Matrix product states for quantum metrology

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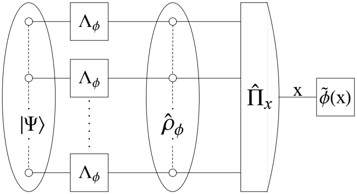
CEQIP

Goals of quantum metrology

- We want to measure some quantities more and more precisely - ultimately with the best possible precision.
- Gravitational waves detection (LIGO, GEO600 etc.).

- Precise measurements of frequencies.
- Atomic clocks.
- Magnetometry.
- Many others...

Scheme



Λ_ϕ
Λ_ϕ
$|\Psi\rangle$
$\hat{\rho}_\phi$
Λ_ϕ
$\hat{\Pi}_x$
x
$\tilde{\phi}$(x)

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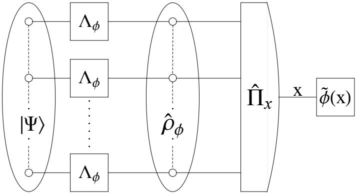

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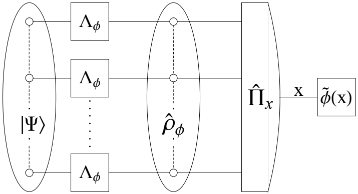

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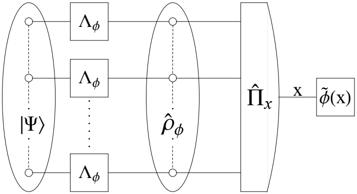

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- System is modeled by some quantum channel Λ_ϕ which acts on $|\Psi\rangle$ and depends on some unknown parameter ϕ which we want to know.
- We make some general quantum measurement (POVM) $\hat{\Pi}_x$ at the output which gives us some value x and then we use estimator $\tilde{\phi}(x)$ to get estimated value $\tilde{\phi}$ of ϕ.

How to calculate the precision of a given observable?

- The most basic and the most common situation - measurement of observable $\hat{A}$ at the output and estimation of ϕ from the average of our outcomes. What is the precision of such estimation procedure?

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where $\Delta\hat{A}$ is defined as usual $\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$.

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- Searching for optimal precision means that we have to optimize the input state $|\Psi\rangle$.

Cramer-Rao bound and quantum Fisher information

Precision is bounded from below by Cramer-Rao inequality

$$\Delta\phi \geq \frac{1}{\sqrt{kF(\phi)}}$$

where $F(\phi)$ is quantum Fisher information (QFI) and k is the number of repetitions of experiment.

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- Shot noise $\Delta\phi = 1/\sqrt{N}$, Heisenberg limit - $\Delta\phi = 1/N$.
- **Fact**: For states $|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle = |\psi\rangle^{\otimes k}$ QFI is equal to $F_\Psi = kF_\psi \to$ only $c/\sqrt{N}$ scaling.

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- We need to know d^N coefficients to describe the state.
- Practically impossible to implement any efficient algorithm of searching $c_{\sigma_1\sigma_2\ldots\sigma_N}$ - exponential scaling of their number with N.

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Is there any other way to **efficiently** describe the state?

YES!

What is MPS?

Answer - **matrix product states** (MPS). They are defined as:

$$|\psi\rangle = \frac{1}{\sqrt{\mathcal{N}}} \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} \mathrm{Tr}(A_{\sigma_1} A_{\sigma_2} \ldots A_{\sigma_N}) |\sigma_1 \sigma_2 \ldots \sigma_N\rangle$$

where A_{σ_i} are some $D \times D$ matrices (D is called **bond dimension**) and $\mathcal{N} = \sum_{\sigma_1 \ldots \sigma_N} \mathrm{Tr}[(A^*_{\sigma_1} \otimes A_{\sigma_1}) \ldots (A^*_{\sigma_N} \otimes A_{\sigma_N})]$ is the normalization factor.

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- Only dD^2N coefficients needed to describe any MPS with bond dimension D.
- Any state can be described by MPS, perhaps with large bond dimension D.

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- Easy evaluation of average values of single particle operators.

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- $$|\mathrm{N00N}\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle) =$$
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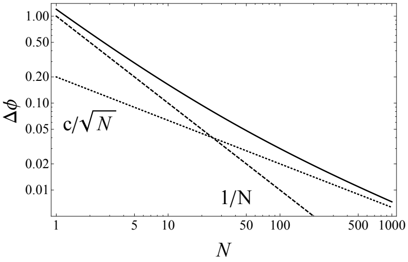
- This state is MPS with minimal bond dimension $D = 2$ and matrices
$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

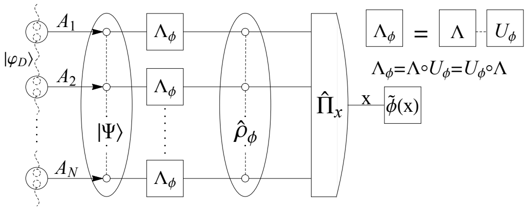
Why MPS? - Intuition

Fact: in the presence of noise asymptotically we have only SQL-like scaling $\Delta\phi \sim c/\sqrt{N} \to$ the same as with product states!



Asymptotically optimal state should have structure $|\Psi\rangle = |\psi\rangle^{\otimes k}$ -entanglement only in small groups!

Let's consider one of the most common cases $d = 2$, i.e photons in interferometer, two-level atoms etc. and losses of probes:



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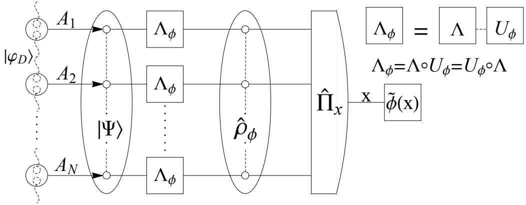

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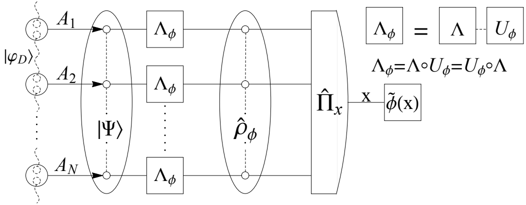

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- We loose each of the probes independently with the probability $1 - \eta$.

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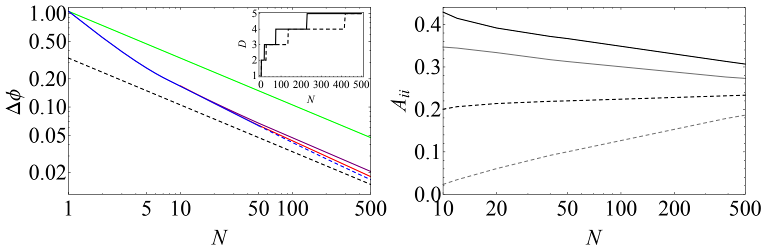
- States are from symmetric subspace $\rightarrow$ trace of any permutation of k matrices A_0 and $N-k$ matrices A_1 is the same $\rightarrow$ diagonal matrices are sufficient$\rightarrow$ only $2D$ parameters for any N.
- Output state is mixed but also from symmetric subspace.

Results



- $\eta = 0.9$.
- Very good approximation for low *D* up to large *N*.
- Insight into the structure of optimal states:
  - A_0, A_1 have the same diagonal elements ordered complementarily - the largest with lowest etc.
  - The higher is *N* the closer are diagonal elements of *A*'s.

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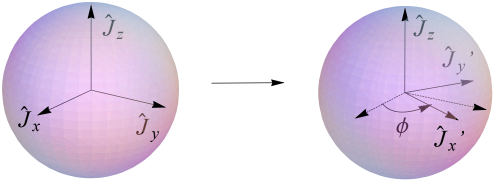

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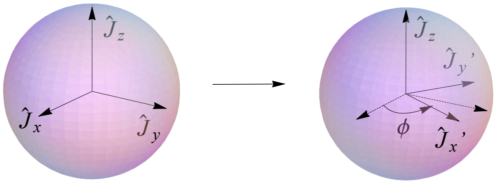

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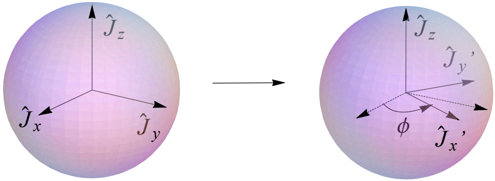

- Optimal angle is $\phi = 0$.
- Precision:

$$\Delta\phi = \sqrt{\frac{\Delta^2 \hat{J}_x}{\langle \hat{J}_y \rangle^2} + \frac{1-\eta}{\eta} \frac{N}{4\langle \hat{J}_y \rangle^2}}.$$

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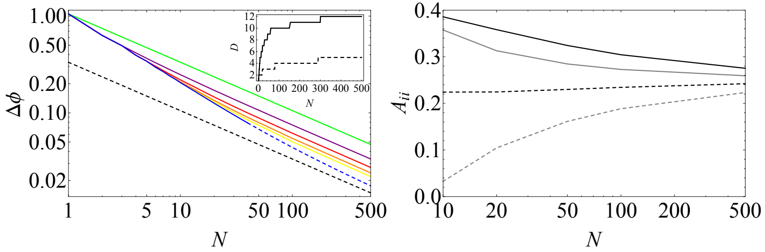

- Larger *D* than previously but still good approximation.
- Insight into the structure of optimal states:
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Conclusions

- Matrix product states are feasible for numerical optimization in quantum metrology.
- We have insight into the structure of optimal states.

Thank You!