The Volume of the Universe after Inflation and de Sitter Entropy

Giovanni Villadoro
(CERN)

Outline

• Basics of inflation and eternal inflation

• Quantum Gravity and de Sitter space
  • Analogies and differences w/ black hole physics
  • An 'holographic' bound
  • Definition of the probability distribution of the volume

• Calculating $\rho(V)$
  • From inflation to bacteria
  • From bacteria back to inflation: a non-linear differential eq.
  • Solving the equation
  • Results, systematics and corrections

• Discussion
The Universe is accelerating today...

...and probably it did so also in the past
The Universe is accelerating today...

...and probably it did so also in the past

**Standard Solution**

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} = \Lambda g_{\mu\nu} + \ldots \]

**CC contribution dominates**

\[ ds^2 \simeq -dt^2 + e^{2Ht} d\bar{x}^2 \]

approx. de Sitter
Slow roll inflation...

\[ \Delta t = H^{-1} \]

\[ a(t) = e^{H\Delta t} = e \]
Slow roll inflation...

\[ \Delta t = H^{-1} \]

every 1 e-folding

\[ a(t) = e^{H\Delta t} = e \]

Hubble Expansion

\[ V(t + \Delta t) = V(t)e^3 \]
Slow roll inflation...

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Random Walk

\[
\frac{\partial P(\phi, t)}{\partial t} = \frac{H^3}{8\pi^2} \frac{\partial^2 P(\phi, t)}{\partial \phi^2} + \phi \frac{\partial P(\phi, t)}{\partial \phi}
\]

Quantum Fluctuations

\[ \langle \delta \phi^2 \rangle = \frac{H^3}{4\pi^2} \Delta t \approx H^2 \]

Starobinsky, Linde 86

Vilenkin, Ford, Linde, Starobinsky 82
Slow roll inflation...

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Quantum Fluctuations

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...and Eternal Inflation

\[ \delta \phi_{q} \gtrsim \delta \phi_{cl} = \dot{\phi} \Delta t = \frac{\dot{\phi}}{H} \]

\[ \frac{\dot{\phi}}{H^{2}} \lesssim 1 \]

Linde, Goncharov, Mukhanov 86-87

i.e. when quantum fluctuations win against classical rolling inflation never ends globally
Slow roll, eternal inflation and de Sitter

\[ \varepsilon_c \ll \varepsilon \ll 1 \]

\[ \sim dS \]

\[ = \text{FRW} \]

\[ \varepsilon \leq \varepsilon_c \]

\[ = dS \]
Slow roll, eternal inflation and de Sitter

The difference is fundamental for defining quantum gravity
Quantum Gravity is non-local at the non-perturbative level from:
metric fluctuates, Bekenstein bound, black hole physics...
(Quantum) Gravity in de Sitter space

Quantum Gravity is non-local at the non-perturbative level

from:
metric fluctuates,
Bekenstein bound,
black hole physics...

indeed known description of QG are defined on boundaries:
String Theory via S-matrix on Mink
AdS/CFT defined on the boundary of AdS
Analogies with black holes...

- **horizon physics**, finite temperature, Hawking radiation...
- and in particular a **finite Entropy** ($S=A/4$)
- Metastability (CdL, HM, Poicare recurrence)

...and differences

- de Sitter is an infinite space (finite entropy?)
- eternal inflation never ends globally
- analogue of an information paradox?
Analogies with black holes...

- **horizon physics**, finite temperature, Hawking radiation...
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Lessons from black hole physics:

- Complementarity governs the global description of black-hole geometry
- AdS/CFT says that black hole evaporation process is **unitary**
- \[ \Gamma > e^{-S_{dS}} \]
- EFT have no problem in describing local physics
- but **EFT breaks down** for global **IR** quantities
  which are sensitive to **non-perturbative** effects
Applying the black hole lesson...

\[ N \sim M_{Pl}^2 R_S^2 \sim S_{bh} \]

\[ t \sim M_{Pl}^2 R_S^3 \sim S_{bh} R_S \]

\( N \)-point function: when non-perturbative effects become of \( O(1) \)

EFT breaks down when \# d.o.f. seen start becoming larger than \( S_{bh} \).
applying the black hole lesson...

**N-point function:** when non-perturbative effects become of \( O(1) \)

\[
N \sim M_{Pl}^2 R_S^2 \approx S_{bh}
\]

EFT breaks down when # d.o.f. seen start becoming larger than \( S_{bh} \)

...to de Sitter

Using inflation as a regulator of de Sitter

After inflation one observer can see \( \sim e^{3N} \) independent Hubble patches

\[
e^{3N} \text{ should be bounded by } e^{S_{dS}}
\]

\[
N \lesssim S_{dS} \quad \Rightarrow \quad t \lesssim S_{dS} H^{-1}
\]
indeed...

In any theory of inflation (satisfying the NEC)

\[ \delta \phi_q \lesssim \delta \phi_{cl} \Rightarrow \frac{\dot{H} M^2_{Pl}}{H^4} \gtrsim 1 \Rightarrow \frac{dS_{ds}}{dN} \gtrsim 1 \]

\[ N \lesssim S_{ds} \]

no eternal inflation

Arkani-Hamed et al. 07
In any theory of inflation (satisfying the NEC)

\[ \delta \phi_q \lesssim \delta \phi_{cl} \Rightarrow \frac{\dot{H} M^2_{Pl}}{H^4} \gtrsim 1 \Rightarrow \frac{dS_{dS}}{dN} \gtrsim 1 \]

\[ N \lesssim S_{dS} \]

Is there a "sharp" bound?

1) It exists a sharp bound for the phase transition to slow-roll eternal inflation

\[ \Omega \equiv \frac{2\pi^2}{3} \frac{\dot{\phi}^2}{H^4} > 1 \]

\[ N_c < \frac{1}{12} S_{dS} \]
1) It exists a sharp bound for the phase transition to slow-roll eternal inflation

\[ \delta \phi_q \lesssim \delta \phi_{el} \Rightarrow \frac{\dot{H} M^2_{Pl}}{H^4} \gtrsim 1 \Rightarrow \frac{dS_{dS}}{dN} \gtrsim 1 \]

\[ N \lesssim S_{dS} \]

Arkani-Hamed et al. 07

2) Quantum fluctuations require to look at the probability distribution of \( N \)

\[ \Omega \equiv \frac{2 \pi^2}{3} \frac{\dot{\phi}^2}{H^4} > 1 \]

\[ N_c < \frac{1}{12} S_{dS} \]

Creminelli et al. 08

Is there a "sharp" bound?

In any theory of inflation (satisfying the NEC)

no eternal inflation

Indeed...
The probability distribution of the Volume of the Universe after Inflation:

\[ \rho(V; \phi) \]

probability that starting with the inflaton at the position \( \phi \) (and a volume \( V_0 = H^{-3} \)) the Universe will have a finite volume \( V \) at \( t \to \infty \), or equivalently that the reheating surface will have volume \( V \).
Slow Roll Inflation as a Bacteria Model

quantum fluctuations

end of inflation

random walk

dead bacteria
Slow Roll Inflation as a Bacteria Model

quantum fluctuations

end of inflation

# of dead bacteria = # Hubble patches

random walk
dead bacteria
From the Bacteria to the Fokker-Planck equation

\[ P(j, n + 1) = (1 - p)P(j - 1, n) + p P(j + 1, n) \]
From the Bacteria to the Fokker-Planck equation

Matching bacteria with inflaton

\[ j = \frac{\phi}{\Delta \phi}, \quad n = \frac{t}{\Delta t}. \]

\[ N_r = e^{3H \Delta t} \simeq 1 + 3H \Delta t \]

Random Walk + Drift

\[ (1 - 2p) \frac{\Delta \phi}{\Delta t} = \dot{\phi} \quad \Rightarrow \quad p = \frac{1}{2} + \sqrt{6\pi^2 \Omega \frac{\Delta \phi}{H}}, \]

\[ P(j, n + 1) = (1 - p)P(j - 1, n) + p P(j + 1, n) \]
From the Bacteria to the Fokker-Planck equation

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Random Walk + Drift

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Taking the Continuum Limit

\[ (\Delta \phi)^2 = \frac{H^3}{4\pi^2 \Delta t} \]

\[ \frac{4\pi^2}{H^3} \partial_t P(\bar{\phi}, t) = \frac{1}{2} \partial_{\bar{\phi}}^2 P(\bar{\phi}, t) + \frac{2\sqrt{6\pi^2 \Omega}}{H} \partial_{\bar{\phi}} P(\bar{\phi}, r), \]
...taking into account the replication: the generating function

\[ f^{(n)}_i(s_j) = \sum_{k_1 \ldots k_L} p^{(n)}_{i;k_0 \ldots k_L} s_0^{k_0} \cdots s_L^{k_L} \]

probability to have after \( n \)-steps

\( k_0 \) bact. in site 0

\( k_1 \) bact. in site 1

e tc..

starting from site \( i \)
...taking into account the replication: the generating function

\[ f_i^{(n)}(s_j) = \sum_{k_1...k_L} p_{i,k_0...k_L}^{(n)} s_0^{k_0} \cdots s_L^{k_L}, \]

where \( i \) is the starting point.

Probability to have after \( n \)-steps:
- \( k_0 \) bact. in site 0
- \( k_1 \) bact. in site 1
- etc..

Starting from site \( i \)

1-step generating function:

\[
\begin{align*}
 f_0^{(1)}(s_0, \ldots, s_L) &= s_0, \\
 f_1^{(1)}(s_0, \ldots, s_L) &= (1 - p)s_2 + p s_0)^N, \\
 & \vdots \\
 f_i^{(1)}(s_0, \ldots, s_L) &= (1 - p)s_{i+1} + p s_{i-1})^N, \\
 & \vdots \\
 f_L^{(1)}(s_0, \ldots, s_L) &= (1 - p)s_L + p s_{L-1})^N, 
\end{align*}
\]
...taking into account the replication: the generating function

\[ f_i^{(n)}(s_j) = \sum_{k_1 \ldots k_L} p_i^{(n)}_{k_1 \ldots k_L} s_0^{k_0} \ldots s_L^{k_L} , \]

where:
- \( i \) – starting point
- \( n \) steps
- \( s_0 \) bact. in site 0
- \( s_1 \) bact. in site 1
- etc..

1-step generating function

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\begin{align*}
 f_0^{(1)}(s_0, \ldots, s_L) &= s_0, \\
 f_1^{(1)}(s_0, \ldots, s_L) &= ((1 - p)s_2 + p s_0)^{N_r}, \\
 &\vdots \\
 f_i^{(1)}(s_0, \ldots, s_L) &= ((1 - p)s_{i+1} + p s_{i-1})^{N_r}, \\
 &\vdots \\
 f_L^{(1)}(s_0, \ldots, s_L) &= ((1 - p)s_L + p s_{L-1})^{N_r}.
\end{align*}
\]

Iterating \( n \)-times

\[ F_{n+1} = F_1(F_n) . \]

\( n \rightarrow \infty \) limit

\[ F_1(F_\infty) = F_\infty . \]

Asymptotic equations

\[
\begin{align*}
 f_0^{(\infty)}(s_0) &= s_0, \\
 &\vdots \\
 f_i^{(\infty)}(s_0) &= \left( (1 - p)f_{i+1}^{(\infty)}(s_0) + p f_{i-1}^{(\infty)}(s_0) \right)^{N_r}, \\
 &\vdots \\
 f_L^{(\infty)}(s_0) &= \left( (1 - p)f_L^{(\infty)}(s_0) + p f_{L-1}^{(\infty)}(s_0) \right)^{N_r}.
\end{align*}
\]
...performing the continuum limit

\[
\frac{1}{2} \frac{\partial^2}{\partial \phi^2} f^{(\infty)}(\phi; s_0) - \frac{2\pi \sqrt{6} \Omega}{H} \frac{\partial}{\partial \phi} f^{(\infty)}(\phi; s_0) + \frac{12\pi^2}{H^2} f^{(\infty)}(\phi; s_0) \log [f^{(\infty)}(\phi; s_0)] = 0 ,
\]

\[
f^{(\infty)}(0; s_0) = s_0 ,
\]

\[
\frac{\partial}{\partial \phi} f^{(\infty)}(\phi; s_0) \bigg|_{\phi_b} = 0 .
\]
...performing the continuum limit

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\frac{1}{2} \frac{\partial^2}{\partial \phi^2} f^{(\infty)}(\phi; s_0) - \frac{2\pi \sqrt{6\Omega}}{H} \frac{\partial}{\partial \phi} f^{(\infty)}(\phi; s_0) + \frac{12\pi^2}{H^2} f^{(\infty)}(\phi; s_0) \log \left[ f^{(\infty)}(\phi; s_0) \right] = 0 ,
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f^{(\infty)}(0; s_0) = s_0 ,
\]

\[
\left. \frac{\partial}{\partial \phi} f^{(\infty)}(\phi; s_0) \right|_{\phi_b} = 0 .
\]

\[
f_j^{(\infty)}(s_0) = \sum_{k=0}^{\infty} p_{j,k} s_0^k .
\]

\[
f^{(\infty)}(\phi; s_0) = \int_0^{\infty} dV \rho(\phi, V) s_0^V .
\]

\[
\rho(\phi, V) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d(-\log(s_0)) f^{(\infty)}(\phi; s_0) e^{-V \log(s_0)}
\]

\(k\) dead bacteria \(\Leftrightarrow\) reheating volume \(V\)
\[ f(\tau; z) \equiv f^{(\infty)}(\phi; s_0), \]
\[ \tau \equiv 2\pi \sqrt{\frac{6}{H}} = 6\sqrt{\Omega} N_c, \]
\[ z \equiv -\log(s_0), \]

\[ \ddot{f}(\tau; z) - 2\sqrt{\Omega} \dot{f}(\tau; z) + f(\tau; z) \log[f(\tau; z)] = 0, \]

\[ f(0; z) = s_0 = e^{-z}, \]
\[ \dot{f}(\tau_b; z) = 0, \]
The Mechanical Problem

\[ f(\tau; z) \equiv f^{(\infty)}(\phi; s_0), \]
\[ \tau \equiv 2\pi \sqrt{\frac{6}{H}} = 6 \sqrt{\Omega N_c}, \]
\[ z \equiv -\log(s_0), \]

\[ \ddot{f}(\tau; z) - 2\sqrt{\Omega} \dot{f}(\tau; z) + f(\tau; z) \log[f(\tau; z)] = 0, \]

\[ f(0; z) = s_0 = e^{-z}, \]
\[ \dot{f}(\tau_b; z) = 0, \]

\[ U(f) = \frac{f^2}{4} \left( \log f^2 - 1 \right) \]

\[ \rho(V, \tau) = \frac{1}{2\pi i} \int_{0+2\pi i}^{0+i\infty} dz f(\tau; z) e^{zV}, \]
Extinction Probability and Eternal Transition

\[ P_{\text{ext}} \equiv \int_0^\infty dV \rho(V, \tau) = f(\tau; 0). \]
Extinction Probability and Eternal Transition

\[
P_{\text{ext}} \equiv \int_0^\infty dV \rho(V, \tau) = f(\tau; 0).
\]

\[
\ddot{f} - 2\sqrt{\Omega} \dot{f} + f - 1 = 0,
\]

linearized approx.

\[
f = 1 - e^{\sqrt{\Omega} \tau} \left( Ae^{\sqrt{\Omega - 1} \tau} + Be^{-\sqrt{\Omega - 1} \tau} \right).
\]
Extinction Probability and Eternal Transition

\[ P_{\text{ext}} \equiv \int_0^\infty dV \rho(V, \tau) = f(\tau; 0). \]

\[ f(\tau; z) \]

\[ \dot{f} - 2\sqrt{\Omega} \frac{df}{d\tau} + f - 1 = 0, \]

linearized approx.

\[ f = 1 - e^{\sqrt{\Omega}\tau}\left(Ae^{\sqrt{\Omega-1}\tau} + Be^{-\sqrt{\Omega-1}\tau}\right). \]
Calculating the Moments of the distribution

\[
\langle V^n \rangle = \int_0^\infty dV \, V^n \rho(V, \tau) = (-1)^n \frac{\partial^n f(\tau; z)}{\partial z^n} \bigg|_{z=0}
\]
Calculating the Moments of the distribution

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\]

\[
\ddot{f}' - 2\sqrt{\Omega} \dot{f}' + f' + f' \log f = 0,
\]

\[
f_0'(0) = -1, \quad \dot{f}_0'(\tau_b) = 0.
\]
Calculating the Moments of the distribution

\[ \langle V^n \rangle = \int_0^\infty dV \ V^n \rho(V, \tau) = (-1)^n \frac{\partial^n f(\tau; z)}{\partial z^n} \bigg|_{z=0} \]

\[ \ddot{f'} - 2\sqrt{\Omega} \dot{f'} + f' + f' \log f = 0, \quad f'_0(0) = -1, \quad \dot{f'}_0(\tau_0) = 0. \]

\[ \omega_\pm \equiv \sqrt{\Omega} \pm \sqrt{\Omega - 1} \]

\[ \langle V \rangle = -f'_0(\tau) = \frac{e^{\omega_+ \tau + \omega_- \tau_0} - \omega_+^2 e^{\omega_+ \tau + \omega_+ \tau_0}}{e^{\omega_- \tau_0} - \omega_-^2 e^{\omega_+ \tau_0}} \]

\[ \lim_{\tau_0 \to -\infty} \langle V \rangle = e^{\omega_0 - \tau_0} = e^{3Nc \frac{2}{1 + \sqrt{1 - 1/\Omega}}}. \]

\[ <V> \]

\[ e^{6Nc} \]

\[ e^{3Nc} \]

\[ 0 \]

\[ \Omega \]
\[
\langle V^2 \rangle = f''_0(\tau) = \frac{\omega_+^6 e^{2\tau + 2\tau_b \omega_+}}{(\omega_+^2 - 2) (e^{\tau_b/\omega_+} - e^{\tau_b \omega_+ + \omega_+^2})^2} - \frac{2\omega_+^4 e^{2\tau_b \omega_+} \left( e^{\frac{\tau_b}{\omega_+} + \tau \omega_+} - e^{\frac{\tau}{\omega_+} + \tau_b \omega_+ + \omega_+^2} \right)}{(\omega_+^2 - 2) (e^{\tau_b/\omega_+} - e^{\tau_b \omega_+ + \omega_+^2})^3} \\
+ \frac{4\omega_+^2 e^{\omega_+ + \tau_b + \frac{\tau_b}{\omega_+}} \left( e^{\frac{\tau_b}{\omega_+} + \tau \omega_+} - e^{\frac{\tau}{\omega_+} + \tau_b \omega_+ + \omega_+^2} \right)}{(e^{\tau_b/\omega_+} - e^{\tau_b \omega_+ + \omega_+^2})^3} + \frac{2\omega_+^2 e^{2\tau_b + \frac{2\tau_b}{\omega_+}} \left( e^{\tau \omega_+} - e^{\tau/\omega_+} \right) (\omega_+^2 - 1)^2 (\omega_+^2 + 1)}{(e^{\tau_b/\omega_+} - e^{\tau_b \omega_+ + \omega_+^2})^3 (2\omega_+^4 - 5\omega_+^2 + 2)} \\
- \frac{2\tau_b}{e^{\omega_+ + 2\tau \omega_+}} \frac{2\omega_+^2}{(2\omega_+^2 - 1) (e^{\tau_b/\omega_+} - e^{\tau_b \omega_+ + \omega_+^2})^2},
\]
\[ \langle V^2 \rangle = f''(\tau) = \frac{\omega_+^6 e^{2\tau_0 + 2\tau_+}}{(\omega_+^2 - 2) (e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2})^2} - \frac{2\omega_+^4 e^{2\tau_0 \omega_+} \left( e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2} \right)}{(\omega_+^2 - 2) (e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2})^3} \]

\[ + \frac{4\omega_+^2 e^{\omega_+ + \tau_0 + \tau_+} \left( e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2} \right)}{(e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2})^3} \]

\[ + \frac{8\omega_+^2 e^{2\omega_+ \tau_0 + 2\tau_0} (e^{\tau_0 / \omega_+} - e^{\tau_0 / \omega_+}) (\omega_+^2 - 1)^2 (\omega_+^2 + 1)}{(e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2}) (2\omega_+^4 - 5\omega_+^2 + 2)} + \frac{2\omega_+^2 e^{\omega_+ + \tau_0 + \tau_0 + \tau_0} \left( e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2} \right)}{(e^{\tau_0 / \omega_+} - e^{\tau_0 \omega_+ + \omega_+^2})^2} \]

\[ \langle V^2 \rangle \xrightarrow{\tau_0 \gg 1} \frac{\omega_+^2}{\omega_+^2 - 2} \left( 1 - 2 \frac{e^{\omega_+ - \tau_0}}{\omega_+^2} \right) e^{2\omega_+ - \tau_0} + \frac{8(\omega_+^2 - 1)^2 (\omega_+^2 + 1)}{\omega_+^4 (2\omega_+ - 1)(2 - \omega_+^2)} e^{-(\omega_+^2 - 2)\omega_+ - \tau_0 + \omega_+ + \tau_0} \]

...analogously for higher moments

for \( \tau_0 \to \infty \) the \( n \)-th moment diverges at

\[ \Omega = \frac{(n + 1)^2}{4n} \]
Reconstructing $\rho(\phi;V)$

$\Omega > 1$ case

\[ f \approx f_g = e^{-\frac{(\tau + \tau_1)^2}{4}}, \]

\[ f_{\text{lin}} \approx 1 - e^{\omega_-(\tau + \tau_0)}. \]
Reconstructing $\rho(\phi;V)$

$\Omega > 1$ case

\[ f \approx f_g = e^{-\frac{(\tau+\tau_1)^2}{4}}, \]

\[ f_{\text{lin}} \approx 1 - e^{\omega_-(\tau+\tau_0)}. \]

\[ \rho(V, \tau) \approx \mathcal{N}e^{-\frac{1}{4}\Omega \left(1 + \sqrt{1 - \frac{1}{\Omega}}\right)^2 \left[\log\left(\frac{V}{\mathcal{V}}\right)\right]^2} = \mathcal{N}e^{-\Omega \left[\frac{3N}{2} \left(1 + \sqrt{1 - \frac{1}{\Omega}}\right) - 3N\right]^2}, \quad V \lesssim \mathcal{V} \]

for $V \gtrsim \mathcal{V}^{\frac{\omega_+}{\omega_-}}$
Reconstructing $\rho(\phi;V)$: $\Omega = 1-\epsilon$ case, the phase transition

$$f_{\text{lin}}(\tau; z) = 1 - \sigma e^{\sqrt{\Omega}(\tau + \tau_0)} \cos \left( \sqrt{\Omega - 1}(\tau + \tau_0) \right) \approx 1 - \sigma e^{\tau + \tau_0} \cos \left( \sqrt{\epsilon}(\tau + \tau_0) \right)$$
Reconstructing $\rho(\phi;V)$: $\Omega = 1 - \epsilon$ case, the phase transition

\[ f_{\text{lin}}(\tau; z) = 1 - \sigma e^{\sqrt{\Omega(\tau + \tau_0)}} \cos \left( \frac{\sqrt{\Omega - 1(\tau + \tau_0)}}{\sqrt{\epsilon(\tau + \tau_0)}} \right) \approx 1 - \sigma e^{\tau + \tau_0} \cos \left( \sqrt{\epsilon(\tau + \tau_0)} \right) \]

\[ V_{\epsilon} \equiv e^{\frac{\pi}{2\sqrt{\epsilon}}} \]

\[ 1 < \Omega \lesssim 1 - \left( \frac{\pi}{6N} \right)^2 \]

\[ \Omega = 1 - \epsilon \quad (V_{\epsilon} > \bar{V}) \]

\[ V > V_{\epsilon} \quad V_{<\epsilon} < V \leq \bar{V} \]

\[ z_{\text{cut}} \sim -\frac{\sigma \sqrt{\epsilon}}{e V_{\epsilon}} \]

\[ \rho(V) \sim \frac{\bar{V}}{V^2} \quad \bar{V} < V < V_{\epsilon} \]

\[ \rho(V) \sim \frac{\bar{V}}{V_{\epsilon} V} e^{z_{\text{cut}} V} = \frac{\bar{V}}{V_{\epsilon} V} e^{-\frac{\sigma}{\epsilon} \sqrt{\epsilon V/V_{\epsilon}}} \quad V_{\epsilon} < V \]
\( \Omega = 1 - \varepsilon \) case: inside eternal inflation

\[ \rho(V, \tau) \approx N e^{-\frac{1}{4} \left( \frac{\tau - \frac{\pi}{2 \sqrt{c}}}{2 \sqrt{c}} \right)^2 - \varepsilon \sqrt{c} V/V_\varepsilon}, \quad V \gtrsim V_\varepsilon, \]
Special limits and cross-checks

The Classical Limit: $\Omega >> 1$

\[ \tau = 2\sqrt{\Omega \tilde{\tau}} \]
\[ \tilde{\tau} = 3N_c \]

\[ \frac{1}{4\Omega} \frac{\partial^2 f}{\partial \tau^2} - \frac{1}{\tilde{\tau}} \frac{\partial f}{\partial \tilde{\tau}} + f \log f = 0 \]

\[ f = e^{-z} e^{\theta} \]

\[ \rho(V, \tau) = \delta(V - e^{3N_c}) \]
Special limits and cross-checks

The Classical Limit: $\Omega >> 1$

$$\tau = 2\sqrt{\Omega \tilde{\tau}}$$
$$\tilde{\tau} = 3N_c$$

$$\frac{1}{4\Omega} \frac{\partial^2 f}{\partial \tau^2} - \frac{\partial f}{\partial \tau} + f \log f = 0$$

$$f = e^{-z} e^\tau$$

$$\rho(V, \tau) = \delta(V - e^{3N_c})$$

Deep inside Eternal Inflation: $\Omega \rightarrow 0$

$$f(\tau; z) = e^{\frac{1}{2} - \frac{1}{4} (\tau + \sqrt{2 + 4z})^2}$$

$$\rho(V, \tau) = \frac{\tau}{\sqrt{4\pi (V-1)^{3/2}}} e^{-\frac{V-1}{2} - \frac{V}{V-1} \frac{\pi^2}{4}}$$

$$\int_0^\infty dV \rho(V, \tau) = e^{-\frac{\tau^2}{4} - \frac{\tau}{\sqrt{2}}} = f(\tau; 0)$$
Approximations, corrections and other effects

Finite Barrier Effects

a) Suppression of the large volume tail

\[ \rho(V) \propto \frac{1}{V\omega^2 + 1} \rightarrow e^{2cutV} \]
Approximations, corrections and other effects

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w/o barrier \hspace{1cm} \text{w/ barrier}

b) Shift of the phase transition point

\[ T = \frac{2\pi}{\sqrt{1 - \Omega}} \]

\[ \tau_b < T/4 \]

\[ \tau_b = \frac{\pi}{2\sqrt{1 - \Omega_c}} \Rightarrow \Omega_c = 1 - \left(\frac{\pi}{2\tau_b}\right)^2 \]
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Slow-Roll corrections

Slow-roll parameter

\[ \frac{\dot{H}}{H^2} \sim \Omega \frac{H^2}{M_{\text{Pl}}^2} \rightarrow 0 \]

Friedman equation

\[ M_{\text{Pl}}^2 H \Delta H \sim V' \Delta \phi \]

\[ \frac{\Delta H}{H} \sim \frac{V' \Delta \phi}{M_{\text{Pl}}^2 H^2} \sim \Omega \frac{H^2}{M_{\text{Pl}}^2} N_c \ll 1 \]

\[ \tau = 6 \int \sqrt{\Omega} dN_c \]

\[ \lim_{\tau_b \to \infty} \langle V \rangle = e^{\int \omega_- d\tau} \]

\[ 3N = \int \omega_- d\tau \quad \Rightarrow \quad \partial_\tau \Omega \ll \Omega \]
Summarizing the $\rho(V)$ shape

- Gaussian
- Power-law
- Exponential tail

$V_\epsilon$, $\bar{V}$, $V_b$, $e^{S_{as}}$, $V$
Summarizing the $\rho(V)$ shape

from gaussian distribution of random walk eq. at small times

$$e^{-\frac{(\phi - \bar{\phi})^2}{2\sigma^2}} \sim e^{-k(N - \overline{N})^2}$$
Summarizing the $\rho(V)$ shape

From Gaussian distribution of random walk eq. at small times:

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From exponential distribution of random walk eq. at large times:

$$e^{-k\frac{(\phi-\bar{\phi})^2}{\Delta\phi}} \sim e^{-k(N-\bar{N})}$$
Summarizing the $\rho(V)$ shape

- For small times, the distribution is from a Gaussian distribution of random walk equation:
  \[ e^{-\frac{(\phi - \bar{\phi})^2}{2\sigma^2}} \sim e^{-k(N - \bar{N})^2} \]

- For large times, the distribution is from an exponential distribution of random walk equation:
  \[ e^{-k\frac{\Delta\phi}{\Delta \phi}} \sim e^{-k(N - \bar{N})} \]

- The curve shows a power-law behavior in the middle range.

- The distribution has an exponential tail at large values of $V$.

- The barrier cut-off is denoted by $V_b$.
Summarizing the $\rho(V)$ shape

From Gaussian distribution of random walk eq. at small times:

$$e^{-(\phi-\bar{\phi})^2/2\sigma^2} \sim e^{-k(N-\bar{N})^2}$$

From exponential distribution of random walk eq. at large times:

$$e^{-k(\phi-\bar{\phi})^2/\Delta\phi} \sim e^{-k(N-\bar{N})}$$

Exp. suppressed prob. from barrier cut-off:

$$e^{-kV/V_b}$$

Exp. suppressed in eternal inflation regime.
Summarizing the $\rho(V)$ shape

From Gaussian distribution of random walk eq. at small times:
$$e^{-k\frac{(\phi - \bar{\phi})^2}{\Delta\phi^2}} \sim e^{-k(N - \bar{N})^2}$$

Power-law from exponential distribution of random walk eq. at large times:
$$e^{-k\frac{(\phi - \bar{\phi})^2}{\Delta\phi^2}} \sim e^{-k(N - \bar{N})}$$

Exp. suppressed prob. from barrier cut-off:
$$e^{-kV/V_b}$$

Exp. suppressed in eternal inflation regime:
$$V_b = e^{3N_b} < e^{S_{dS}/2}$$
Discussion

Two main results:

1) The probability distribution itself; non trivial informations on the phase transition to eternal inflation.

2) Confirmation of the bound at the quantum level; the bound can be made 'sharp' in the following sense:

The probability for slow-roll inflation to produce a finite volume larger than $e^{S_{\text{ds}}/2}$, where $S_{\text{ds}}$ is de Sitter entropy at the end of the inflationary stage, is suppressed below the uncertainty due to non-perturbative quantum gravity effects.
Discussion

Two main results:

1) The probability distribution itself; non trivial informations on the phase transition to eternal inflation.

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Open problems and further extensions:

- Does the value "½" posses a deeper meaning (e.g. as the ½ in the Page argument for black holes), is it universal?
- Is the result robust against modification of the setting: - multi field inflation (more species) - non slow-roll inflation - different number of dimensions - etc...
- Is the bound associated with complementarity? How?