ***independent*** we cannot overcome the ***shot noise***.
  Asymptotically (*N*→∞) the error can maximally scale as:

$$\Delta\tilde{\varphi}_N = \frac{1}{\sqrt{N}}\Delta\tilde{\varphi}$$ **Shot Noise Limit (SNL)**

- **Quantum mechanically**, input can be ***entangled*** and measurement can be ***non-local***.
  Asymptotically (*N*→∞) the error can maximally scale as:

$$\Delta\tilde{\varphi}_N = \frac{1}{N}\Delta\tilde{\varphi}$$ **Heisenberg limit (HL)**

[*Giovannetti et al, Science 306, 2004*]

Channels considered that asymptotically achieve **Heisenberg Limit,** $\Delta\tilde{\varphi}_N = \frac{1}{N}\Delta\tilde{\varphi}$, are **unitary:**

$$\Lambda_\varphi\left[\varrho_{in}\right] = U(\varphi)\varrho_{in}U^\dagger(\varphi)$$

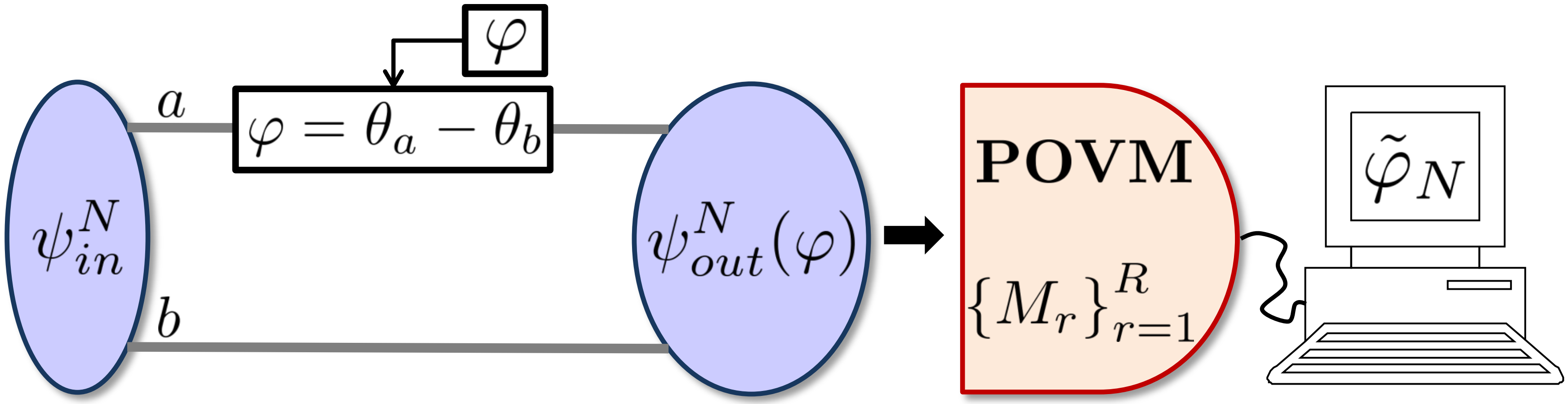
Do the **realistic physical channels**, which include **losses/decoherence**, also achieve **Heisenberg Limit**?

$$\Lambda_\varphi\left[\varrho_{in}\right] = \sum_{k=1}^{K} K_k(\varphi)\varrho_{in}K_k^\dagger(\varphi)$$

LET US INVESTIGATE SOME EXAMPLES ...

EXAMPLE 1A: OPTICAL INTERFEROMETER

N independent **realisations** of the channel $\leftrightarrow$ ***N*** **photon** pure input state



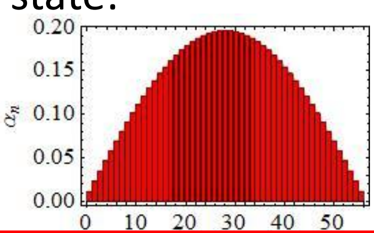
pure ***N*** photon state | output state | set of POVM's | estimator

$$\left|\psi_{in}^N\right\rangle = \sum_{n=0}^N \alpha_n \left|n\right\rangle_a \otimes \left|N-n\right\rangle_b \Rightarrow \left|\psi_{out}^N\right\rangle = \sum_{n=0}^N \alpha_n \mathrm{e}^{in\varphi} \left|n\right\rangle_a \otimes \left|N-n\right\rangle_b$$

SOLUTIONS IN TWO SCENARIOS:

- **No knowledge** of the estimated φ.
- **Flat *"a priori"* distribution** , $p(\varphi) = \frac{1}{2\pi}$.
- Optimal (**entangled**) input state:

$$\alpha_n = \sqrt{\frac{2}{N+2}} \sin\left(\frac{(n+1)\pi}{N+2}\right)$$



- Error scaling:

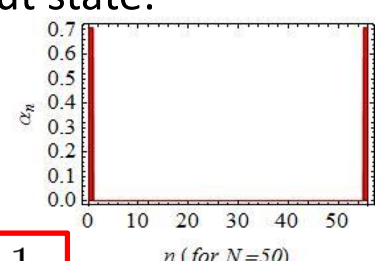
$$\Delta\tilde{\varphi}_N = \frac{\pi}{N+2}, \quad \Delta\tilde{\varphi}_{N\to\infty} = \frac{1}{N}$$

[*Berry and Wiseman, PRL 85, 5098 (2000)*]

- **Highest sensitivity** to changes from φ_0 .
- **Delta *"a priori"* distribution** , $p(\varphi) \approx \delta(\varphi_0)$
- Optimal (**entangled**) input state:

$$\left|\psi_{in}^N\right\rangle = \frac{1}{\sqrt{2}}(\left|N0\right\rangle + \left|0N\right\rangle)$$

NOON state:



- Error scaling:

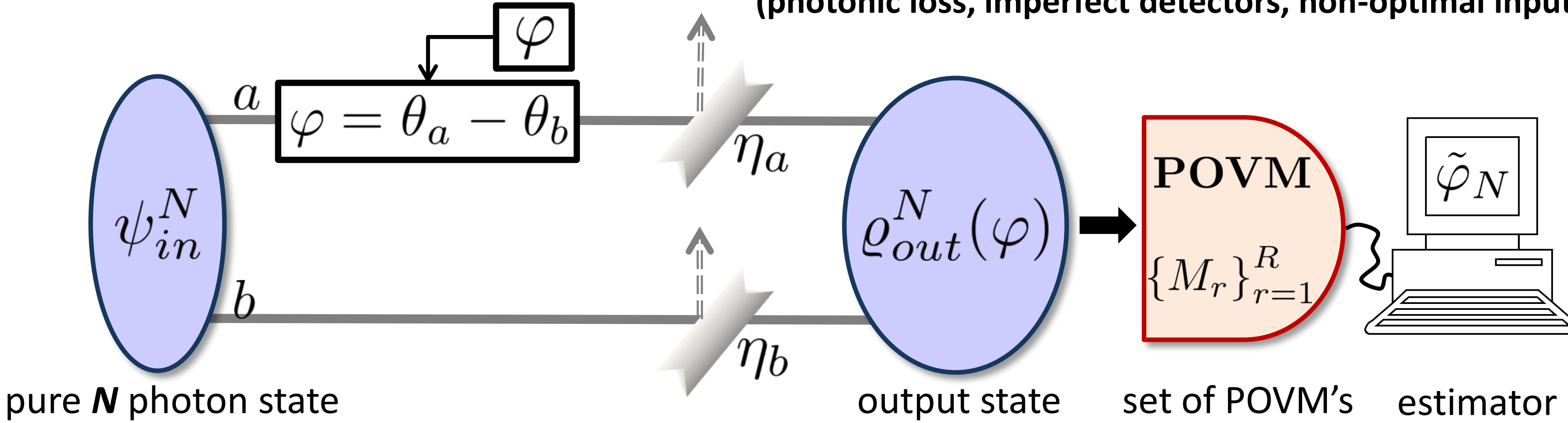
$$\Delta\tilde{\varphi}_N\big|_{\tilde{\varphi}\approx\varphi_0} = \frac{1}{N}$$

Heisenberg Limit !
(unitary channel)

[*J. P. Dowling, Phys. Rev. A 57, 4736 (1998)*]

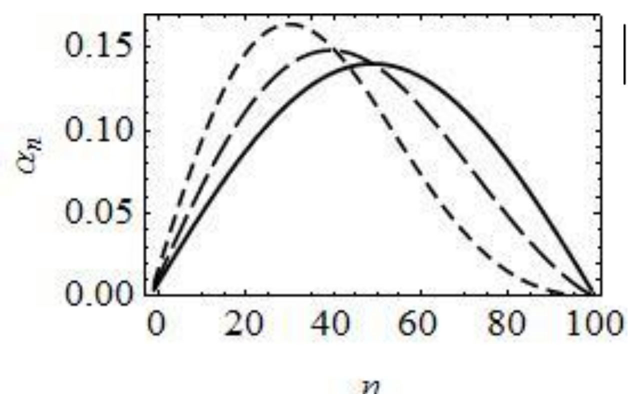
EXAMPLE 1B: **OPTICAL INTERFEROMETER WITH LOSS**

(photonic loss, imperfect detectors, non-optimal input)

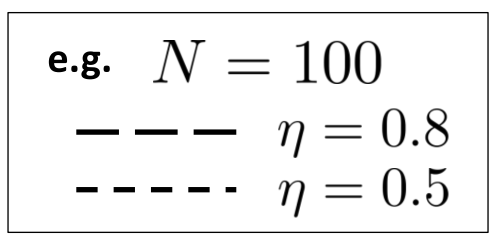


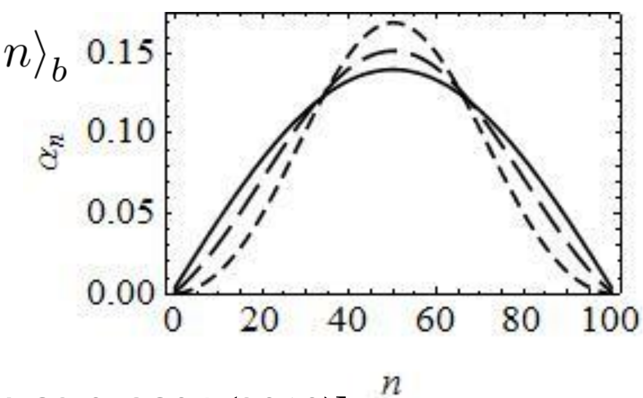
- **Flat "*a priori*" distribution** of the estimated φ, $p(\varphi) = \frac{1}{2\pi}$.
- Channel is **no longer unitary** → mixed output $\varrho_{out}^N(\varphi)$.
- **For finite N**: optimal input state's coefficients, α_n , are found *numerically*:

Loss in **one** arm: $\eta_a = \eta,\ \eta_b = 1$	Loss in **both** arms: $\eta_a = \eta_b = \eta$



$$\left|\psi_{in}^N\right\rangle = \sum_{n=0}^N \alpha_n \left|n\right\rangle_a \left|N-n\right\rangle_b$$



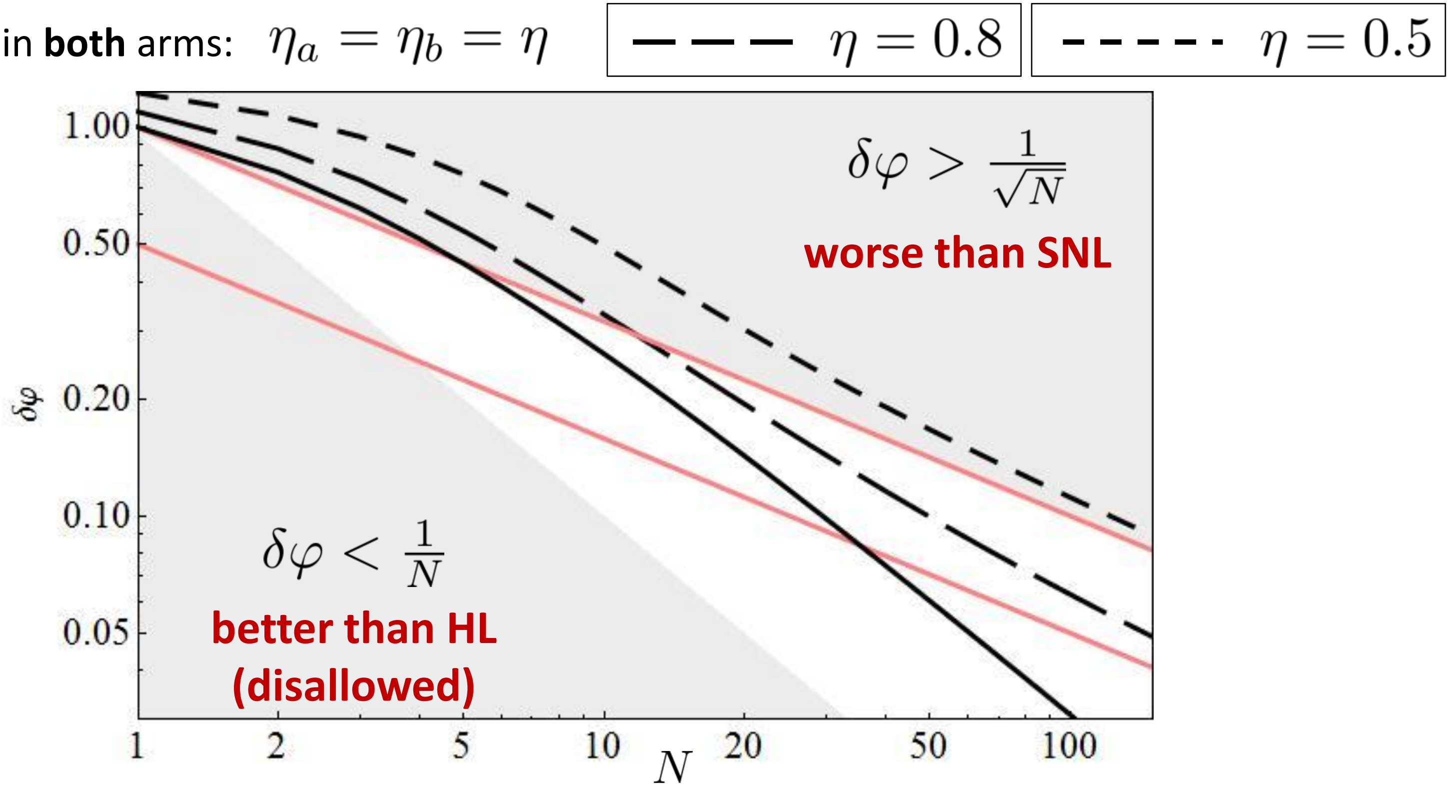


[*Kolodynski and Demkowicz-Dobrzanski, PRA 82,053804 (2010)*]

EXAMPLE 1B: **OPTICAL INTERFEROMETER WITH LOSS**

SCALING OF THE ERROR , $\Delta\tilde{\varphi}_N$, WITH THE NUMBER OF PHOTONS, N.

e.g. Loss in **both** arms: $\eta_a = \eta_b = \eta$



- For any type of loss we provide a **lower bound** on error with $\eta = \min(\eta_a, \eta_b)$

$$\Delta\tilde{\varphi}_N \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}} + O(\frac{1}{N})$$

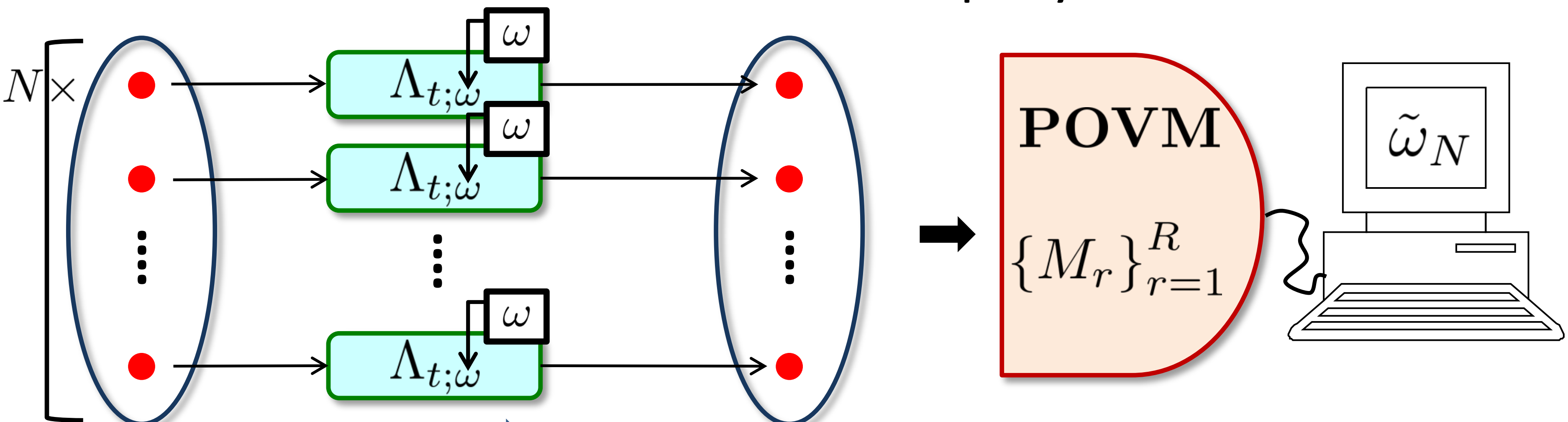
Shot Noise Limit !!!

ALREADY INFINITESMAL AMOUNT OF LOSS DESTROYS THE ASYMPTOTIC HEISENBERG SCALING

[*Kolodynski and Demkowicz-Dobrzanski, PRA 82,053804 (2010)*]

Example 2A: Atomic Spectroscopy

N two-level atoms (**qubits**) evolving for *fixed* time, t, oscillating *independently* with same estimated **transition frequency** ω.

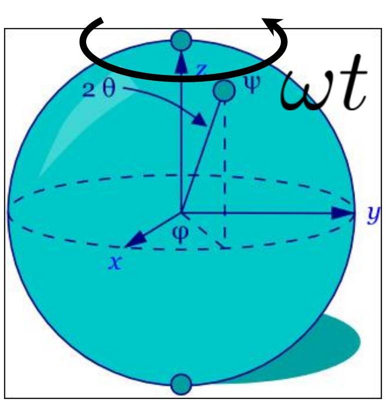


ϱ_{in}^{N}

$$\varrho_t^N(\omega) = U_{\omega t}^{\otimes N}\,\varrho_{in}^N\,U_{\omega t}^{\dagger\otimes N}$$

Aim: Best possible estimate of the frequency, ω.

Bloch sphere picture:



- An optimal (**entangled**) input state – **GHZ** state:

$$\varrho_{in}^{N} = |\psi_N\rangle\langle\psi_N|\,,\ |\psi_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$$

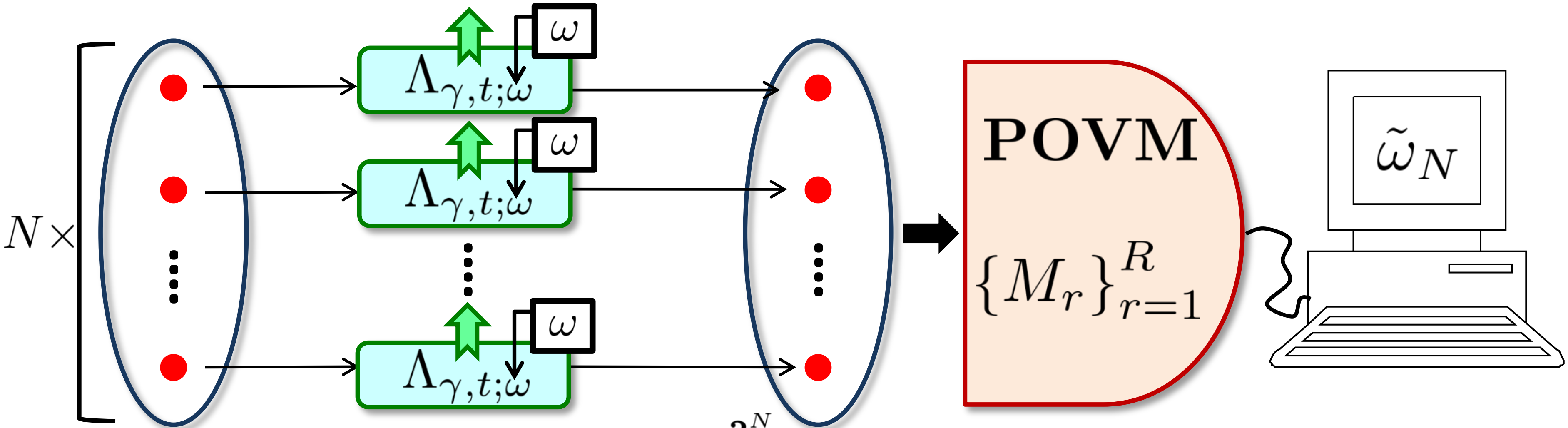
- Error scaling:

$$\Delta\tilde{\omega}_N = \frac{1}{\sqrt{t}}\cdot\frac{1}{N}$$

Heisenberg Limit !!!

[*Huelga et al, PRL 79, 38653868 (1997)*]

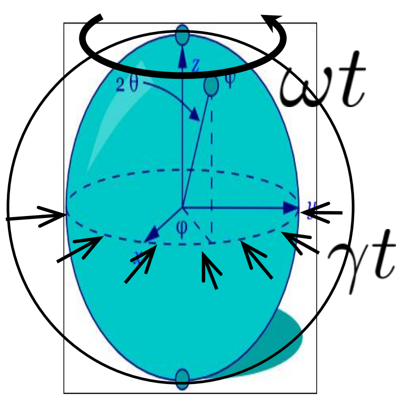
EXAMPLE 2B: **ATOMIC SPECTROSCOPY WITH DEPHASING**



ϱ_{in}^{N} time, *t*

$$\varrho_{out}^{N}(\omega)=\sum_{\mathbf{k}=\mathbf{1}^N}^{\mathbf{2}^N} K_{\mathbf{k}}(\gamma,t;\omega)\,\varrho_{in}^{N}\,K_{\mathbf{k}}^{\dagger}(\gamma,t;\omega)$$

Bloch sphere picture:



WHAT IS *NOW* THE ASYMPTOTIC SCALING OF THE ERROR OF ESTIMATED FREQUENCY, ω, WITH THE NUMBER OF ATOMS N?

[*Escher et al, Nature Phys. 7, 406 (2011)*]

- Asymptotic error is given by

$$\Delta\tilde{\omega}_{N\to\infty}=\sqrt{\frac{2\gamma}{t}}\cdot\frac{1}{\sqrt{N}}$$

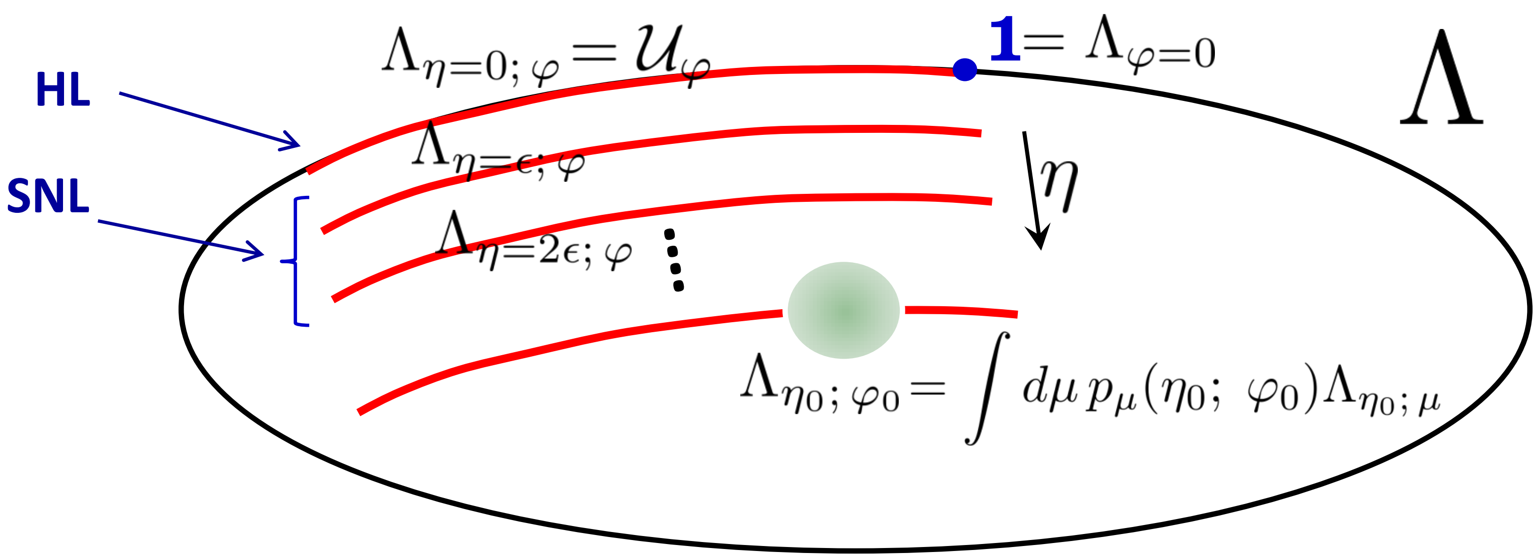
Shot Noise Limit !!!

ALREADY INFINITESMAL AMOUNT OF DEPHASING DESTROYS THE ASYMPTOTIC HEISENBERG SCALING

Are there any channels that include ***decoherence*** and preserve the ***Heisenberg scaling***?

Geometric Intuition

- Consider the **convex set of all quantum channels** mapping a general input density matrix onto an output one: $\Lambda: \ \varrho_{in} \in B\left(\mathcal{H}_{d_{in}}\right) \longrightarrow \varrho_{out} \in B\left(\mathcal{H}_{d_{out}}\right)$
- Consider the **unitary subset parameterised by the estimated parameter**: Λ_φ
- Consider the **family of subsets for different strengths of the decoherence** model: $\Lambda_{\eta;\,\varphi}$



- **Heisenberg Limit** for **extremal ($\supset$ unitary)** channels at the border.
- **Shot Noise Limit** for ones inside the convex set.

What is so special about the non-extremal channels?

We could try to move the **estimated parameter dependence** into the "Mixing" probability!

NON-EXTREMAL CHANNELS WITH SQL SCALING

IDEA OF MADALIN GUTA, UNIVERSITY OF NOTTINGHAM

For the **channel to possess SQL scaling** it is enough to show that the channel has the **following properties**:

1. It has to be **non-extremal** – must **not** lie on the (non-flat) boundary of the convex set of all channels:

$$\Lambda = \frac{1}{2}(\Lambda_1 + \Lambda_2) \;\text{ or in general }\; \Lambda = \int d\mu\, p_\mu \Lambda_\mu$$

2. The **estimated parameter dependence** can be **moved** into the **"mixing" probability distribution**:

$$\boxed{\Lambda_\varphi = \int d\mu\, p_\mu(\varphi)\Lambda_\mu}$$

3. The **"mixing" probability has to be regular** w.r.t. φ' at the estimated φ:

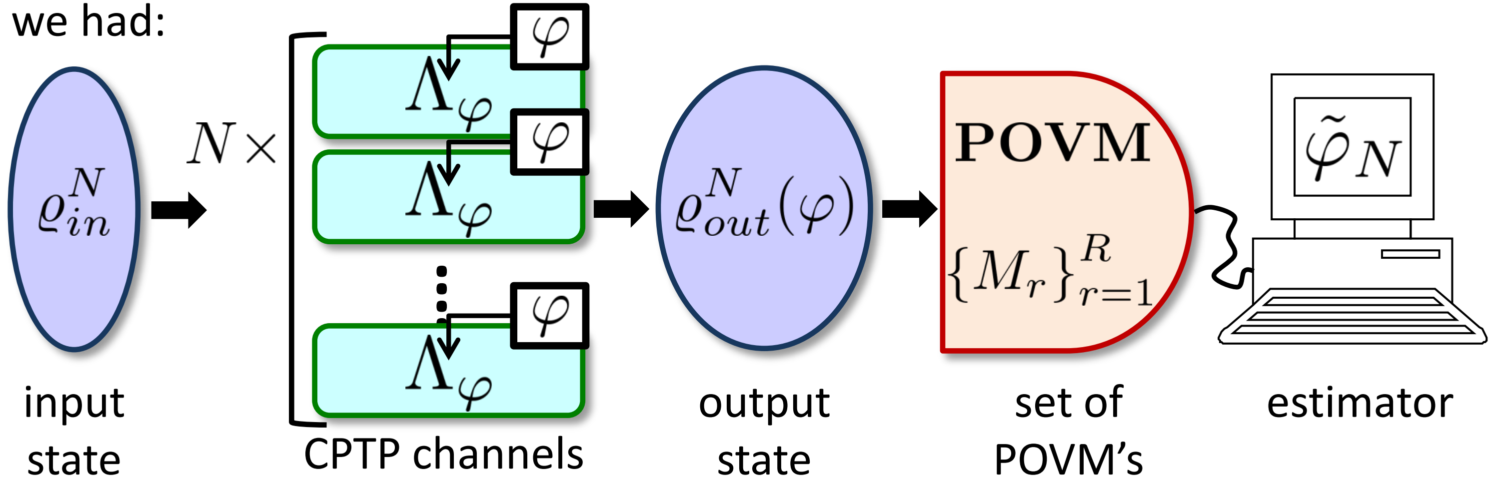
$$\langle \partial_{\varphi'} \ln p_\mu(\varphi') \rangle_{\varphi'=\varphi} = \int d\mu p_\mu(\varphi')\, \partial_{\varphi'} \ln p_\mu(\varphi') \Big|_{\varphi'=\varphi} = 0$$

e.g. not a *Dirac delta* function, $p_\mu(\varphi) \neq \delta(\varphi)$

[*Keiji Matsumoto, arXiv:1006.0300, 2010*]

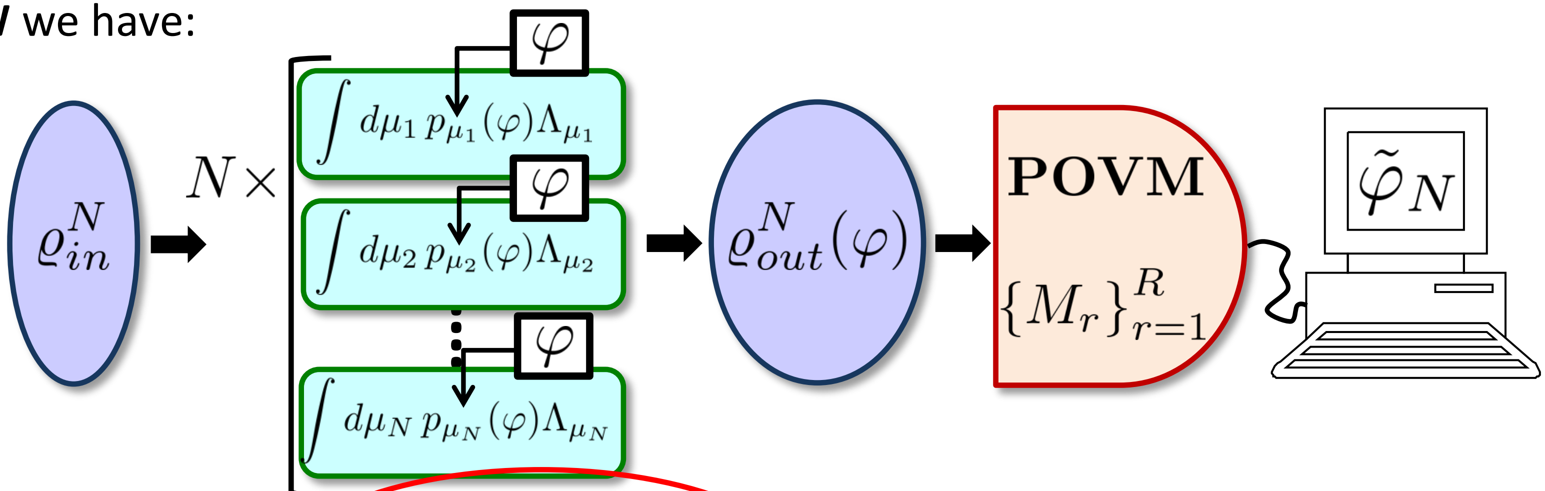
PROOF

Originally we had:



Markov chain **of information about the parameter:** $\varphi \longrightarrow \Lambda \longrightarrow \varrho_{out}^N \longrightarrow \tilde{\varphi}$

But **NOW** we have:



independent var^s

$$\varphi \xrightarrow{p_{\mu_i}} \{\mu_1, \mu_2, \ldots, \mu_N\} \cancel{\longrightarrow \Lambda^N \longrightarrow \varrho_{out}^N \longrightarrow \tilde{\varphi}}$$

$$\hookrightarrow \tilde{\varphi}$$

classical !!!

SQL!!! ⟹ **Q.E.D.**

$$\Lambda_\varphi = \int d\mu \, p_\mu(\varphi) \Lambda_\mu$$

IN CASE OF OUR EXAMPLES...

EXAMPLE 2B: **ATOMIC SPECTROSCOPY WITH <u>DEPHASING</u>**

- **one independent** channel – **one atom (qubit)** evolving for time t.
- The channel, $\Lambda_{\gamma,t;\,\omega}$, is a **<u>qubit-qubit</u>** one. Due to dephasing it is **not unitary** and possesses two **Kraus operators** :

$$\Lambda_{\gamma,t;\,\omega}: \left\{ \begin{array}{c} K_1 = \sqrt{\frac{1+\mathrm{e}^{-\gamma t}}{2}} U_{\omega t} = \\ = \sqrt{\frac{1+\mathrm{e}^{-\gamma t}}{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\omega t}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \end{pmatrix} \end{array}, \begin{array}{c} K_2 = \sqrt{\frac{1-\mathrm{e}^{-\gamma t}}{2}} \hat{\sigma}_z U_{\omega t} = \\ \sqrt{\frac{1-\mathrm{e}^{-\gamma t}}{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\omega t}{2}} & 0 \\ 0 & -\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \end{pmatrix} \end{array} \right\}$$

We construct the required **<u>"mixing" probability distribution</u>**: $\Lambda_\varphi = \int d\mu\, p_\mu(\varphi)\Lambda_\mu$

$$\Lambda_{\gamma,t;\,\omega} = \int \mathrm{d}\mu\, p_\mu\left(\gamma,t;\omega\right) \mathcal{U}_{\mu t} \quad \text{where } \mathcal{U}_{\mu t}\left[\rho\right] = U_{\mu t}\rho U_{\mu t}^\dagger$$

with the smearing probability, $p_\mu\left(\gamma,t;\,\omega\right)$, being a solution to the diffusion equation on the phase circle (group U(1)) :

$$\frac{\partial}{\partial t} p_\mu\left(\gamma,t;\omega\right) = \gamma \frac{\partial^2}{\partial\omega^2} p_\mu\left(\gamma,t;\omega\right) \text{ with } p_\mu\left(\gamma,t=0;\omega\right) = \delta\left(\omega\right)$$

$$p_\mu\left(\gamma,t;\,\omega\right) = \frac{1}{2\pi}\left(1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\gamma t}\cos\left[n\left(\mu t - \omega t\right)\right]\right)$$

<u>"MIXING" PROBABILITY DISTRIBUTION</u> CONSTRUCTED→ *PROVED SQL SCALING*

Example 1B: **Optical Interferometer with <u>Loss</u>**

- **one independent** channel – **one photon** in the MZ interferometer.
- With **<u>loss</u>** the channel, $\Lambda_{\eta_a,\eta_b;\,\varphi}$, becomes a **<u>qubit-qutrit</u>** one, accounting for the possibility of losing the photon to the vacuum state at the output.
- Hence, the **Kraus operators** :

$$\Lambda_{\eta_a,\eta_b;\,\varphi}:\ \left\{K_1=\sqrt{1-\eta_a}\begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix},\ K_2=\sqrt{1-\eta_b}\begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix},\ K_3=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\varphi}\sqrt{\eta_b}&0\\0&\sqrt{\eta_a}\\0&0\end{pmatrix}\right\}$$

We construct the required **<u>"mixing" probability distribution</u>**: $\Lambda_\varphi=\int d\mu\, p_\mu(\varphi)\Lambda_\mu$

$$\Lambda_{\eta_a,\eta_b;\,\varphi}=(\eta_a+\eta_b-1)\int \mathrm{d}\mu\, p_\mu\left(\eta_a,\eta_b;\,\varphi\right)\Lambda_{\eta_a=1,\eta_b=1;\,\mu}+(1-\eta_a)\,\Lambda_{\eta_a=0,\eta_b=1}+(1-\eta_b)\,\Lambda_{\eta_a=1,\eta_b=0}$$

where the smearing probability, $p_\mu\left(\eta_a,\eta_b;\,\varphi\right)$, is a solution to the diffusion equation on the phase circle (group U(1)) :

$$\frac{\partial}{\partial\gamma}p_\mu\left(\gamma;\,\varphi\right)=\frac{\partial^2}{\partial\varphi^2}p_\mu\left(\gamma;\,\varphi\right)\ \text{ with } p_\mu\left(\gamma=0;\,\varphi\right)=\delta\left(\varphi\right)$$

$$p_\mu\left(\gamma;\,\varphi\right)=\frac{1}{2\pi}\left(1+2\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\gamma}\cos\left[n\left(\mu-\theta\right)\right]\right)\quad \text{with }\ \gamma=\ln\frac{\eta_a+\eta_b-1}{\sqrt{\eta_a\eta_b}}$$

<u>"MIXING" PROBABILITY DISTRIBUTION</u> CONSTRUCTED → *PROVED SQL SCALING*

CONCLUSIONS:

1. OPTICAL INTERFEROMETRY WITH LOSS

Asymptotically, we cannot do better than the **Shot Noise scaling** allows:

$$\Delta\tilde{\varphi}_{N\to\infty} \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$$

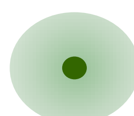
[*JK and RDD, PRA 82,053804 (2010)*]

2. ATOMIC SPECTROSCOPY WITH DEPHASING

Asymptotically, we are follow the **Shot Noise scaling**:

$$\Delta\tilde{\omega}_{N\to\infty} = \sqrt{\frac{2\gamma}{t}}\cdot\frac{1}{\sqrt{N}}$$

3. GENERALISATION:



$$\Lambda_{\eta_0;\,\varphi_0} = \int d\mu\, p_\mu(\eta_0;\ \varphi_0)\Lambda_{\eta_0;\,\mu}$$

IF WE CAN CONSTRUCT THE *"MIXING" PROBABILITY* FOR A **NON-EXTREMAL CHANNEL**, THEN WE TRIVIALLY PROVE ASYMPTOTIC **SHOT NOISE SCALING**.

4. OPEN QUESTION:

HOW DOES THIS METHOD OVERLAPS AND RELATES TO OTHER METHODS OF FINDING ASYMPTOTIC SCALING?

- Considering **purification** of the estimated channel – [*Escher et al, Nat. Phys. 7, 406 (2011)*]
- Considering **extension** of the estimated channel – which proved **SNL** of all ***full rank channels***. [*Fujiwara & Imai, J.Ph.A:Math.Theor.,41, 255304 (2008)*]