5th Workshop on PARTICLE PHYSICS AND COSMOLOGY - Warsaw - Feb. 4 -7, 2009

The Volume of the Universe after Inflation and de Sitter Entropy

Giovanni Villadoro
(CERN)

0812.2246 [hep-th] - with S. Dubovsky and L. Senatore

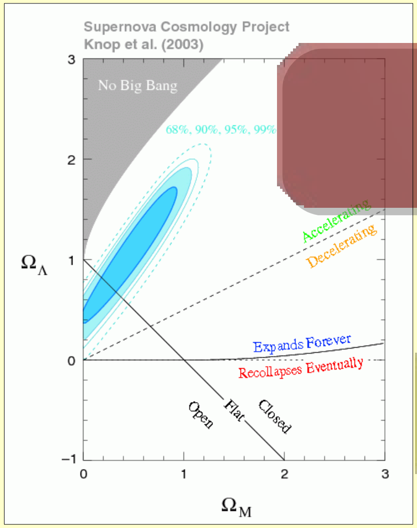
Outline

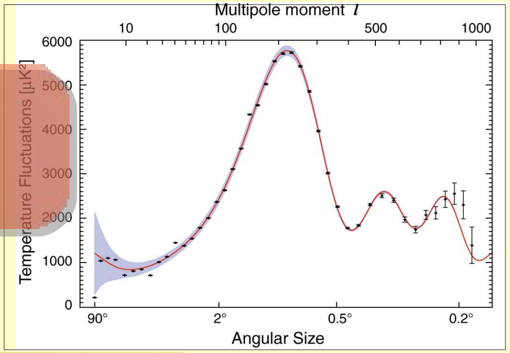
- **Basics of inflation and eternal inflation**

- **Quantum Gravity and de Sitter space**
  - Analogies and differences w/ black hole physics
  - An *'holographic'* bound
  - Definition of the probability distribution of the volume

- **Calculating $\rho(V)$**
  - From inflation to bacteria
  - From bacteria back to inflation: *a non-linear differential eq.*
  - Solving the equation
  - Results, systematics and corrections

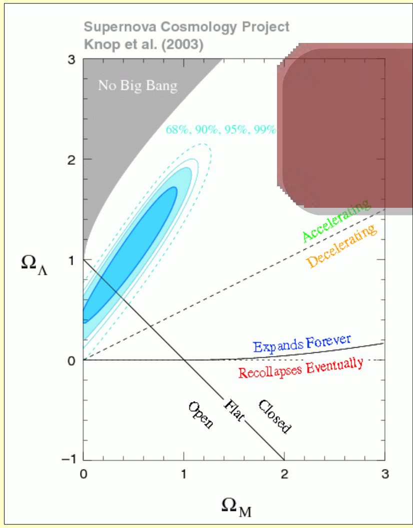
- **Discussion**





The Universe is accellerating today…

…and probably it did so also in the past



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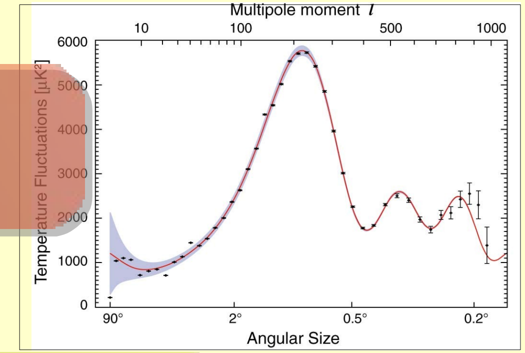

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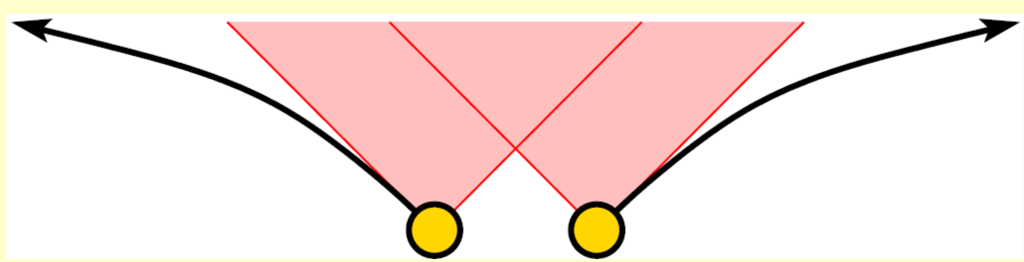
Standard Solution

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} = \Lambda g_{\mu\nu} + \dots$$

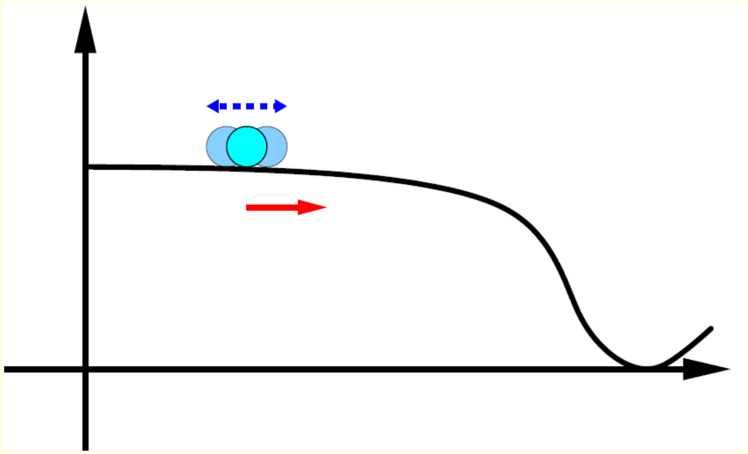
CC contribution dominates

$$ds^2 \simeq -dt^2 + e^{2Ht} d\vec{x}^2$$

approx. de Sitter



Slow roll inflation...

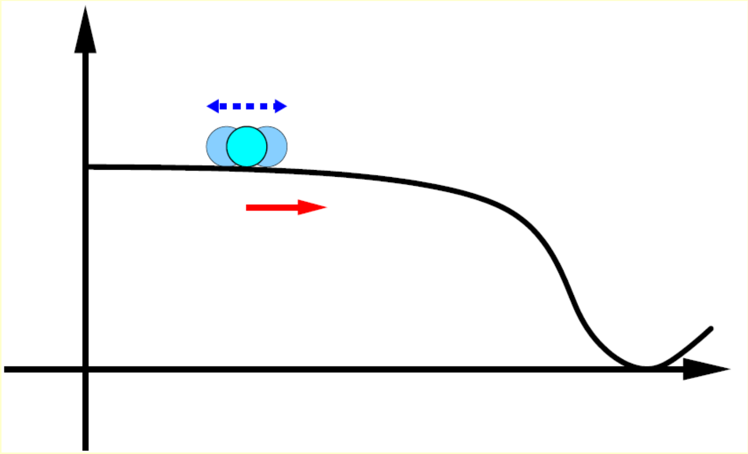


$$\Delta t = H^{-1}$$

every 1 e-folding

$$a(t) = e^{H\Delta t} = e$$

Slow roll inflation...



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Hubble Expansion

$$V(t+\Delta t) = V(t)e^3$$

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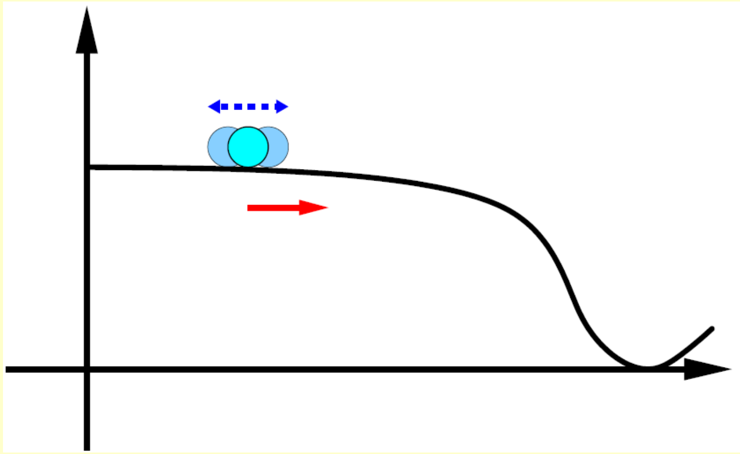

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Random Walk

$$\frac{\partial P(\phi,t)}{\partial t} = \frac{H^3}{8\pi^2}\frac{\partial^2 P(\phi,t)}{\partial \phi^2} + \dot{\phi}\frac{\partial P(\phi,t)}{\partial \phi}$$

Starobinsky, Linde 86

Quantum Fluctuations

$$\langle \delta\phi^2 \rangle = \frac{H^3}{4\pi^2}\Delta t \simeq H^2$$

Vilenkin, Ford, Linde, Starobinsky 82

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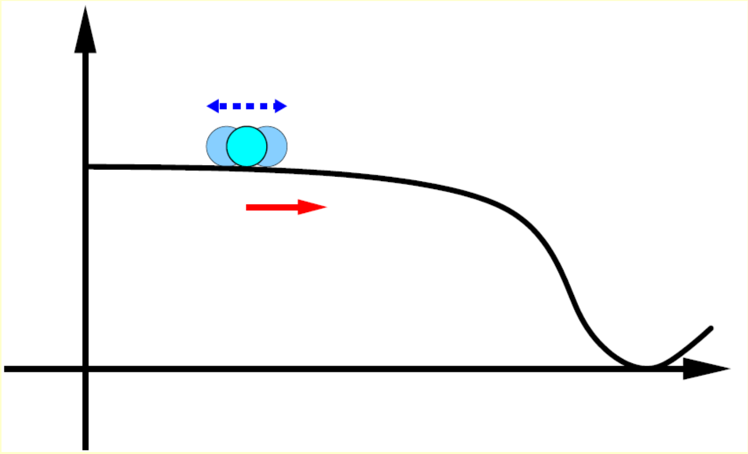

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...and Eternal Inflation

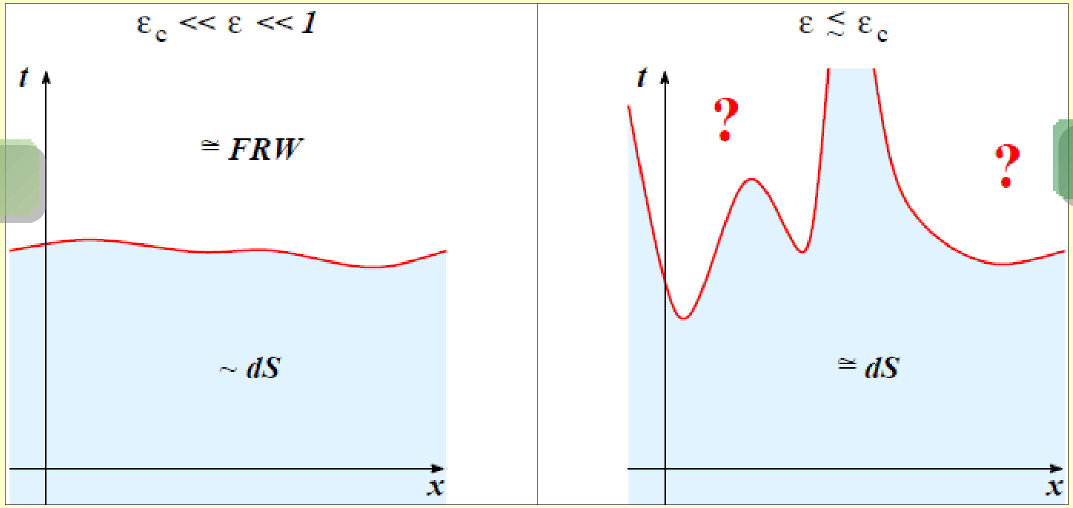
i.e. when quantum fluctuations win against classical rolling inflation never ends globally

Linde, Goncharov, Mukhanov 86-87

$$\delta\phi_q \gtrsim \delta\phi_{cl} = \dot{\phi}\Delta t = \frac{\dot{\phi}}{H} \quad \Rightarrow \quad \frac{\dot{\phi}}{H^2} \lesssim 1$$

Slow roll, eternal inflation and de Sitter

slow roll



eternal inflation

Slow roll, eternal inflation and de Sitter

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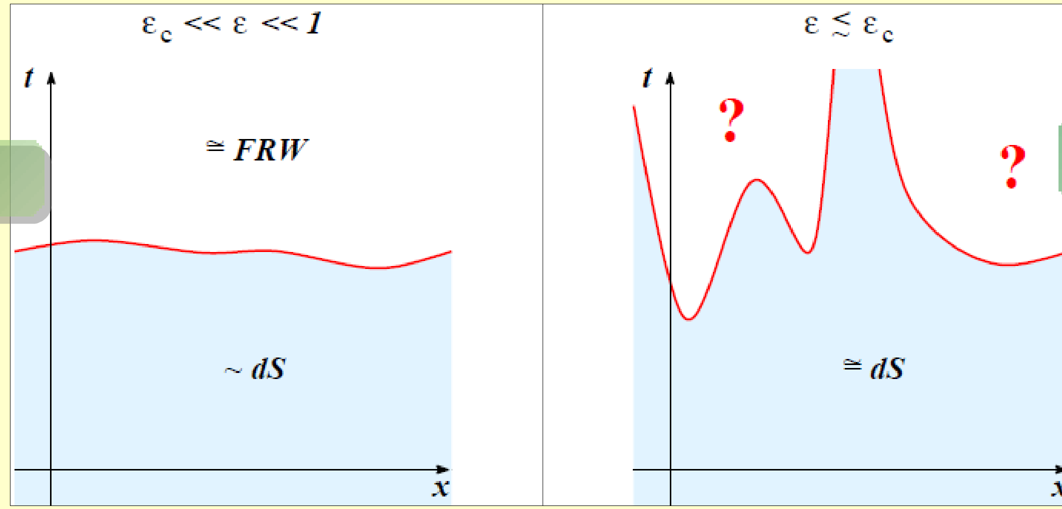
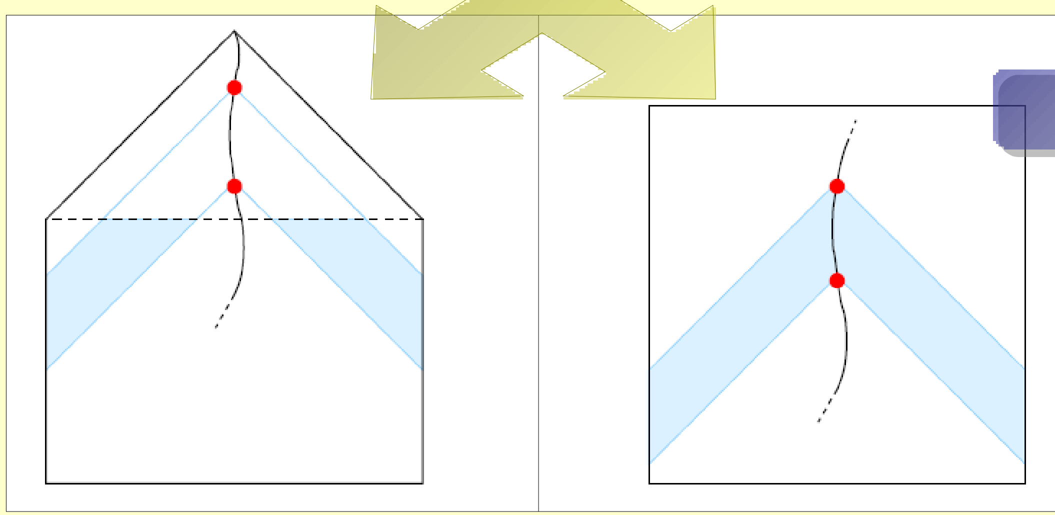


eternal inflation

the difference is fundamental for defining quantum gravity



de Sitter

(Quantum) Gravity in de Sitter space

Quantum Gravity is non-local at the non-perturbative level

⇐

from:
metric fluctuates,
Bekenstein bound,
black hole physics...

(Quantum) Gravity in de Sitter space

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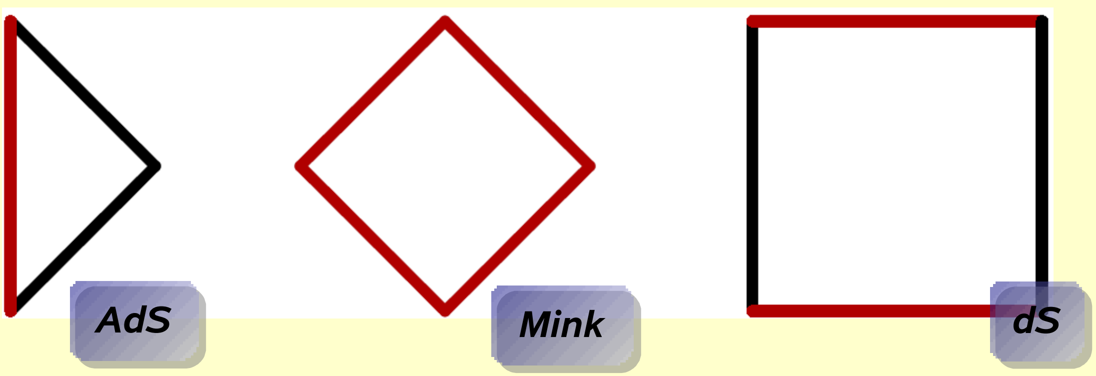
indeed known description of QG are defined on boundaries:
String Theory via S-matrix on Mink
AdS/CFT defined on the boundary of AdS

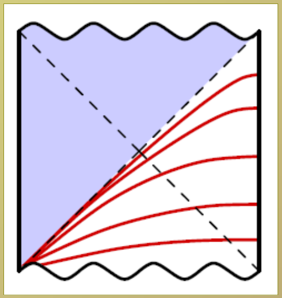


Troubles with de Sitter

<u>**Analogies with black holes...**</u>

- **horizon physics**, finite temperature, Hawking radiation...
- and in particular a **finite Entropy (S=$A/4$)**
- Metastability (CdL, HM, Poicare recurrence)

$$\Gamma > e^{-S_{dS}}$$



<u>**...and differences**</u>

- de Sitter is an infinite space (finite entropy?)
- eternal inflation never ends globally
- analogue of an information paradox?

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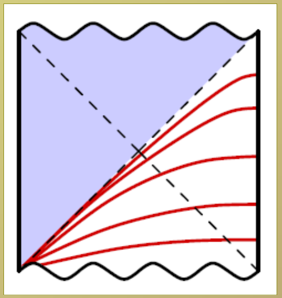

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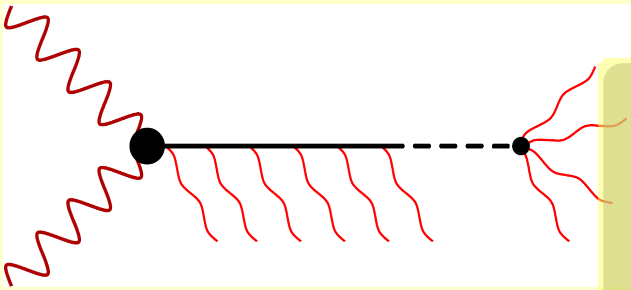
Lessons from black hole physics:

- Complementarity governs the global description of black-hole geometry
- AdS/CFT says that black hole evaporation process is **unitary**
- ⇒ it must be non local
- EFT have no problem in describing local physics
- but **EFT breaks down** for global **IR** quantities
  which are sensitive to ***non-perturbative*** effects

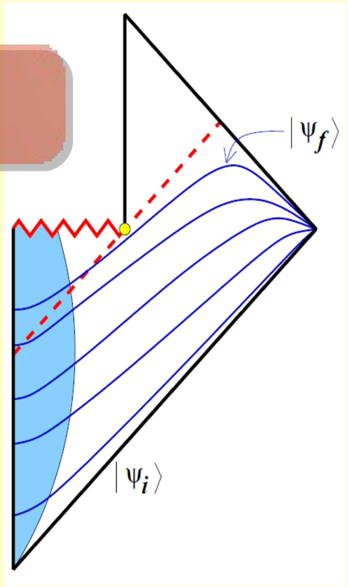
applying the black hole lesson...

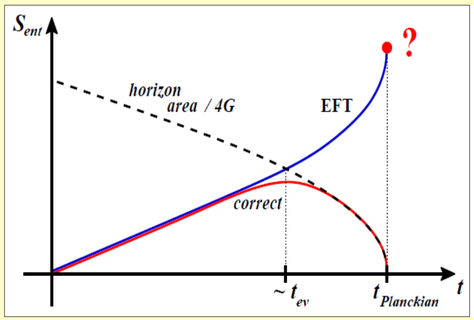


N-point function: when

$$N \sim M_{Pl}^2 R_S^2 \simeq S_{\rm bh}$$

non-perturbative effects become of $O(1)$

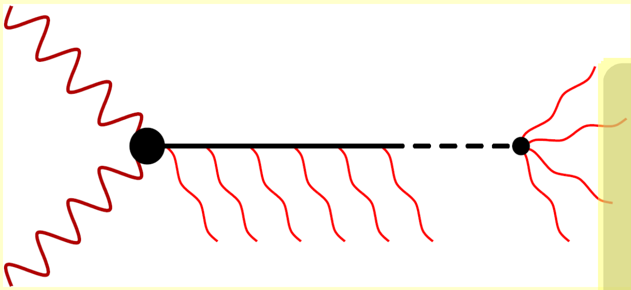




$$t \sim M_{Pl}^2 R_S^3 \simeq S_{\rm bh} R_S$$

EFT breaks down when # d.o.f. seen start becoming larger than S_{bh}

applying the black hole lesson...



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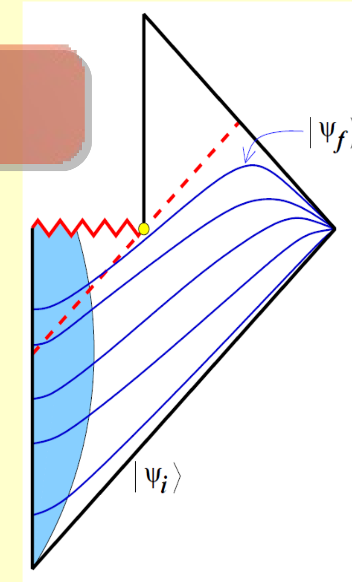

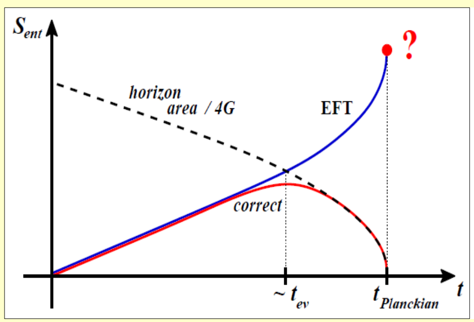

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...to de Sitter

Using inflation as a regulator of de Sitter

After inflation one observer can see ~e^{3N} independent Hubble patches

e^{3N} should be bounded by $e^{S_{ds}}$

$$N \lesssim S_{dS} \quad \Rightarrow \quad t \lesssim S_{dS} H^{-1}$$

indeed...

In any theory of inflation (satisfying the NEC)

no eternal inflation

$$\delta\phi_q \lesssim \delta\phi_{cl} \Rightarrow \frac{\dot{H} M_{Pl}^2}{H^4} \gtrsim 1 \Rightarrow \frac{dS_{dS}}{dN} \gtrsim 1$$

Arkani-Hamed et al. 07

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Is there a "sharp" bound?

1) It exists a sharp bound for the phase transition to slow-roll eternal inflation

Creminelli et al. 08

$$\Omega \equiv \frac{2\pi^2}{3} \frac{\dot{\phi}^2}{H^4} > 1$$

$$N_c < \frac{1}{12} S_{dS}$$

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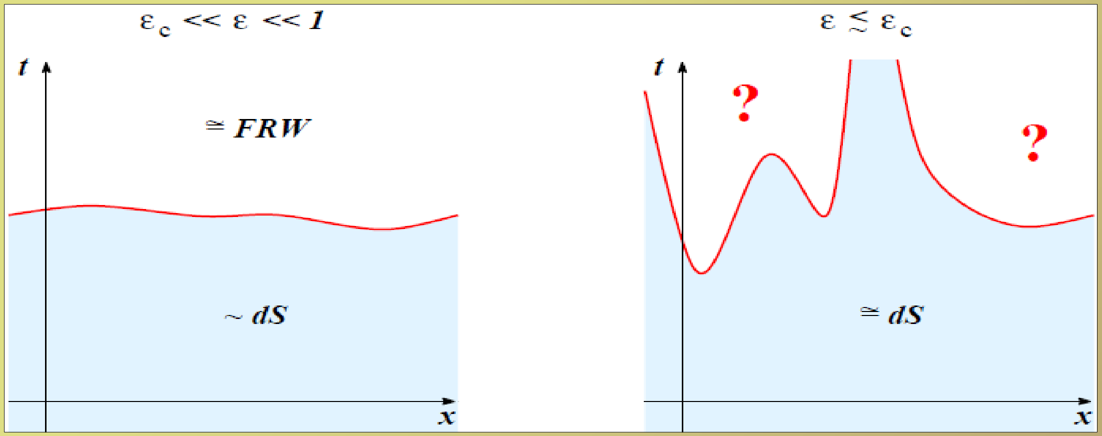
2) Quantum fluctuations require to look at the probability distribution of N

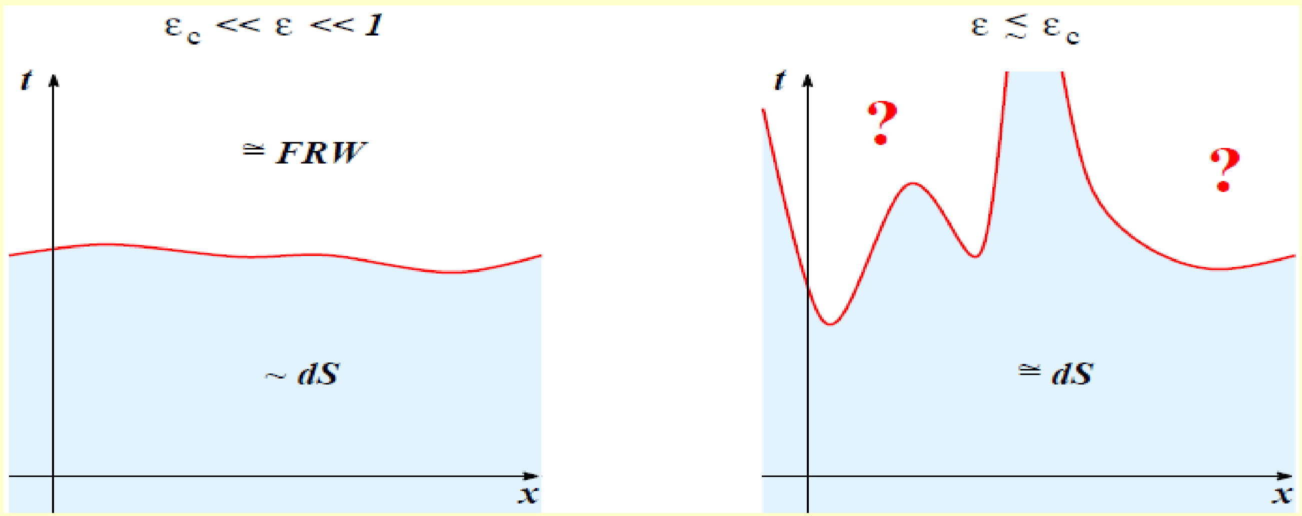


defining the "observable"

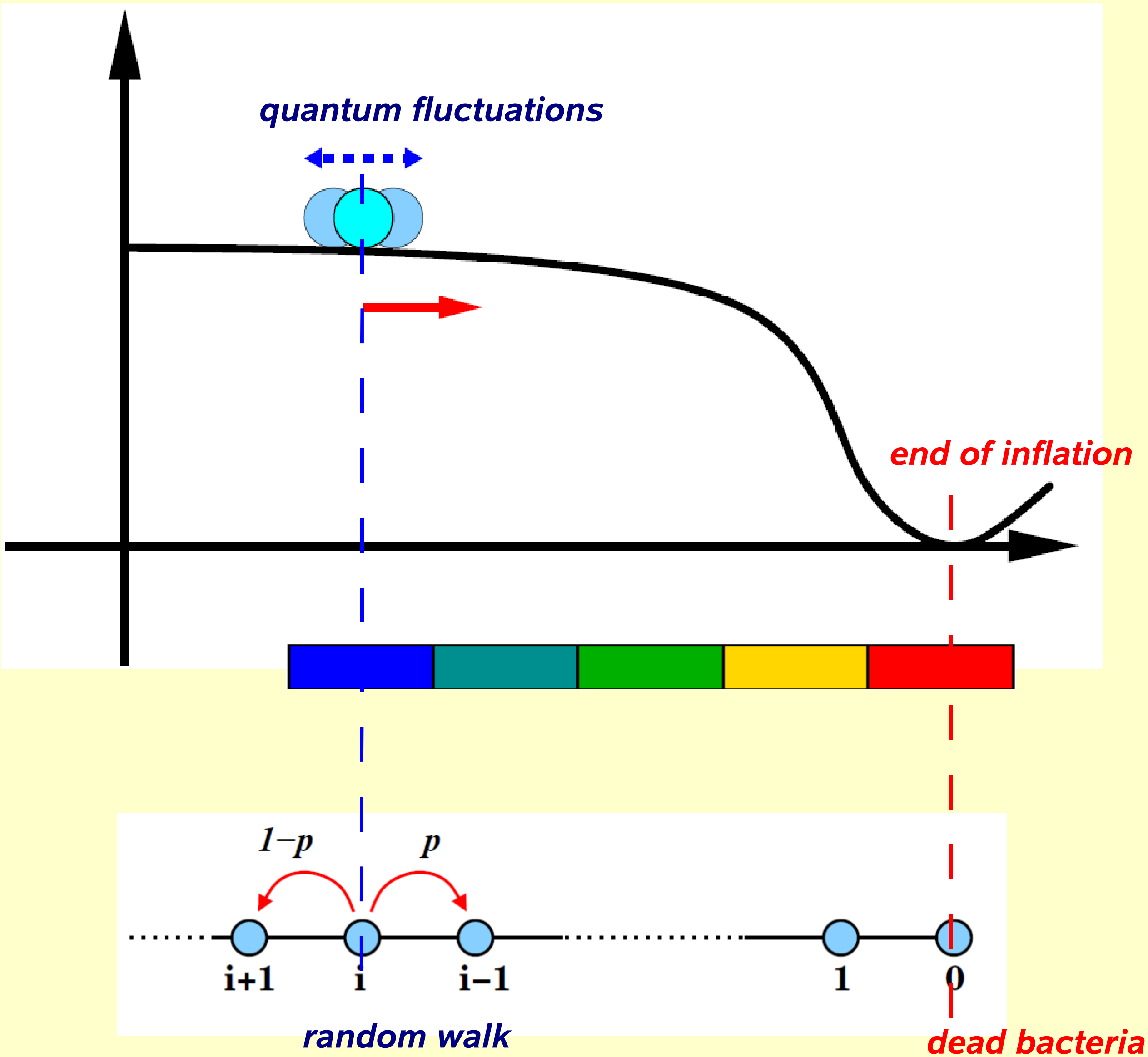
<u>The probability distribution of the Volume of the Universe after Inflation:</u>

$$\rho(V;\phi)$$

probability that starting with the inflaton at the position ϕ (and a volume $V_0 = H^{-3}$)
the Universe will have a finite volume V at $t \to \infty$,
or equivalently that the reheating surface will have volume V



Slow Roll Inflation as a Bacteria Model



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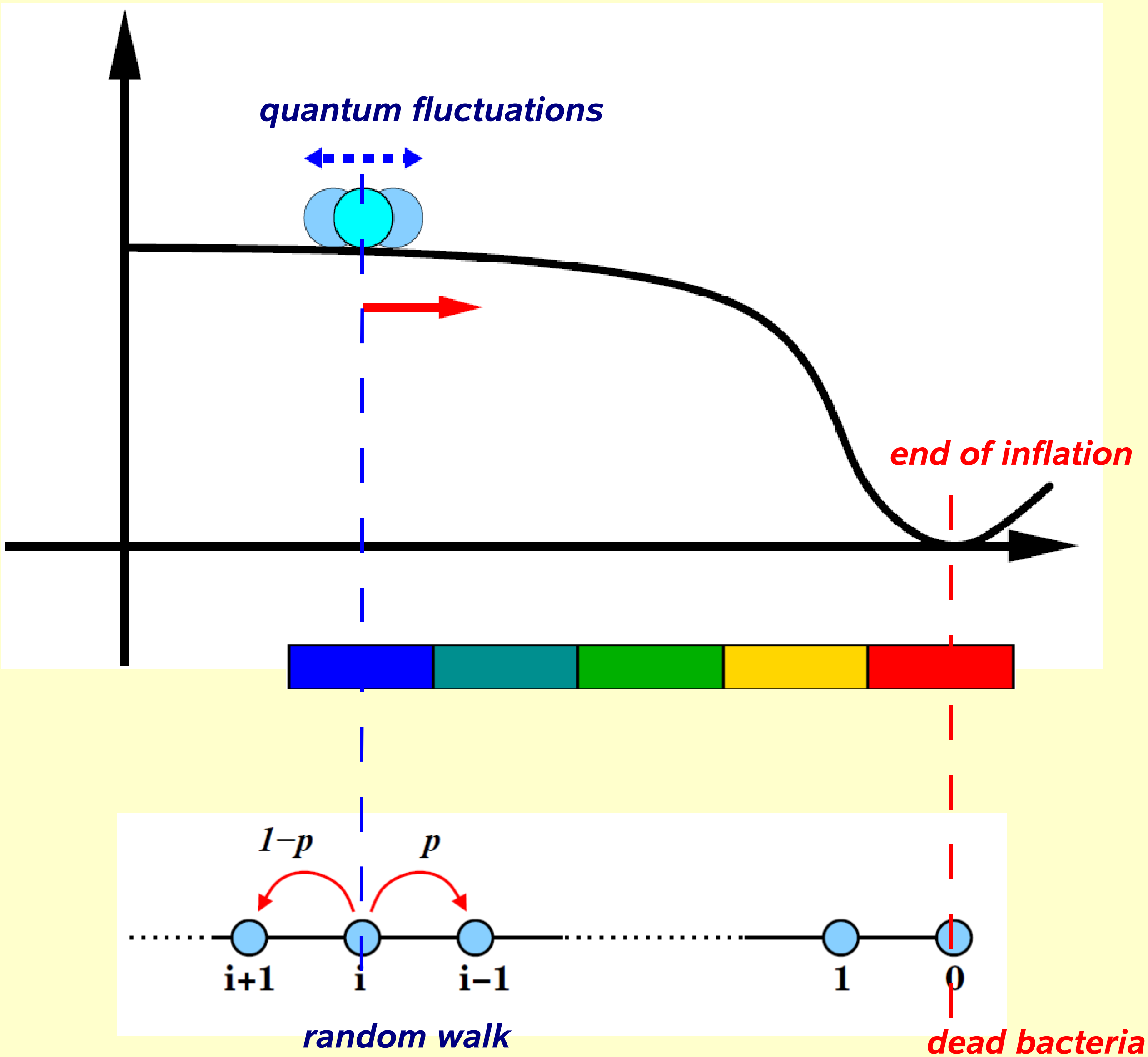
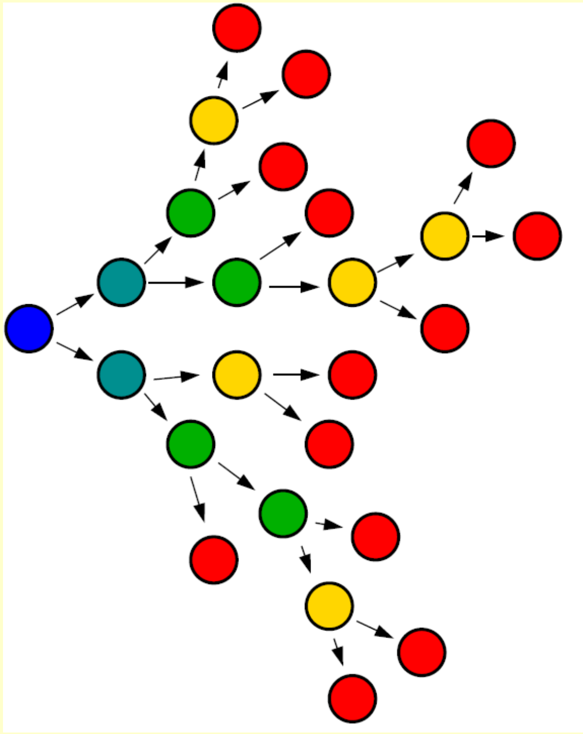
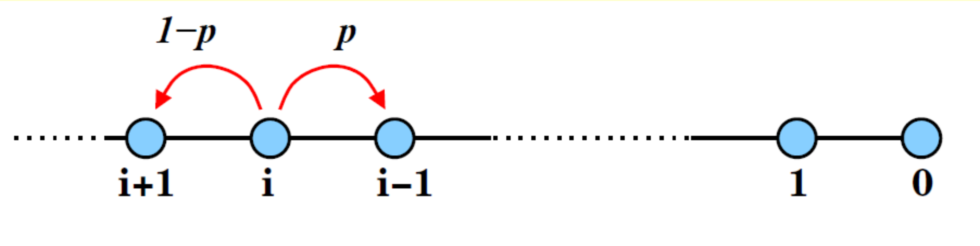




# of dead bacteria
=
Hubble patches

From the Bacteria to the Fokker-Planck equation



$$P(j, n+1) = (1-p)P(j-1, n) + p\, P(j+1, n)$$

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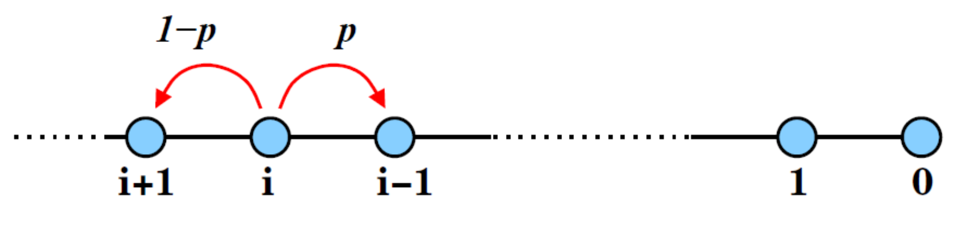

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Matching bacteria with inflaton

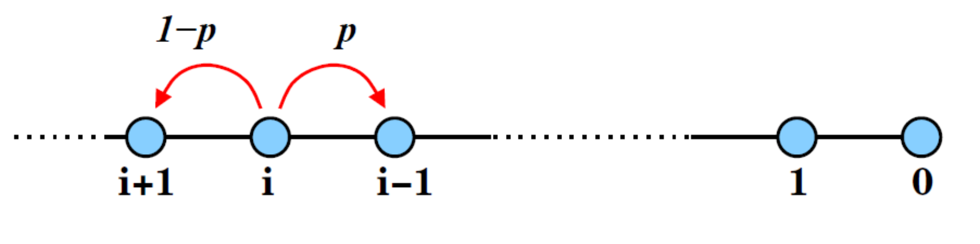
$$j = \frac{\phi}{\Delta\phi}\,, \qquad n = \frac{t}{\Delta t}\,.$$

$$N_r = e^{3H\Delta t} \simeq 1 + 3H\Delta t$$

Random Walk + Drift

$$(1-2p)\frac{\Delta\phi}{\Delta t} = \dot{\phi} \quad \Rightarrow \quad p = \frac{1}{2} + \sqrt{6\pi^2\Omega}\,\frac{\Delta\phi}{H}\,,$$

From the Bacteria to the Fokker-Planck equation



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Taking the Continuum Limit

$$(\Delta\phi)^2 = \frac{H^3}{4\pi^2}\Delta t$$

$$\frac{4\pi^2}{H^3}\partial_t P(\bar\phi,t) = \frac{1}{2}\partial^2_{\bar\phi}P(\bar\phi,t) + \frac{2\sqrt{6\pi^2\Omega}}{H}\partial_{\bar\phi}P(\bar\phi,r)\,,$$

...taking into account the replication: the generating function

i – starting point

$$f_i^{(n)}(s_j) = \sum_{k_1 \dots k_L} p^{(n)}_{i;k_0 \dots k_L} s_0^{k_0} \dots s_L^{k_L} \ ,$$

probability to have after n-steps
k_0 bact. in site 0
k_1 bact. in site 1
etc..
starting from site i

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1-step generating function

$$\begin{aligned}
f_0^{(1)}(s_0, \ldots, s_L) &= s_0 , \\
f_1^{(1)}(s_0, \ldots, s_L) &= ((1-p)s_2 + p\, s_0)^{N_r} , \\
&\vdots \\
f_i^{(1)}(s_0, \ldots, s_L) &= ((1-p)s_{i+1} + p\, s_{i-1})^{N_r} , \\
&\vdots \\
f_L^{(1)}(s_0, \ldots, s_L) &= ((1-p)s_L + p\, s_{L-1})^{N_r} ,
\end{aligned}$$

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\end{aligned}$$

iterating n-times

$$F_{n+1} = F_1(F_n) \,.$$

$n\to\infty$ limit

$$F_1(F_\infty) = F_\infty \,,$$

asymptotic equations

$$\begin{aligned}
f_0^{(\infty)}(s_0) &= s_0 \,, \\
&\vdots \\
f_i^{(\infty)}(s_0) &= \left((1-p)\, f_{i+1}^{(\infty)}(s_0) + p\, f_{i-1}^{(\infty)}(s_0)\right)^{N_r} \,, \\
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f_L^{(\infty)}(s_0) &= \left((1-p)\, f_L^{(\infty)}(s_0) + p\, f_{L-1}^{(\infty)}(s_0)\right)^{N_r} \,.
\end{aligned}$$

...performing the continuum limit

$$\frac{1}{2}\frac{\partial^2}{\partial\phi^2}f^{(\infty)}(\phi;s_0) - \frac{2\pi\sqrt{6\Omega}}{H}\frac{\partial}{\partial\phi}f^{(\infty)}(\phi;s_0) + \frac{12\pi^2}{H^2}f^{(\infty)}(\phi;s_0)\log\left[f^{(\infty)}(\phi;s_0)\right] = 0\ ,$$

$$\begin{aligned} f^{(\infty)}(0;s_0) &= s_0\ , \\ \frac{\partial}{\partial\phi}f^{(\infty)}(\phi;s_0)\bigg|_{\phi_b} &= 0\,. \end{aligned}$$

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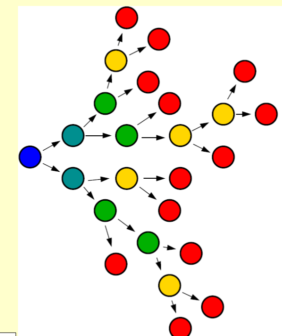

k dead bacteria
$\Leftrightarrow$
reheating volume V

$$f^{(\infty)}(\phi;s_0) = \int_0^\infty dV\,\rho(\phi,V)\,s_0^V\,.$$

$$\rho(\phi,V) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\left(-\log(s_0)\right) f^{(\infty)}(\phi;s_0)e^{-V\log(s_0)}$$

The Mechanical Problem

$$\begin{aligned} f(\tau; z) &\equiv f^{(\infty)}(\phi; s_0)\,, \\ \tau &\equiv 2\pi\sqrt{6}\frac{\phi}{H} = 6\sqrt{\Omega}N_c\,, \\ z &\equiv -\log(s_0)\,, \end{aligned}$$

$$\ddot{f}(\tau; z) - 2\sqrt{\Omega}\dot{f}(\tau; z) + f(\tau; z)\log[f(\tau; z)] = 0\,,$$

$$f(0; z) = s_0 = e^{-z}\,,$$

$$\dot{f}(\tau_b; z) = 0\,,$$

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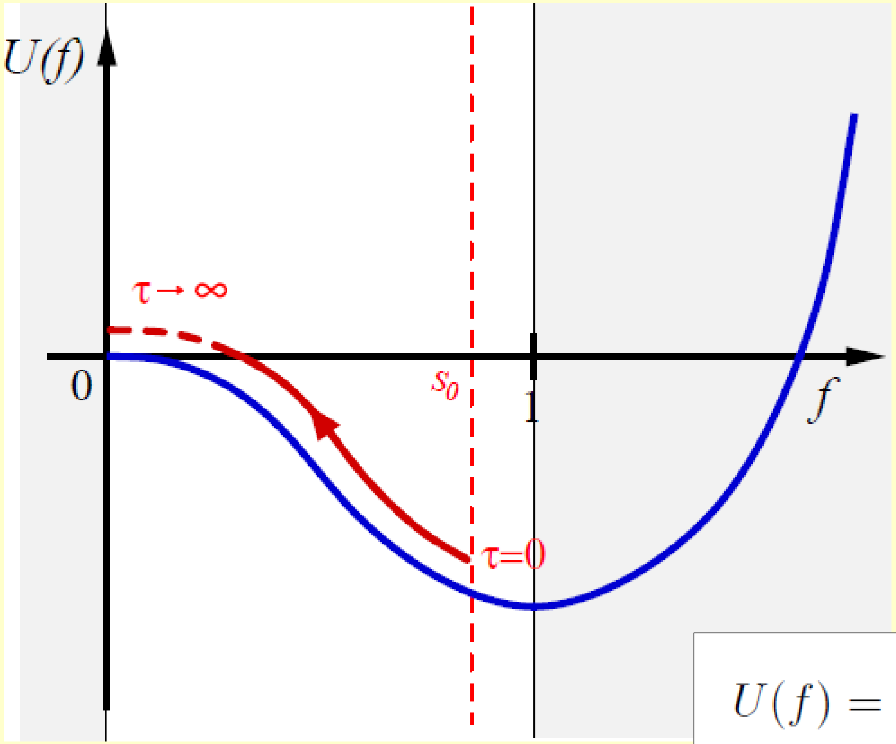

$$\rho(V,\tau) = \frac{1}{2\pi i}\int_{0^+-i\infty}^{0^++i\infty} dz\, f(\tau;z)e^{zV}\,,$$

$$U(f) = \frac{f^2}{4}\left(\log f^2 - 1\right)$$

Extinction Probability and Eternal Transition

$$P_{\mathrm{ext}} \equiv \int_0^{\infty} dV\, \rho(V,\tau) = f(\tau;0)\,.$$

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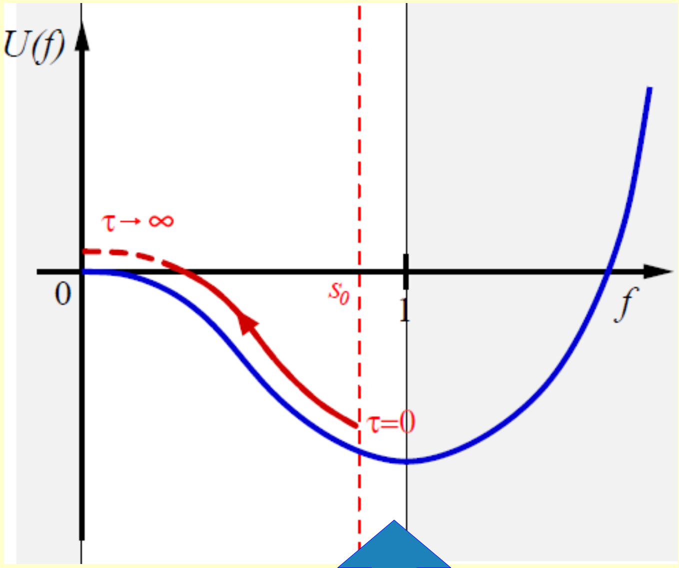

$$\ddot{f} - 2\sqrt{\Omega}\dot{f} + f - 1 = 0\,,$$

linearized approx.

$$f = 1 - e^{\sqrt{\Omega}\tau}\left(Ae^{\sqrt{\Omega-1}\tau} + Be^{-\sqrt{\Omega-1}\tau}\right).$$

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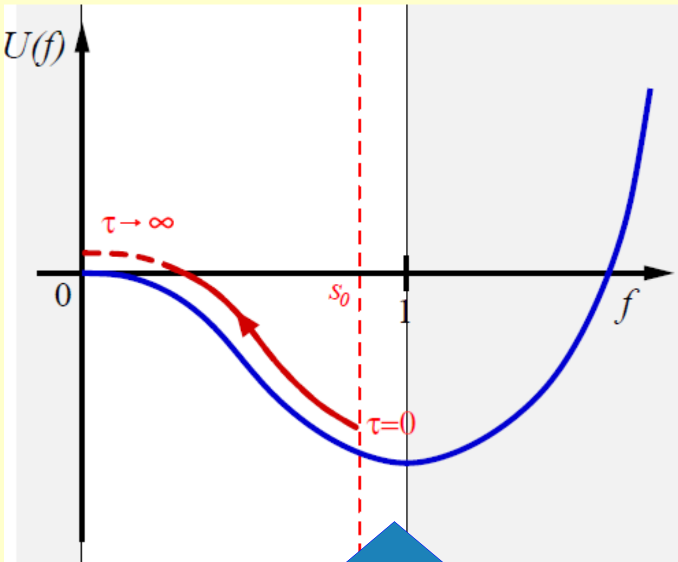

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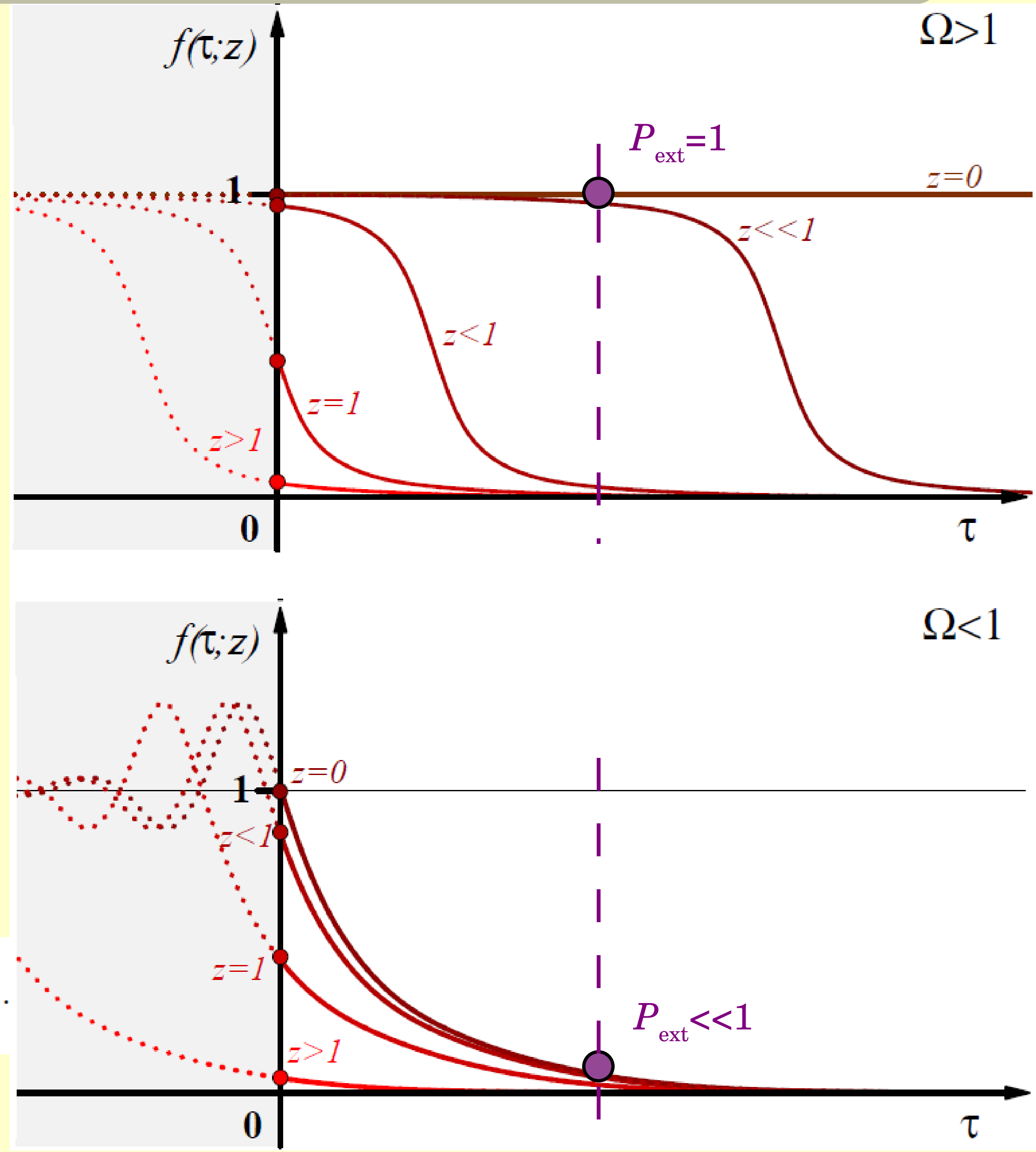

Calculating the Moments of the distribution

$$\langle V^n \rangle = \int_0^\infty dV\, V^n \rho(V,\tau) = (-1)^n \left. \frac{\partial^n f(\tau;z)}{\partial z^n} \right|_{z=0}$$

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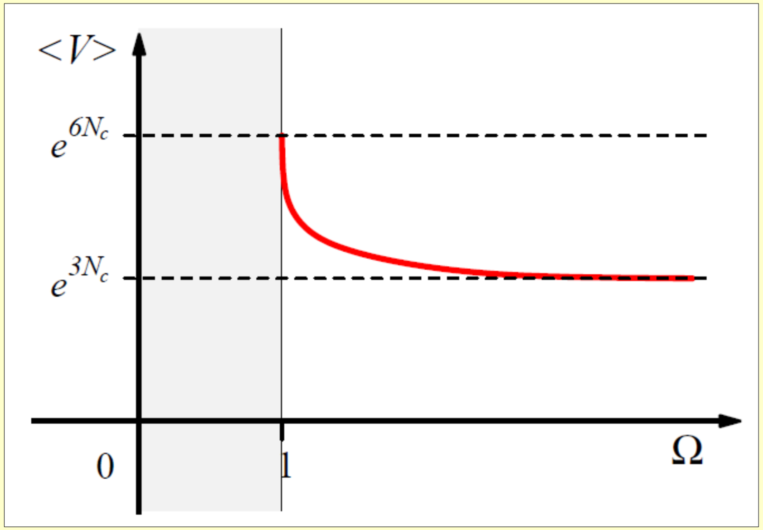
$$\omega_\pm \equiv \sqrt{\Omega} \pm \sqrt{\Omega - 1}$$

$$\langle V \rangle = -f_0'(\tau) = \frac{e^{\omega_+\tau+\omega_-\tau_b} - \omega_+^2 e^{\omega_-\tau+\omega_+\tau_b}}{e^{\omega_-\tau_b} - \omega_+^2 e^{\omega_+\tau_b}}$$

$$\lim_{\tau_b\to\infty} \langle V \rangle = e^{\omega_-\tau} = e^{3N_c \frac{2}{1+\sqrt{1-1/\Omega}}}\,.$$



...analogously for higher moments

$$
\begin{aligned}
\langle V^2 \rangle \;=\; & f_0''(\tau) = \frac{\omega_+^6 e^{\frac{2\tau}{\omega_+}+2\tau_b\omega_+}}{(\omega_+^2-2)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^2} - \frac{2\omega_+^4 e^{2\tau_b\omega_+}\left(e^{\frac{\tau_b}{\omega_+}+\tau\omega_+}-e^{\frac{\tau}{\omega_+}+\tau_b\omega_+}\omega_+^2\right)}{(\omega_+^2-2)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^3} \\
& - \frac{4\omega_+^2 e^{\omega_+\tau_b+\frac{\tau_b}{\omega_+}}\left(e^{\frac{\tau_b}{\omega_+}+\tau\omega_+}-e^{\frac{\tau}{\omega_+}+\tau_b\omega_+}\omega_+^2\right)}{\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^3} + \frac{2\omega_+^2 e^{\frac{2\tau_b}{\omega_+}}\left(e^{\frac{\tau_b}{\omega_+}+\tau\omega_+}-e^{\frac{\tau}{\omega_+}+\tau_b\omega_+}\omega_+^2\right)}{(2\omega_+^2-1)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^3} \\
& + \frac{8\omega_+^2 e^{2\omega_+\tau_b+\frac{2\tau_b}{\omega_+}}\left(e^{\tau\omega_+}-e^{\tau/\omega_+}\right)\left(\omega_+^2-1\right)^2\left(\omega_+^2+1\right)}{\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^3\left(2\omega_+^4-5\omega_+^2+2\right)} + \frac{2\omega_+^2 e^{\omega_+\tau+\frac{\tau}{\omega_+}+\tau_b\omega_++\frac{\tau_b}{\omega_+}}}{\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^2} \\
& - \frac{e^{\frac{2\tau_b}{\omega_+}+2\tau\omega_+}}{(2\omega_+^2-1)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^2} \,,
\end{aligned}
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\begin{aligned}
\langle V^2\rangle \quad = \quad & f_0''(\tau) = \frac{\omega_+^6 e^{\frac{2\tau}{\omega_+}+2\tau_b\omega_+}}{(\omega_+^2-2)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^2} - \frac{2\omega_+^4 e^{2\tau_b\omega_+}\left(e^{\frac{\tau_b}{\omega_+}+\tau\omega_+}-e^{\frac{\tau}{\omega_+}+\tau_b\omega_+}\omega_+^2\right)}{(\omega_+^2-2)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^3} \\
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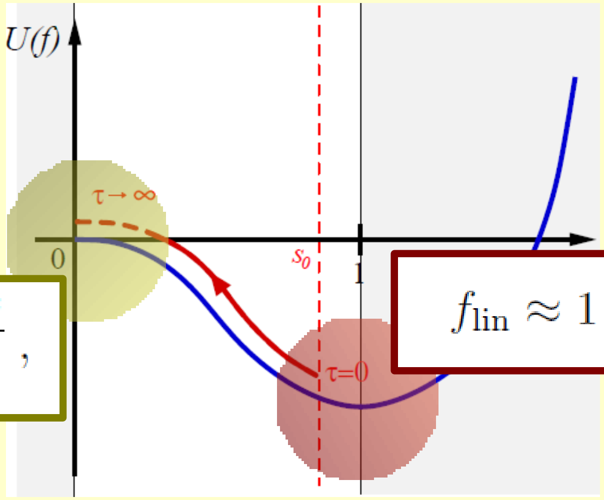
& -\frac{e^{\frac{2\tau_b}{\omega_+}+2\tau\omega_+}}{(2\omega_+^2-1)\left(e^{\tau_b/\omega_+}-e^{\tau_b\omega_+}\omega_+^2\right)^2},
\end{aligned}
$$

$$
\langle V^2\rangle \xrightarrow{\tau_b\gg 1} \frac{\omega_+^2}{\omega_+^2-2}\left(1-2\frac{e^{-\omega_-\tau}}{\omega_+^2}\right)e^{2\omega_-\tau} + \frac{8(\omega_+^2-1)^2(\omega_+^2+1)}{\omega_+^4(2\omega_+-1)(2-\omega_+^2)}e^{-(\omega_+^2-2)\omega_-\tau_b+\omega_+\tau}
$$

for $\tau_b \to \infty$ the n-th moment diverges at $\boxed{\Omega = \dfrac{(n+1)^2}{4n}}$

Reconstructing ρ(ϕ;V)
Ω > 1 case



$$f \approx f_g = e^{-\frac{(\tau+\tau_1)^2}{4}} \,,$$

$$f_{\rm lin} \approx 1 - e^{\omega_-(\tau+\tau_0)} \,.$$

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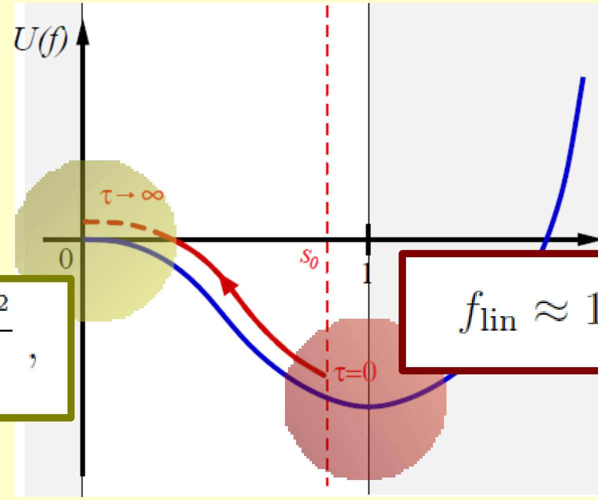

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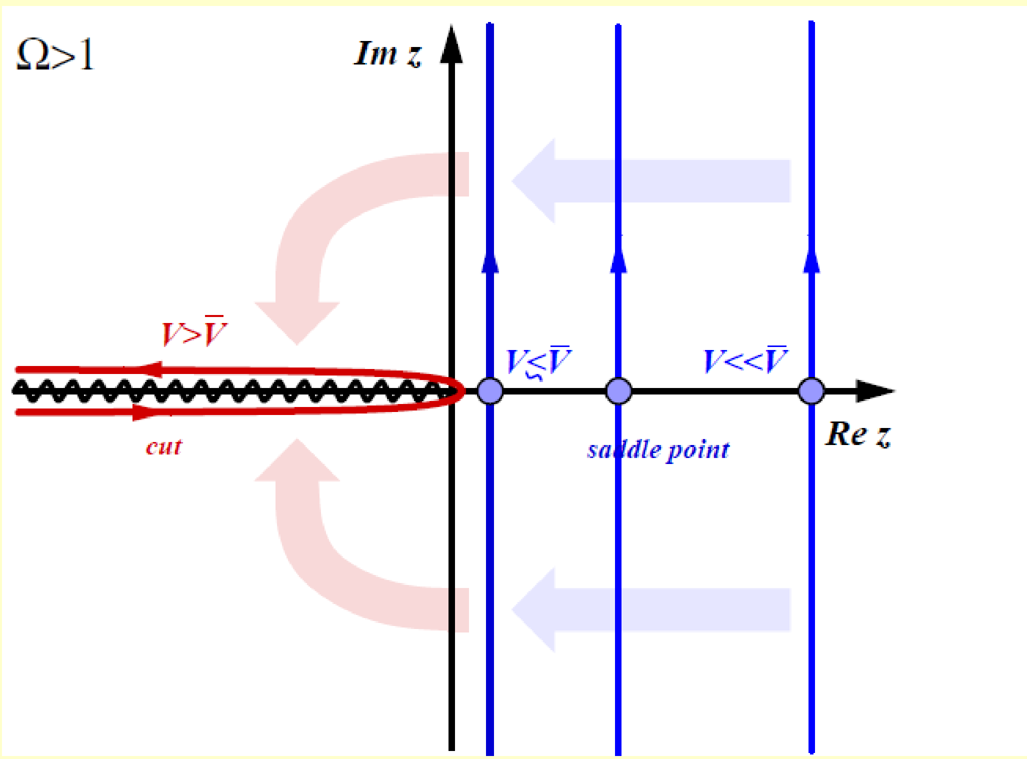

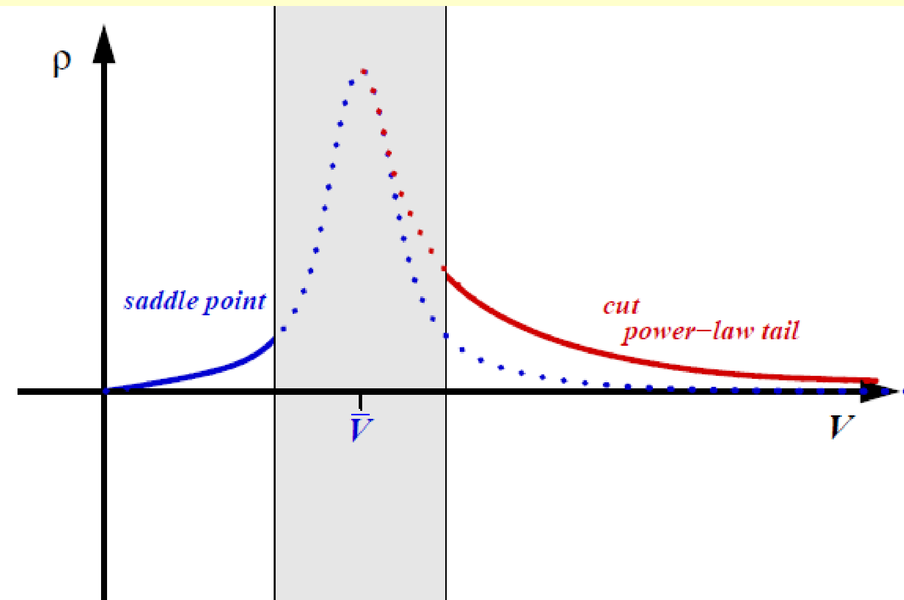

$$\rho(V,\tau) \sim \frac{1}{V}\left(\frac{\overline{V}}{V}\right)^{\frac{\omega_+}{\omega_-}} \quad \text{for} \quad V \gtrsim \overline{V}\frac{\omega_+}{\omega_-}$$

$$\rho(V,\tau) \approx \mathcal{N} e^{-\frac{1}{4}\Omega\left(1+\sqrt{1-\frac{1}{\Omega}}\right)^2\left[\log\left(\frac{V}{\overline{V}}\right)\right]^2} = \mathcal{N} e^{-\Omega\left[\frac{3N}{2}\left(1+\sqrt{1-\frac{1}{\Omega}}\right)-3N_c\right]^2} \,, \qquad V \lesssim \overline{V}$$

Reconstructing ρ(ϕ;*V*): Ω =1-ε case, the phase transition

$$f_{\text{lin}}(\tau;z) = 1 - \sigma e^{\sqrt{\Omega}(\tau+\tau_0)} \cos\left(\sqrt{\Omega - 1}(\tau + \tau_0)\right) \approx 1 - \sigma e^{\tau+\tau_0} \cos\left(\sqrt{\epsilon}(\tau + \tau_0)\right)$$

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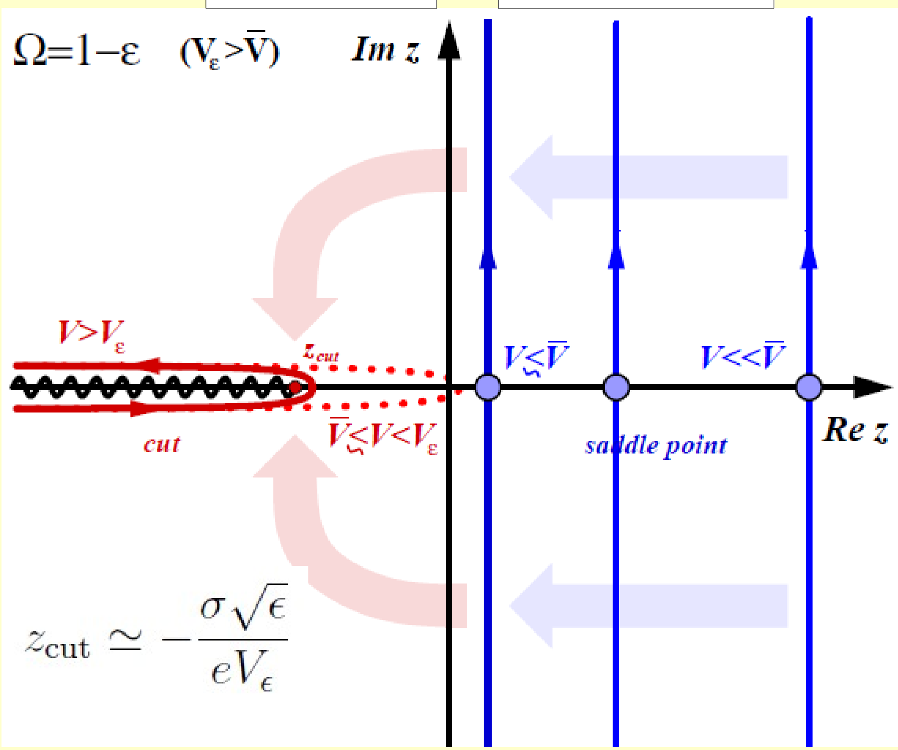
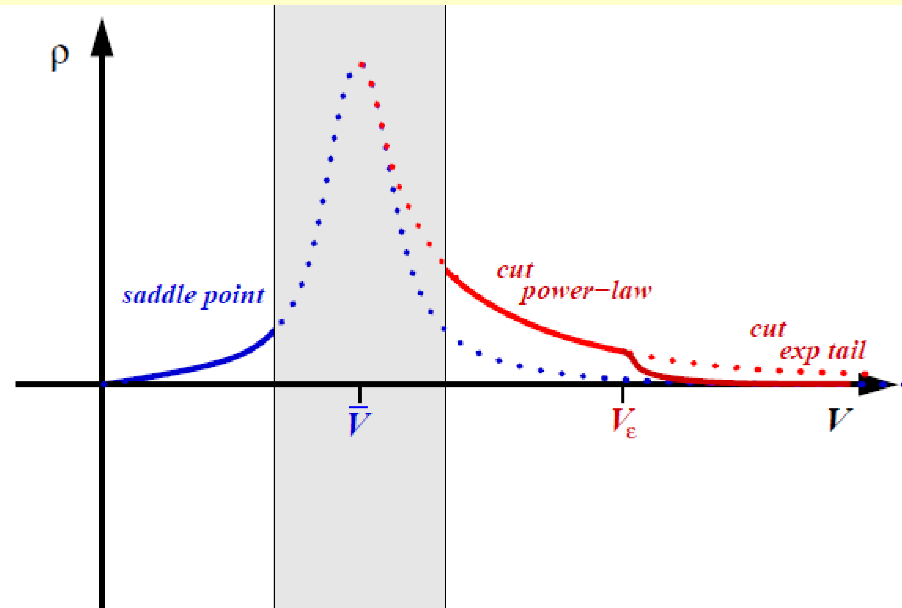
$$V_\epsilon \equiv e^{\frac{\pi}{2\sqrt{\epsilon}}}$$

$$1 < \Omega \lesssim 1 - \left(\frac{\pi}{6\overline{N}}\right)^2$$



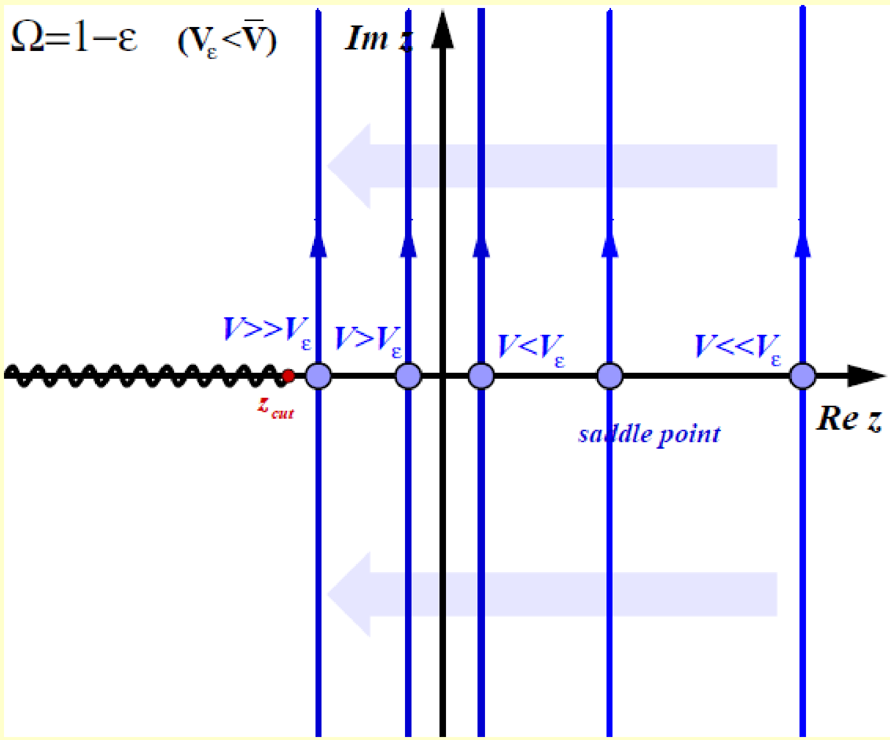
$$z_{\rm cut} \simeq -\frac{\sigma\sqrt{\epsilon}}{eV_\epsilon}$$

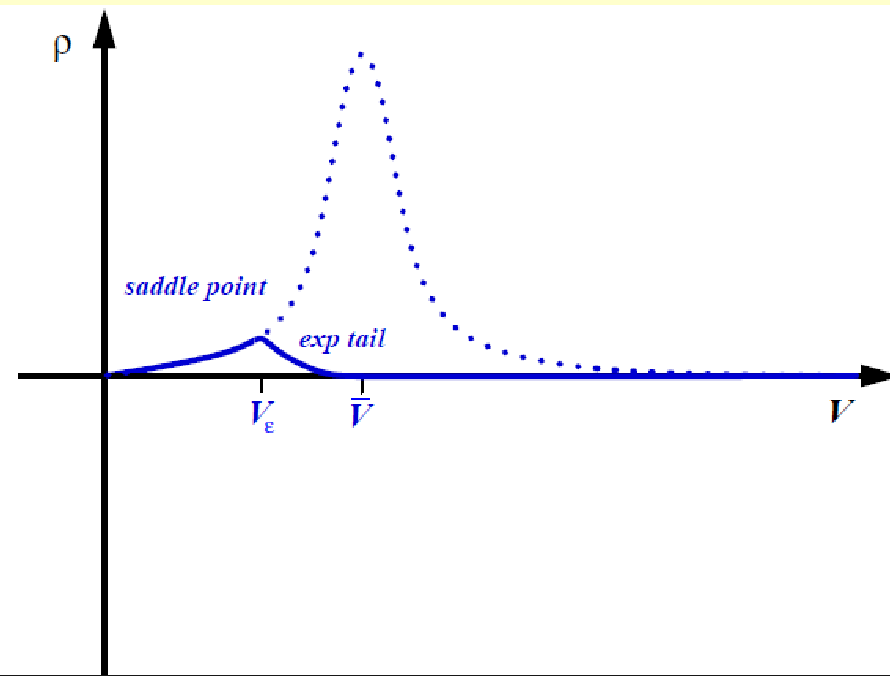


$$\rho(V) \sim \frac{\overline{V}}{V^2} \qquad \overline{V} < V < V_\epsilon$$

$$\rho(V) \sim \frac{\overline{V}}{V_\epsilon}\frac{1}{V}e^{z_{\rm cut}V} = \frac{\overline{V}}{V_\epsilon}\frac{1}{V}e^{-\frac{\sigma}{e}\sqrt{\epsilon}V/V_\epsilon} \qquad V_\epsilon < V$$

$\Omega = 1-\varepsilon$ case: inside eternal inflation





$$\rho(V,\tau) \approx \mathcal{N} e^{-\frac{1}{4}\left(\tau-\frac{\pi}{2\sqrt{\epsilon}}\right)^2-\frac{\sigma}{\epsilon}\sqrt{\epsilon}\, V/V_\epsilon}, \qquad V \gtrsim V_\epsilon\,,$$

Special limits and cross-checks

The Classical Limit: Ω>>1

$$\tau = 2\sqrt{\Omega}\tilde{\tau}$$

$$\tilde{\tau} = 3N_c$$

$$\cancel{\frac{1}{4\Omega}\frac{\partial^2 f}{\partial\tilde{\tau}^2}} - \frac{\partial f}{\partial\tilde{\tau}} + f\log f = 0$$

$$\boxed{f = e^{-z\,e^{\tilde{\tau}}}}$$

$$\rho(V,\tau) = \delta(V - e^{3N_c})$$

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$$\rho(V,\tau) = \delta(V - e^{3N_c})$$

Deep inside Eternal Inflation: Ω → 0

$$f(\tau;z) = e^{\frac{1}{2}-\frac{1}{4}\left(\tau+\sqrt{2+4z}\right)^2}$$

$$\rho(V,\tau) = \frac{1}{2\pi}\int_{0^+-i\infty}^{0^++i\infty} dz\, e^{\frac{1}{2}-\frac{1}{4}\left(\tau+\sqrt{2+4z}\right)^2+zV}$$

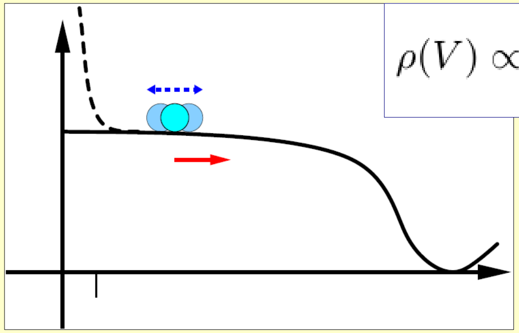
$$\rho(V,\tau) = \frac{\tau}{\sqrt{4\pi}(V-1)^{3/2}}e^{-\frac{V-1}{2}-\frac{V}{(V-1)}\frac{\tau^2}{4}}$$

$$\int_0^\infty dV\,\rho(V,\tau) = e^{-\frac{\tau^2}{4}-\frac{\tau}{\sqrt{2}}} = f(\tau;0)$$

Approximations, corrections and other effects

Finite Barrier Effects

a) Suppression of the large volume tail



$$\rho(V) \propto \frac{1}{V^{\omega_+^2+1}} \rightarrow e^{z_{cut} V}$$

w/o barrier

w/ barrier

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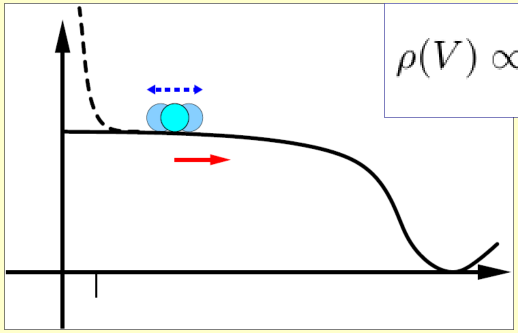

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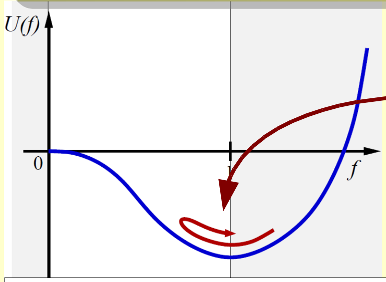
b) Shift of the phase transition point



$$T = \frac{2\pi}{\sqrt{1-\Omega}}$$

$$\tau_b < T/4$$

$$\tau_b = \frac{\pi}{2\sqrt{1-\Omega_c}} \quad \Rightarrow \quad \Omega_c = 1 - \left(\frac{\pi}{2\tau_b}\right)^2$$

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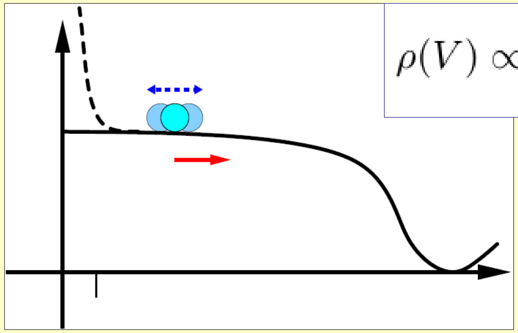

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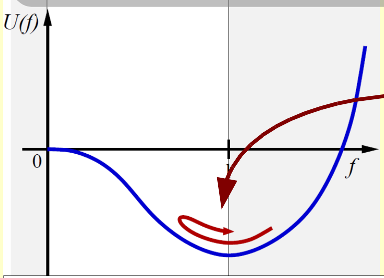

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Slow-Roll corrections

slow-roll parameter

$$\frac{\dot{H}}{H^2} \sim \Omega \frac{H^2}{M_{\rm Pl}^2} \to 0$$

Friedman equation

$$M_{\rm Pl}^2 H \Delta H \sim V' \Delta\phi$$

$$\frac{\Delta H}{H} \sim \frac{V'\Delta\phi}{M_{\rm Pl}^2 H^2} \sim \Omega \frac{H^2}{M_{\rm Pl}^2} N_c \lesssim 1$$

$$\tau \equiv 6 \int \sqrt{\Omega}\, dN_c$$

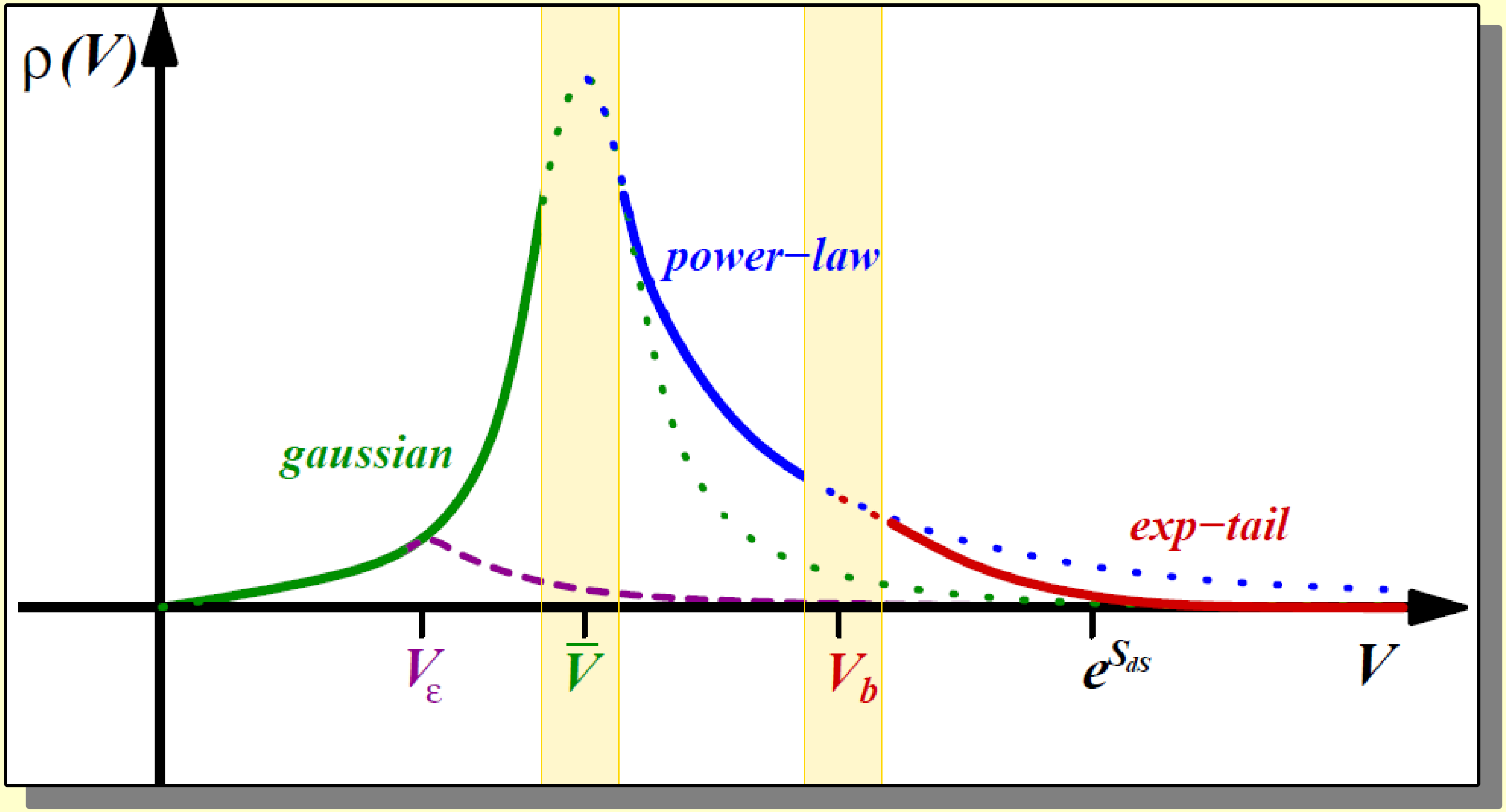
ϕ-dependent diff.eq.

$$\lim_{\tau_b \to \infty} \langle V \rangle = e^{\int \omega_- d\tau}$$

$$3\overline{N} = \int \omega_- d\tau$$

$$\partial_\tau \Omega \ll \Omega$$

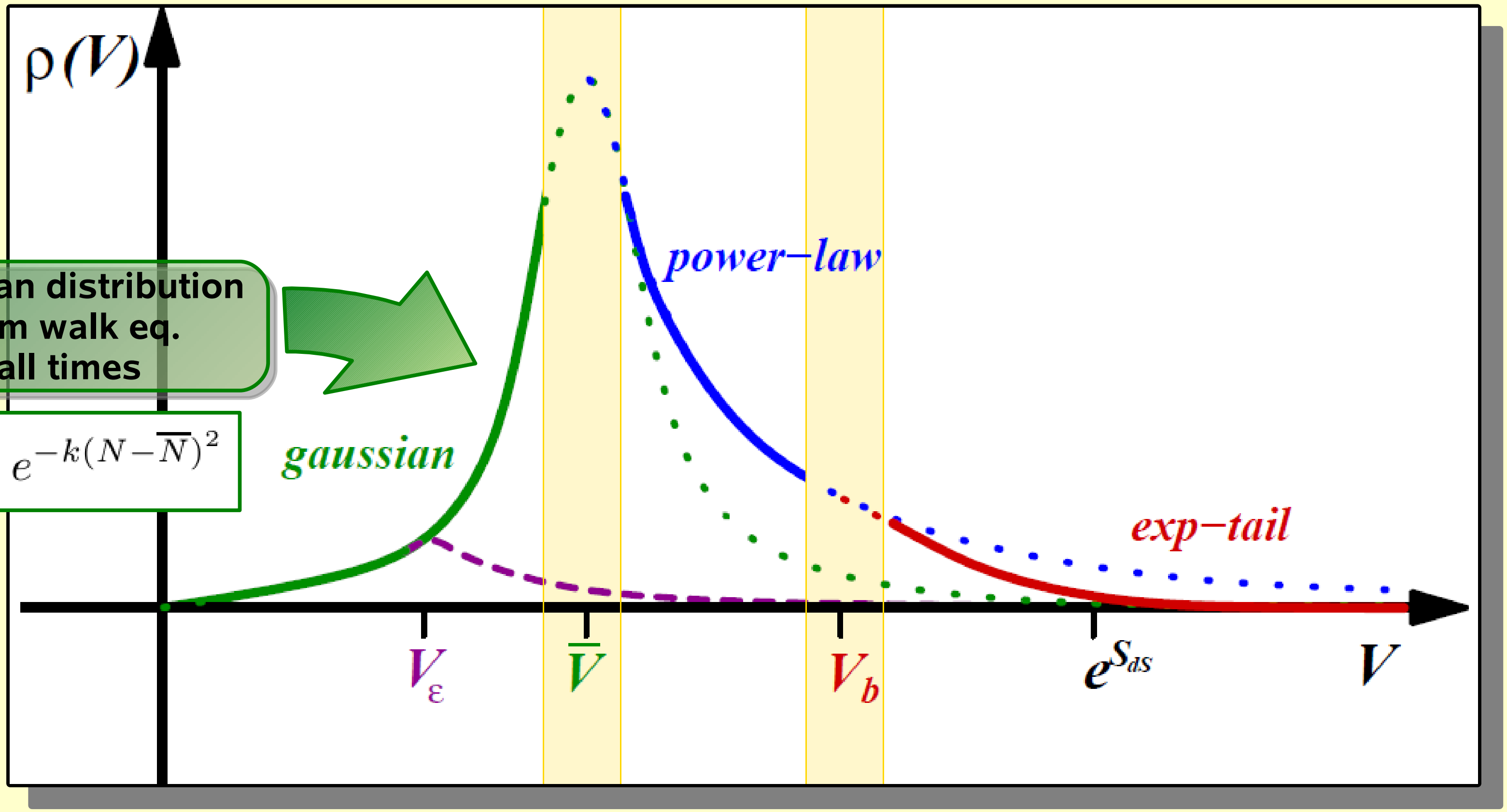
Summarizing the ρ(*V*) shape



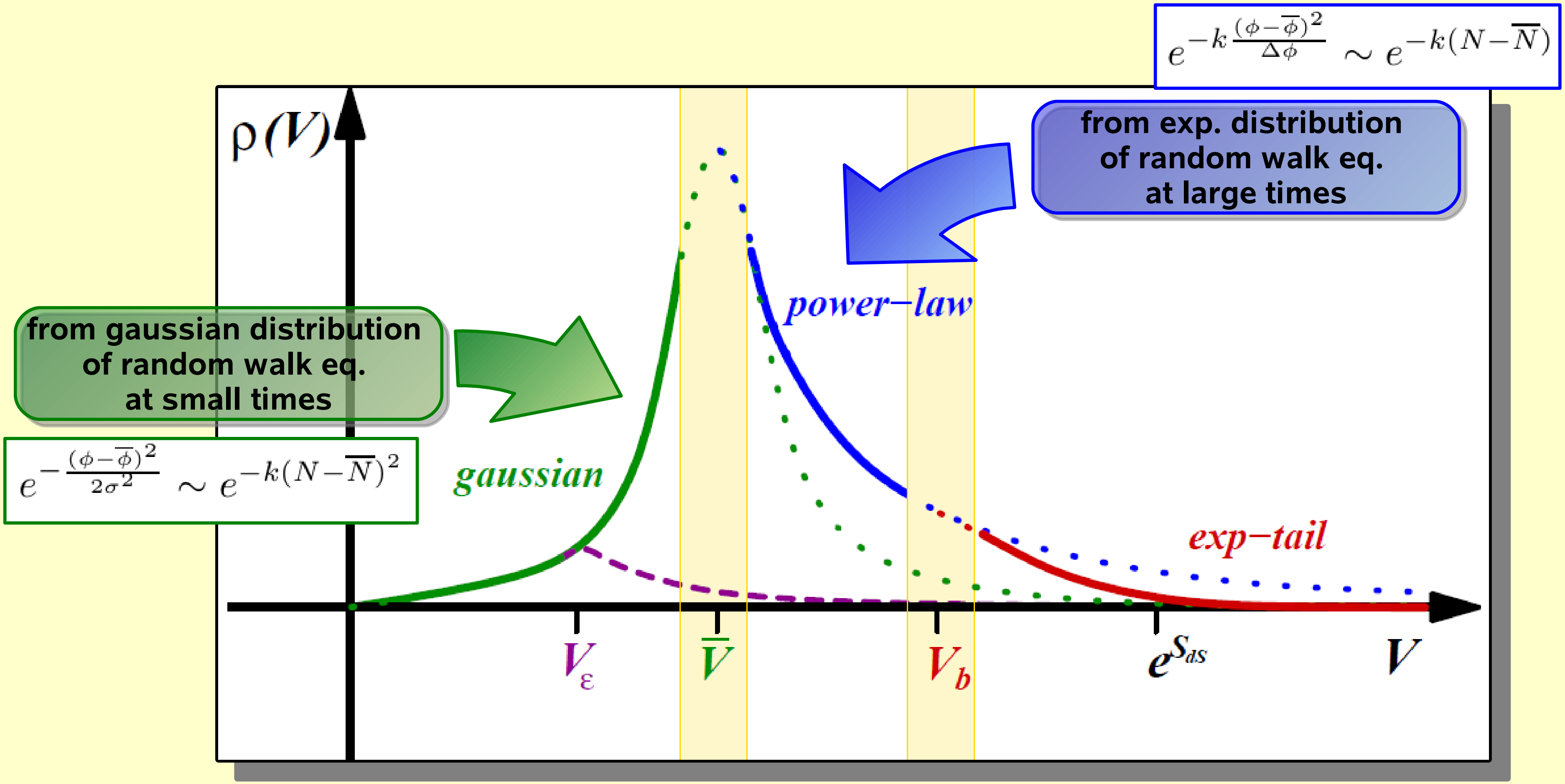
Summarizing the ρ(*V*) shape

from gaussian distribution of random walk eq. at small times

$$e^{-\frac{(\phi-\overline{\phi})^2}{2\sigma^2}} \sim e^{-k(N-\overline{N})^2}$$



Summarizing the ρ(V) shape



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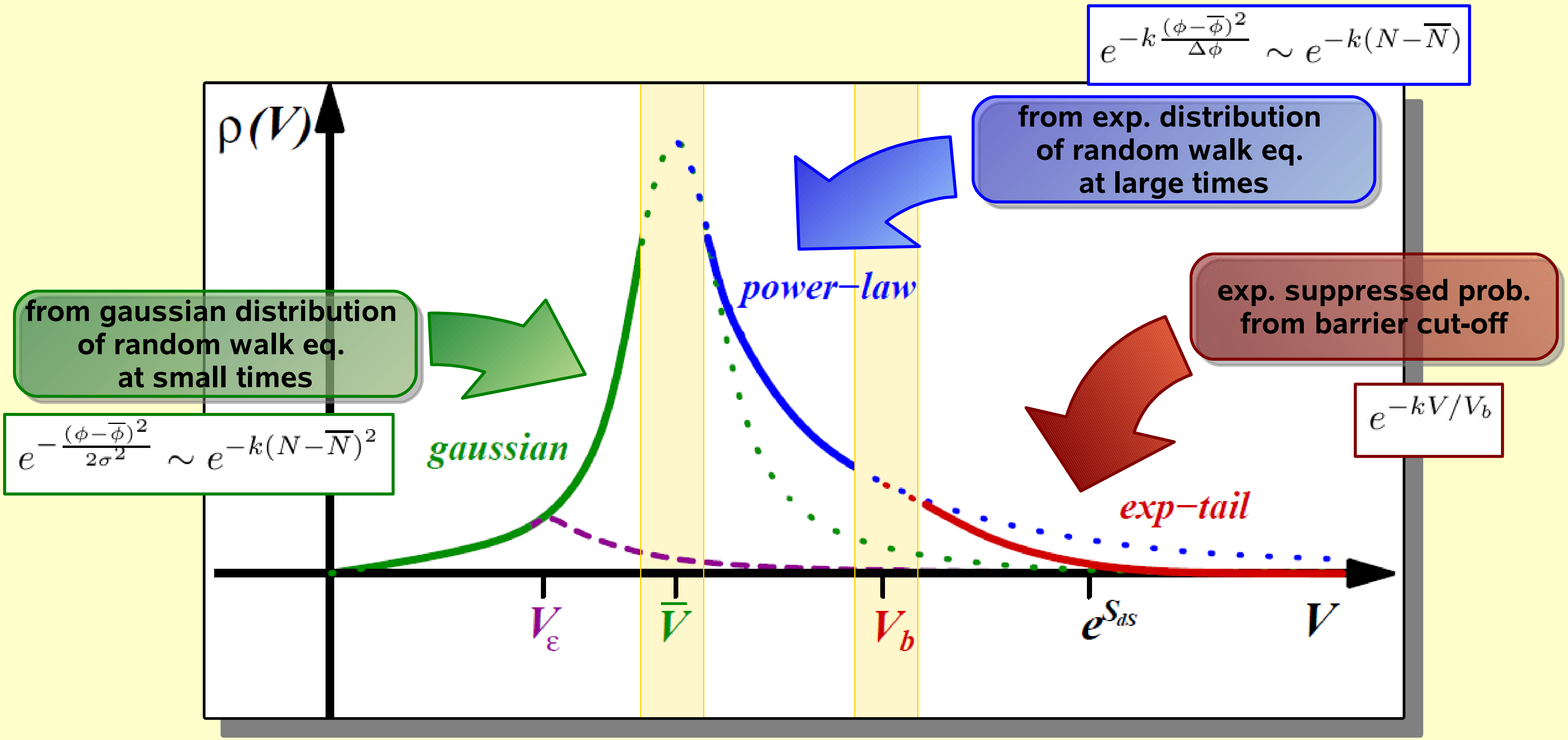

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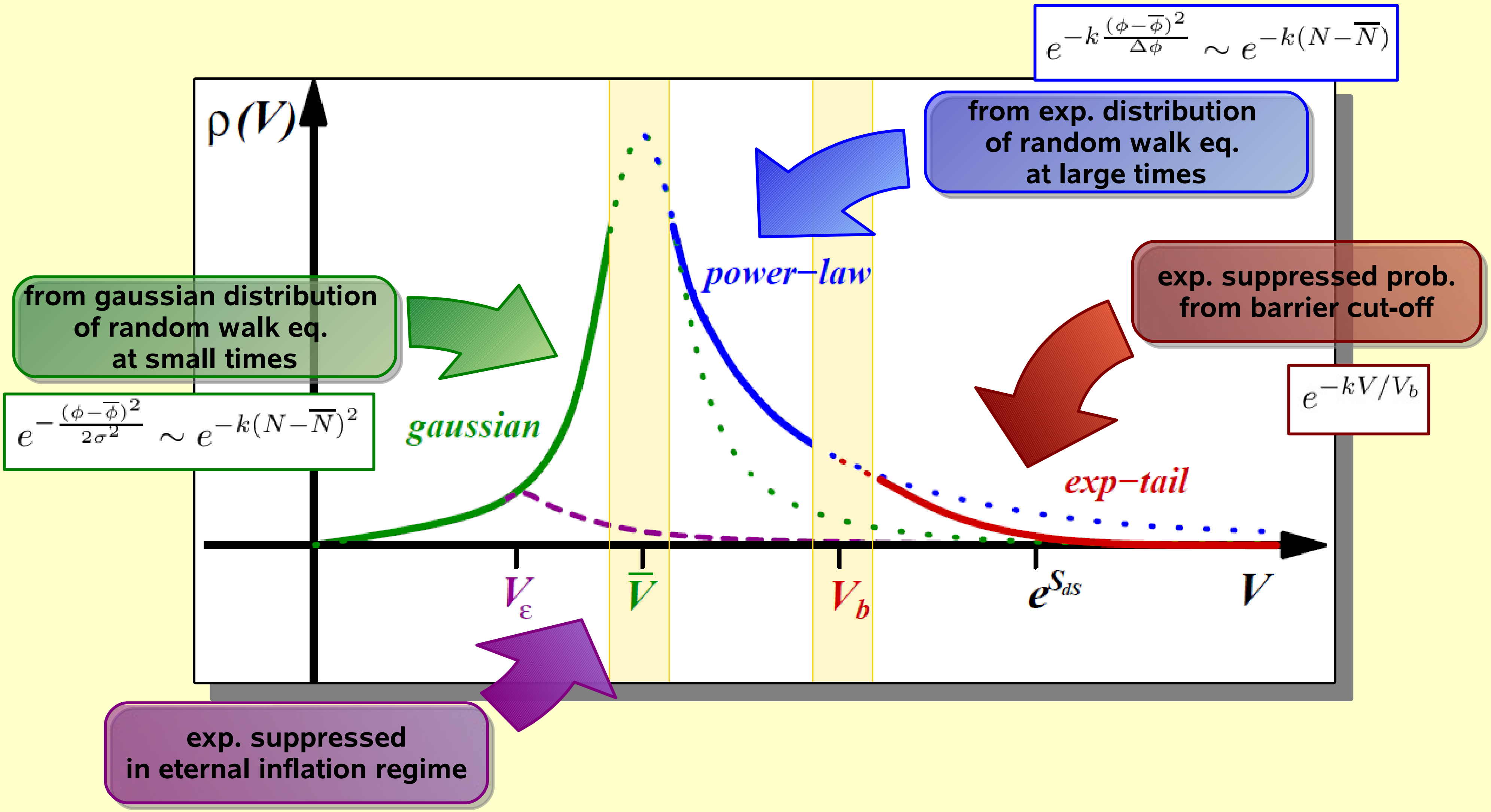

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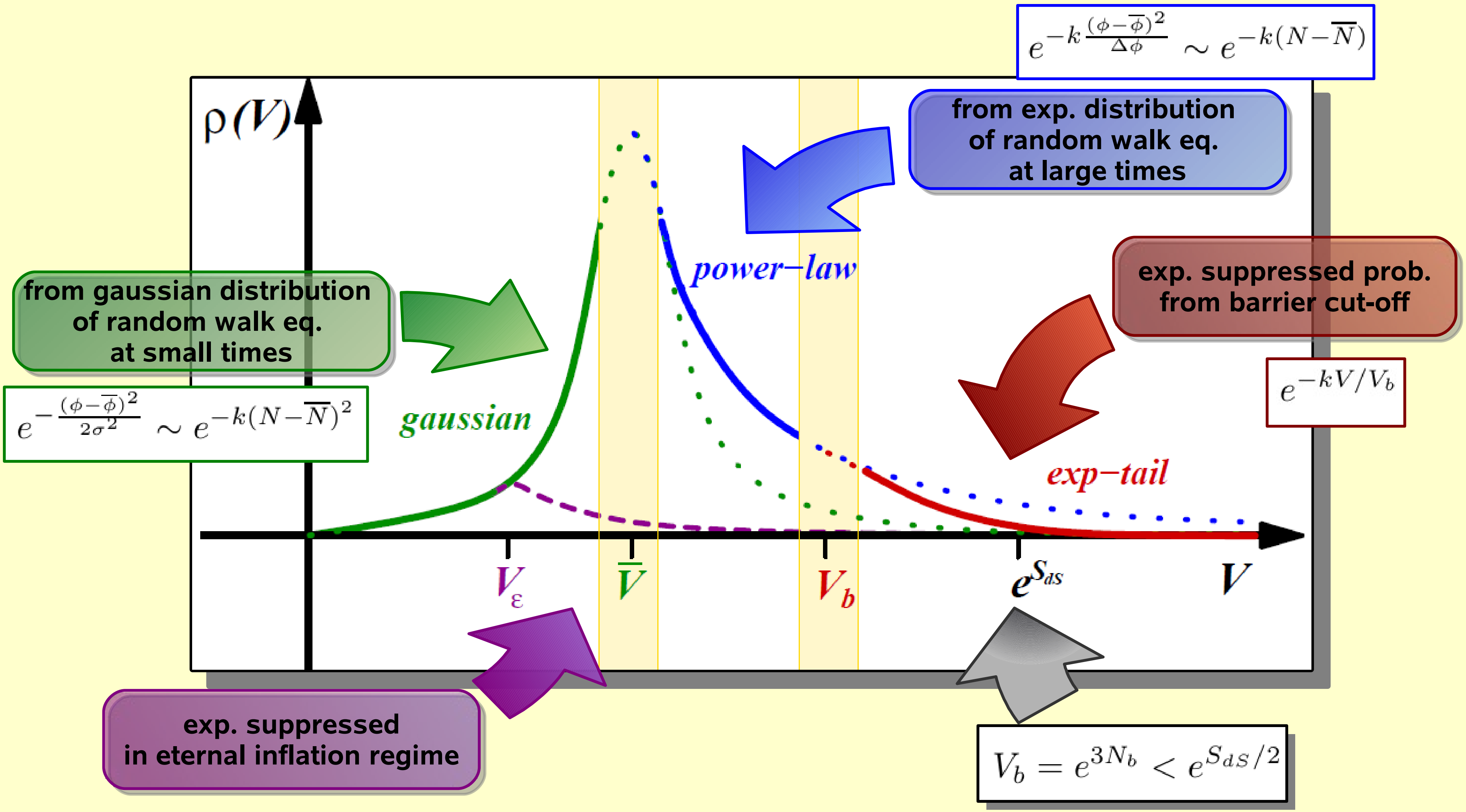

Discussion

Two main results:

1) The probability distribution itself; non trivial informations on the phase transition to eternal inflation.

2) Confirmation of the bound at the quantum level; the bound can be made 'sharp' in the following sense:

The probability for slow-roll inflation to produce a finite volume larger than $e^{S_{dS}/2}$, where S_{dS} is de Sitter entropy at the end of the inflationary stage, is suppressed below the uncertainty due to non-perturbative quantum gravity effects.

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Open problems and further extensions:

- Does the value "½" posses a deeper meaning
  (e.g. as the ½ in the Page argument for black holes),
  is it universal?
- Is the result robust against modification of the setting:
  -multi field inflation (more species)
  -non slow-roll inflation
  -different number of dimensions
  -etc...
- Is the bound associated with complementarity? How?