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The Legendre transformation


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The Legendre transformation

by

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A general geometric definition of the Legendre transformation is given and illustrated by examples from particle dynamics and thermostatics. The definition is based on concepts of symplectic geometry reviewed in the early sections which serve as an introduction.
1. LAGRANGIAN SUBMANIFOLDS
AND SYMPLECTIC DIFFEOMORPHISMS

Let $P$ be a differential manifold. The tangent bundle of $P$ is denoted by $TP$ and $\tau_P : TP \to P$ is the tangent bundle projection. Let $\omega$ be a 2-form on $P$. The form $\omega$ is called a symplectic form if $d\omega = 0$ and if $\langle u \wedge w, \omega \rangle = 0$ for each $u \in TP$ such that $\tau_p(u) = \tau_p(w)$ implies $w = 0$. If $\omega$ is a symplectic form then $(P, \omega)$ is called a symplectic manifold.

**Definition 1.1.** Let $(P, \omega)$ be a symplectic manifold. A submanifold $N$ of $P$ such that $\omega | N = 0$ and $\dim P = 2 \dim N$ is called a Lagrangian submanifold of $(P, \omega)$.

**Definition 1.2.** Let $(P_1, \omega_1)$ and $(P_2, \omega_2)$ be symplectic manifolds. A diffeomorphism $\varphi : P_1 \to P_2$ is called a symplectic diffeomorphism of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$ if $\varphi^* \omega_2 = \omega_1$.

Let $(P_1, \omega_1)$ and $(P_2, \omega_2)$ be symplectic manifolds and let $pr_1$ and $pr_2$ denote the canonical projections of $P_2 \times P_1$ onto $P_1$ and $P_2$ respectively. The 2-form $\omega_2 \otimes \omega_1 = pr_2^* \omega_2 - pr_1^* \omega_1$ is clearly a symplectic form on $P_2 \times P_1$.

**Proposition 1.1.** The graph of a symplectic diffeomorphism $\varphi$ of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$ is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \otimes \omega_1)$.

**Proof.** The graph of $\varphi : P_1 \to P_2$ is the image of $(\varphi, \text{Id}) : P_1 \to P_2 \times P_1$ and $(\varphi, \text{Id})^*(\omega_2 \otimes \omega_1) = \varphi^* \omega_2 - \omega_1 = 0$. Hence $(\omega_2 \otimes \omega_1) | \text{graph } \varphi = 0$. Also $\dim (P_2 \times P_1) = 2 \dim (\text{graph } \varphi)$. Hence $\varphi$ is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \otimes \omega_1)$.

The converse is also true. If the graph of a diffeomorphism $\varphi : P_1 \to P_2$ is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \otimes \omega_1)$ then $\varphi$ is a symplectic diffeomorphism of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$.

2. LOCAL EXPRESSIONS

Let $(P, \omega)$ be a symplectic manifold and let $(x^i, y_j)$, $1 \leq i, j \leq n$ be local coordinates of $P$ such that $\omega = \Sigma dy_i \wedge dx^i$. Coordinates $(x^i, y_j)$ are called canonical coordinates of $(P, \omega)$ [I]. Existence of canonical coordinates is guaranteed by Darboux theorem. A submanifold $N$ of $P$ of dimension $n$ represented locally by $x^i = \xi^i(u^k), y_j = \eta_j(u^k)$, $1 \leq i, j, k \leq n$ is a Lagrangian submanifold of $(P, \omega)$ if and only if

$\omega | N = \Sigma_{i,j,k} \frac{\partial \eta_i}{\partial u^j} \frac{\partial \xi^i}{\partial u^k} du^j \wedge du^k = 0$. 

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This condition is equivalent to \([u^i, u^j] = 0, 1 \leq i, j \leq n\), where
\[
[u^i, u^j] = \sum_k \left[ \frac{\partial \xi^k}{\partial u^i} \frac{\partial \eta_k}{\partial u^j} - \frac{\partial \xi^k}{\partial u^j} \frac{\partial \eta_k}{\partial u^i} \right]
\]
are the Lagrange brackets [2].

Let \((x^i, y^j), 1 \leq i, j \leq n\) and \((x'^i, y'^j), 1 \leq i, j \leq n\) be canonical coordinates of symplectic manifolds \((P_1, \omega_1)\) and \((P_2, \omega_2)\) respectively. The two sets of coordinates are combined into a set \((x'^i, x^k, y^j, y^l), 1 \leq i, j, k, l \leq n\) of local coordinates of \(P_2 \times P_1\). Then \(\omega_2 \ominus \omega_1 = \sum_i (dy^i \wedge dx'^i - dy^i \wedge dx^i)\).

A diffeomorphism \(\varphi : P_1 \to P_2\) represented locally by \(x'^i = \psi^i(x^k, y^l)\), \(y^j = \chi^j(x^k, y^l)\) is a symplectic diffeomorphism of \((P_1, \omega_1)\) onto \((P_2, \omega_2)\) if and only if
\[
\omega_2 \ominus \omega_1 |_{\text{graph } \varphi} = \sum_i \left[ \frac{\partial \chi^i}{\partial x'^k} \frac{\partial \psi^i}{\partial x^j} \ dx^j \wedge dx^k + \frac{\partial \chi^i}{\partial y^j} \frac{\partial \psi^i}{\partial y^k} \ dy^j \wedge dy^k \right] - \sum_i (dy^i \wedge dx^i) = 0.
\]

This condition is equivalent to \([x^i, x^j] = 0, [x^i, y^j] = \delta^i_j, [y^i, y^j] = 0, 1 \leq i, j \leq n\).

3. SPECIAL SYMPLECTIC MANIFOLDS AND GENERATING FUNCTIONS [4]

Let \(Q\) be a manifold, let \(TQ\) denote the tangent bundle of \(Q\) and \(\tau_Q : TQ \to Q\) the tangent bundle projection. The cotangent bundle of \(Q\) is denoted by \(T^*Q\) and \(\tau_{T^*Q} : T^*Q \to Q\) is the cotangent bundle projection. The canonical 1-form \(\theta_Q\) on \(T^*Q\) is defined by
\[
\left< u, \theta_Q \right> = \left< T\tau_Q(u), \tau_{T^*Q}(u) \right>
\]
for each \(u \in TT^*Q\).

The canonical 2-form \(\omega_Q = d\theta_Q\) is known to be a symplectic form. Hence \((T^*Q, \omega_Q)\) is a symplectic manifold.

Let \(F\) be a differentiable function on the manifold \(Q\). The 1-form \(dF\) is a section \(dF : Q \to T^*Q\) of the cotangent bundle. The image \(N\) of \(dF\) is a submanifold of \(T^*Q\), the mapping \(\rho = \pi_Q : N \to Q\) is a diffeomorphism and \(\theta_Q \mid N = \rho^*dF\). Hence \(\omega_Q \mid N = 0\) and \(N\) is a Lagrangian submanifold of \((T^*Q, \omega_Q)\).

The above construction of Lagrangian submanifolds is generalized in the following proposition.

**Proposition 3.1.** — Let \(K\) be a submanifold of \(Q\) and \(F\) a function on \(K\). The set
\[
N = \{ p \in T^*Q : \pi_Q(p) \in K \text{ and } \left< u, p \right> = \left< u, dF \right> \text{ for each } u \in TK \subset TQ \text{ such that } \tau_Q(u) = \pi_Q(p) \}
\]
is a Lagrangian submanifold of \((T^*Q, \omega_Q)\).

Proof. — Using local coordinates it is easily shown that $N$ is a submanifold of $T^*Q$ of dimension equal to $\dim Q$. The submanifold $K$ is the image of $N$ by $\pi_Q$. Let $\rho : N \to K$ be the mapping defined by the commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\text{injection}} & T^*Q \\
\rho \downarrow & & \downarrow \pi_Q \\
K & \xrightarrow{\text{injection}} & Q
\end{array}
$$

Then $\langle u, \rho^*dF \rangle = \langle T\rho(u), dF \rangle = \langle T\rho(u), \tau_{T^*Q}(u) \rangle = \langle u, \beta_Q \rangle$ for each vector $u \in TN \subset T^*Q$. Hence $\beta_Q|N = \rho^*dF$, $\omega_Q|N = 0$ and $N$ is a Lagrangian submanifold of $(T^*Q, \omega_Q)$.

**Definition 3.1.** — The function $F$ in Proposition 3.1 is called a generating function of the Lagrangian submanifold $N$. The Lagrangian submanifold $N$ is said to be generated by $F$.

There is a canonical submersion $\kappa$ of $\pi_Q^{-1}(K)$ onto $T^K$ and the Lagrangian submanifold $N$ is given by $N = \kappa^{-1}(dF(K))$. The Lagrangian submanifold $N$ can also be characterized as the maximal submanifold $N$ of $T^*Q$ such that $\pi_Q(N) = K$ and $\beta_Q|N = \rho^*dF$, where $\rho : N \to K$ is the mapping defined in the proof of Proposition 3.1.

In many applications of symplectic geometry it is convenient to consider symplectic manifolds which are not directly cotangent bundles but are isomorphic to cotangent bundles.

**Definition 3.2.** — Let $(P, Q, \pi)$ be a differential fibration and $\beta$ a 1-form on $P$. The quadruple $(P, Q, \pi, \beta)$ is called a special symplectic manifold if there is a diffeomorphism $\alpha : P \to T^*Q$ such that $\pi = \pi_Q \circ \alpha$ and $\beta = \alpha^*\beta_Q$.

If the diffeomorphism $\alpha$ exists it is unique. If $(P, Q, \pi, \beta)$ is a special symplectic manifold then $(P, \omega) = (P, d\beta)$ is a symplectic manifold called the underlying symplectic manifold of $(P, Q, \pi, \beta)$.

If $(P, Q, \pi, \beta)$ is a special symplectic manifold, $K$ a submanifold of $Q$ and $F$ a function on $K$ then the set $N = \{ p \in P; \pi(p) \in K \text{ and } \langle u, \beta \rangle = \langle T\pi(u), dF \rangle \text{ for each } u \in T^P \text{ such that } T\pi(u) = p \text{ and } T\pi(u) \in TK \subset TD \}$ is a Lagrangian submanifold of $(P, d\beta)$ said to be generated with respect to $(P, Q, \pi, \beta)$ by the function $F$. The function $F$ is called a generating function of $N$ with respect to $(P, Q, \pi, \beta)$. The diffeomorphism $\alpha : P \to T^*Q$ maps the Lagrangian submanifold $N$ onto the Lagrangian submanifold of $(T^*Q, \omega_Q)$ generated by $F$.

Let $(P_1, Q_1, \pi_1, \beta_1)$ and $(P_2, Q_2, \pi_2, \beta_2)$ be special symplectic manifolds and let $\beta_2 \oplus \beta_1$ denote the 1-form $pr_2^*\beta_2 - pr_1^*\beta_1$, where $pr_1$ and $pr_2$ are the canonical projections of $P_2 \times P_1$ onto $P_1$ and $P_2$ respectively.

**Proposition 3.2.** — The quadruple $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \beta_2 \oplus \beta_1)$ is a special symplectic manifold.
Proof. — Let $\alpha_1 : P_1 \to T^*Q_1$ and $\alpha_2 : P_2 \to T^*Q_2$ be diffeomorphisms such that $\pi_1 = \pi_{Q_1} \circ \alpha_1$, $\pi_2 = \pi_{Q_2} \circ \alpha_2$, $\theta_1 = \alpha_1^*\theta_{Q_1}$ and $\theta_2 = \alpha_2^*\theta_{Q_2}$. Then the mapping $\alpha_{21} : P_2 \times P_1 \to T^*(Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1 : (p_2, p_1) \mapsto (\alpha_2(p_2), - \alpha_1(p_1))$ is a diffeomorphism such that $\pi_2 \times \pi_1 = (\pi_{Q_2} \times \pi_{Q_1}) \circ \alpha_{21}$ and $\theta_2 \ominus \theta_1 = \alpha_{21}^*(\theta_{Q_2} \ominus \theta_{Q_1})$.

The identification $T^*(Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1$ implies the identification of $\theta_{Q_2} \ominus \theta_{Q_1} = pr_2^*\theta_{Q_2} + pr_1^*\theta_{Q_1}$ with $\theta_{Q_2 \times Q_1}$. Hence $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)$ is a special symplectic manifold.

If $(P_1, \omega_1)$ and $(P_2, \omega_2)$ are underlying symplectic manifolds of $(P_1, Q_1, \pi_1, \theta_1)$ and $(P_2, Q_2, \pi_2, \theta_2)$ then $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ is the underlying symplectic manifold of $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)$. Let $\varphi$ be a symplectic diffeomorphism of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$.

Definition 3.3. — If the graph of the diffeomorphism $\varphi : P_1 \to P_2$ is generated with respect to the special symplectic structure $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)$ by a function $G$ on a submanifold $M$ of $Q_2 \times Q_1$ then $\varphi$ is said to be generated with respect to $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)$ by the function $G$ and $G$ is called a generating function of $\varphi$ with respect to $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)$.

If $N_1$ is a Lagrangian submanifold of $(P_1, \omega_1)$ and $\varphi$ is a symplectic diffeomorphism of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$ then $N_2 = \varphi(N_1)$ is a Lagrangian submanifold of $(P_2, \omega_2)$. Let $(P_1, \omega_1)$ and $(P_2, \omega_2)$ be underlying symplectic manifolds of special symplectic manifolds $(P_1, Q_1, \pi_1, \theta_1)$ and $(P_2, Q_2, \pi_2, \theta_2)$ respectively and let $N_1$, $\varphi$ and $N_2$ be generated by functions $F_1$, $G$ and $F_2$ defined on submanifolds $K_1 \subset Q_1$, $M \subset Q_2 \times Q_1$ and $K_2 \subset Q_2$ respectively.

Proposition 3.3. — Let $K_{21}$ denote the image of $N_1$ by $(\pi_2 \times \pi_1) \circ (\varphi, \text{Id}) : P_1 \to Q_2 \times Q_1$.

Then $K_{21} = \{ (q_2, q_1) \in Q_2 \times Q_1 ; q_1 \in K_1, (q_2, q_1) \in M \land \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0 \text{ for each } v_1 \in T_{q_1}K_1 \text{ such that } (v_2, v_1) \in T_{q_2}Q_2 \times T_{q_1}Q_1 \text{ and } v_2 = 0 \}$.

Proof (for $P_1 = T^*Q_1$ and $P_2 = T^*Q_2$). — If $(q_2, q_1) \in K_{21}$ then $q_1 \in K_1$, $(q_2, q_1) \in M$ and there is a covector $p_1 \in N_1$ such that $\pi_1(p_1) = q_1$ and $\pi_2(\varphi(p_1)) = q_2$. It follows that $\langle v_1, p_1 \rangle = \langle v_1, dF_1 \rangle$ and $\langle v_2, v_1 \rangle = \langle v_2, v_1 \rangle, dG \rangle$ and finally $\langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0 \text{ for each } v_1 \in T_{q_1}K_1 \text{ such that } (v_2, v_1) \in T_{(q_2,q_1)}M \text{ and } v_2 = 0$. Conversely if $q_1 \in K_1$ and $(q_2, q_1) \in M$ then there are covectors $p'_1 \in P_1$, $p''_1 \in P_1$ and $p''_2 \in P_2$ such that $\pi_1(p'_1) = q_1$, $\pi_2(p''_2) = q_2$, $p'_1 \in N_1$ and $p''_2 = \varphi(p''_1)$.
Consequently
\[ \langle u_1, p_1' \rangle = \langle u_1, dF_1 \rangle \quad \text{for each} \quad u_1 \in T_{q_1}K_1 \]
and
\[ \langle w_2, p_2' \rangle - \langle w_1, p_2' \rangle = \langle (w_2, w_1), dG \rangle \quad \text{if} \quad (w_2, w_1) \in T_{(q_2, q_1)}M. \]
If in addition \( \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0 \) for each \( v_1 \in T_{q_1}K_1 \) such that \( (v_2, v_1) \in T_{(q_2, q_1)}M \) and \( v_2 = 0 \) then \( \langle v_1, p_1' - p_1'' \rangle = 0 \) for each \( v_1 \) satisfying the same conditions. It follows from a simple algebraic argument that there are covectors \( p_1 \in P_1 \) and \( p_2 \in P_2 \) such that \( \pi_1(p_1) = q_1, \pi_2(p_2) = q_2, \langle u_1, p_1 \rangle = \langle u_1, p_1' \rangle = \langle u_1, dF_1 \rangle \) for each \( u_1 \in T_{q_1}K_1 \) and
\[ \langle w_2, p_2 \rangle - \langle w_1, p_1 \rangle = \langle w_2, p_2'' \rangle - \langle w_1, p_1'' \rangle = \langle (w_2, w_1), dG \rangle \]
for each \( (w_2, w_1) \in T_{(q_2, q_1)}M \). Hence \( p_1 \in N_1, p_2 = \varphi(p_1) \) and \( (q_2, q_1) \in K_{21} \).

The following proposition is an immediate consequence of the definition of \( K_{21} \).

**Proposition 3.4.** — The submanifold \( K_2 \) is the set
\[ \{ q_2 \in Q_2 ; \exists_{q_1 \in K_1} (q_2, q_1) \in K_{21} \}. \]

**Proposition 3.5.** — If \( (q_2, q_1) \in K_{21}, v_1 \in T_{q_1}K_1, v_2 \in T_{q_2}K_2 \) and \( (v_2, v_1) \in T_{(q_2, q_1)}M \) then \( \langle v_2, dF \rangle = \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle \).

**Proof (for \( P_1 = T^*Q_1 \) and \( P_2 = T^*Q_2 \)).** — If \( (q_2, q_1) \in K_{21} \) then there are covectors \( p_1 \in P_1 \) and \( p_2 \in P_2 \) such that \( \pi_1(p_1) = q_1, \pi_2(p_2) = q_2, p_1 \in N_1, p_2 \in N_2 \) and \( p_2 = \varphi(p_1) \). It follows that \( \langle u_1, p_1 \rangle = \langle u_1, dF_1 \rangle \) for each \( u_1 \in T_{q_1}K_1 \), \( \langle u_2, p_2 \rangle = \langle u_2, dF_2 \rangle \) for each \( u_2 \in T_{q_2}K_2 \) and
\[ \langle w_2, p_2 \rangle - \langle w_1, p_1 \rangle = \langle (w_2, w_1), dG \rangle \]
for each \( (w_2, w_1) \in T_{(q_2, q_1)}M \). Hence \( \langle v_2, dF_2 \rangle = \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle \) for each \( (v_2, v_1) \in T_{(q_2, q_1)}M \) such that \( v_1 \in T_{q_1}K_1 \) and \( v_2 \in T_{q_2}K_2 \).

Let \( G_{q_2} \) denote the function defined by \( G_{q_2}(q_1) = G(q_2, q_1) \). Then for each \( q_2 \in Q_2 \) the function \( G_{q_2} + F_1 \) is defined on the set \( \{ q_1 \in K_1 ; (q_2, q_1) \in M \} \).

The following two propositions are simplified versions of Propositions 3.3 and 3.4 valid under the additional assumption that for each \( q_2 \in Q_2 \) the set \( \{ q_1 \in K_1 ; (q_2, q_1) \in M \} \) is a submanifold of \( Q_1 \).

**Proposition 3.3'.** — The set \( K_{21} \) is the subset of \( Q_2 \times Q_1 \) such that \( (q_2, q_1) \in K_{21} \) if and only if \( q_1 \) is a critical point of \( G_{q_2} + F_1 \).

**Proposition 3.4'.** — The set \( K_2 \) is the subset of \( Q_2 \) such that \( q_2 \in K_2 \) if and only if \( G_{q_2} + F_1 \) has critical points.

For each \( q_2 \in K_2 \) the set of critical points of \( G_{q_2} + F_1 \) is the set \( \{ q_1 \in K_1 ; (q_2, q_1) \in K_{21} \} \). The following proposition holds if for each \( q_2 \in K_2 \) the set of critical points of \( G_{q_2} + F_1 \) is a connected submanifold of \( Q_1 \).
PROPOSITION 3.5’. — The function $F_2$ defined on $K_2$ by setting $F_2(q_2)$ equal to the (unique) critical value of $G_{q_2} + F_1$ is a generating function of $N_2$.

We write $F_2(q_2) = \text{Stat}_{q_1}(G(q_2, q_1) + F_1(q_1))$ meaning that $F_2(q_2)$ is equal to the function $G_{q_2} + F_1$ evaluated at a point $q_1$ at which it is stationary, that is at a critical point, and that $F_2(q_2)$ is not defined if no critical points of $G_{q_2} + F_1$ exist.

4. LOCAL EXPRESSIONS

Let $(x^i), 1 \leq i \leq n$ be local coordinates of a manifold $Q_1$. We use coordinates $(x^i, y^j), 1 \leq i, j \leq n$ of $P_1 = T^*Q_1$ such that $\theta_1 = \theta_{Q_1} = \sum y^i dx^i$. Let a Lagrangian submanifold $N_1$ of $(P_1, \omega_1)$ be generated by a function $F_1$ defined on a submanifold $K_1$ of $Q_1$. If the submanifold $K_1$ is described locally by equations $U^\kappa(x^i) = 0, 1 \leq \kappa \leq k$ and if $\overline{F}_1(x^i)$ is the local expression of an arbitrary (local) continuation $\overline{F}_1$ of the function $F_1$ to $Q_1$ then the Lagrangian submanifold $N_1$ is described by the equation

$$\sum_\kappa \lambda_\kappa dx^i = d(\overline{F}_1(x^i) + \sum_\kappa \lambda_\kappa U^\kappa(x^i)) = \sum_\kappa \left( \frac{\partial \overline{F}_1}{\partial x^i} + \sum_\kappa \lambda_\kappa \frac{\partial U^\kappa}{\partial x^i} \right) dx^i + \sum_\kappa U^\kappa(x^i) d\lambda_\kappa$$

equivalent to the system

$$y_i = \frac{\partial \overline{F}_1}{\partial x^i} + \sum_\kappa \lambda_\kappa \frac{\partial U^\kappa}{\partial x^i}, 1 \leq i \leq n$$

$$U^\kappa(x^i) = 0, 1 \leq \kappa \leq k.$$ We note that $F_1(x^i) = \text{Stat}_{(x^i, \overline{F}_1(x^i) + \sum_\kappa \lambda_\kappa U^\kappa(x^i))}$ is the local expression of $F_1$ for values of coordinates $(x^i), 1 \leq i \leq n$ satisfying $U^\kappa(x^i) = 0, 1 \leq \kappa \leq k$. In the special case of $K_1 = Q_1$ we have the equation

$$\sum_\kappa \lambda_\kappa dx^i = dF_1(x^i)$$
equivalent to $y_i = \frac{\partial F_1}{\partial x^i}, 1 \geq i \geq n$.

Let $(x'^i), 1 \leq i \leq n$ be local coordinates of a manifold $Q_2$ and let $(x'^i, y'^j), 1 \leq i, j \leq n$ be coordinates of $P_2 = T^*Q_2$ such that $\theta_2 = \theta_{Q_2} = \sum y'^i dx'^i$. We use coordinates $(x'^i, x^j), 1 \leq i, j \leq n$ for $Q_2 \times Q_1$ and coordinates $(x'^i, y'^j, x^k, y^l), 1 \leq i, j, k, l \leq n$ for $P_2 \times P_1$. The local expression of the form $\theta_2 \ominus \theta_1$ is $\theta_2 \ominus \theta_1 = \sum (y'^i dx'^i - y^l dx^i)$. Let a symplectic diffeomorphism $\phi$ of $(P_1, \omega_1)$ onto $(P_2, \omega_2)$ be generated by a function $G$ defined on a submanifold $M$ of $Q_2 \times Q_1$. Let the submanifold $M$ be described locally by equations $W^\mu(x'^i, x^i) = 0, 1 \leq \mu \leq m$ and let $\overline{G}(x'^i, x^i)$ be the local expression of an arbitrary continuation $\overline{G}$ of the function $G$ to $Q_2 \times Q_1$. An implicit description of the diffeomorphism $\phi$ is given by the equation

\[\Sigma_i (y'_i dx'^i - y_i dx^i) = d(\overline{G}(x'^i, x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j))\] equivalent to the system

\[
y'_i = \frac{\partial \overline{G}}{\partial x'^i} + \Sigma_\mu v_\mu \frac{\partial W^\mu}{\partial x'^i}, \quad 1 \leq i \leq n
\]

\[
y_i = -\frac{\partial \overline{G}}{\partial x^i} + \Sigma_\mu v_\mu \frac{\partial W^\mu}{\partial x^i}, \quad 1 \leq i \leq n
\]

\[W^\mu(x'^i, x^j) = 0, \quad 1 \leq \mu \leq m.\]

The local expression of \(G\) for values of coordinates \((x'^i, x^j), 1 \leq i, j \leq n\) satisfying \(W^\mu(x'^i, x^j) = 0, \quad 1 \leq \mu \leq m\) is obtained from

\[G(x'^i, x^j) = \text{Stat}(v_\mu) [G(x'^i, x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j)].\]

If \(M = Q_2 \times Q_1\) then we have the equation \(\Sigma_i (y'_i dx'^i - y_i dx^i) = dG(x'^i, x^j)\) equivalent to \(y'_i = \frac{\partial G}{\partial x'^i}, \quad y_i = -\frac{\partial G}{\partial x^i}, \quad 1 \leq i \leq n\).

If the Lagrangian submanifold \(N_2 = \varphi(N_1)\) is generated by a generating function then \(N_2\) is described by the equation

\[\Sigma_i y'_i dx'^i = d(\overline{G}(x'^i, x^j) + F_1(x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j) + \Sigma_\nu \lambda_\nu U^\nu(x^j)).\]

Hence the local expression of a generating function \(F_2\) of \(N_2\) is

\[F_2(x'^i) = \text{Stat}_{(x^i, \lambda_\nu, v_\mu)}[\overline{G}(x'^i, x^j) + F_1(x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j) + \Sigma_\nu \lambda_\nu U^\nu(x^j)].\]

If \(K_1 = Q_1\) and \(M = Q_2 \times Q_1\) then \(F_2(x'^i) = \text{Stat}(\varphi)[G(x'^i, x^j) + F_1(x^j)].\)

The following simple example illustrates composition of generating functions. Let \(Q_1\) and \(Q_2\) be manifolds of dimension 2. The submanifold \(N_1\) of \(P_1\) described locally by equations \(y_1 = 2x_1(1 - y_2), \quad x_2 = (x_1)^2\) is a Lagrangian submanifold of \((P_1, \omega_1)\). The mapping \(\varphi : P_1 \to P_2\) described locally by equations \(x'^1 = x^1, \quad x'^2 = -y_2, \quad y'_1 = y_1, \quad y'_2 = x^2\) is a symplectic diffeomorphism of \((P_1, \omega_1)\) onto \((P_2, \omega_2)\). The Lagrangian submanifold \(N_1\) is generated by a function \(F_1\) on a submanifold \(K_1\) of \(Q_1\). The submanifold \(K_1\) is described by \(U(x^1, x^2) = x^2 - (x^1)^2 = 0\) and \(\overline{F}_1(x^1, x^2) = (x^1)^2\) is the local expression of a continuation of \(F_1\) to \(Q_1\). The symplectic diffeomorphism \(\varphi\) is generated by a function \(G\) defined on a submanifold \(M\) of \(Q_2 \times Q_1\). The submanifold \(M\) is described locally by

\[W(x'^1, x'^2, x^1, x^2) = x'^1 - x^1 = 0 \quad \text{and} \quad \overline{G}(x'^1, x'^2, x^1, x^2) = x'^2 x^2\]

is the local expression of a continuation of \(G\) to \(Q_2 \times Q_1\). The Lagrangian submanifold \(N_2\) is generated by a function \(F_2\) defined on \(Q_2\). The local expression of \(F_2\) is \(F_2(x'^1, x'^2) = (x'^1)^2(1 + x'^2)^2\). The relation

\[F_2(x'^1, x'^2) = \text{Stat}_{(x^1, x^2, v_\mu)}[\overline{G}(x'^1, x'^2, x^1, x^2) + \overline{F}_1(x^1, x^2) + \nu W(x'^1, x'^2, x^1, x^2) + \lambda U(x^1, x^2)]\]

is easily verified.
5. THE LEGENDRE TRANSFORMATION

Let \((P, \omega)\) be the underlying symplectic manifold of two special symplectic manifolds \((P, Q_1, \pi_1, \theta_1)\) and \((P, Q_2, \pi_2, \theta_2)\). Lagrangian submanifolds of \((P, \omega)\) may be generated by generating functions with respect to both special symplectic structures.

**Definition 5.1.** — The transition from the representation of Lagrangian submanifolds of \((P, \omega)\) by generating functions with respect to \((P, Q_1, \pi_1, \theta_1)\) to the representation by generating functions with respect to \((P, Q_2, \pi_2, \theta_2)\) is called the Legendre transformation from \((P, Q_1, \pi_1, \theta_1)\) to \((P, Q_2, \pi_2, \theta_2)\).

Let the identity mapping of \(P\) be generated with respect to \((P \times P, Q_2 \times Q_1, \pi_2 \times \pi_1, \theta_2 \ominus \theta_1)\) by a generating function \(E_{21}\) defined on a submanifold \(I_{21}\) of \(Q_2 \times Q_1\).

**Definition 5.2.** — The function \(E_{21}\) is called a generating function of the Legendre transformation from \((P, Q_1, \pi_1, \theta_1)\) to \((P, Q_2, \pi_2, \theta_2)\).

If \(F_1\) is a generating function of a Lagrangian submanifold \(N\) of \((P, \omega)\) with respect to \((P, Q_1, \pi_1, \theta_1)\) and if the special conditions assumed at the end of Section 3 hold then the Legendre transformation leads to a function \(F_2\) satisfying \(F_2(q_2) = \text{Stat}_{q_1} [E_{21}(q_2, q_1) + F_1(q_1)]\).

Physicists use the term Legendre transformation also in a different sense. Let \(\Delta : P \to P \times P\) denote the diagonal mapping. If the image \(K_{21}\) of \(N\) by the mapping \((\pi_2 \times \pi_1) \circ \Delta : P \to Q_2 \times Q_1\) is the graph of a mapping \(\kappa_{21} : Q_1 \to Q_2\) then \(\kappa_{21}\) is called the Legendre transformation of \(Q_1\) into \(Q_2\) corresponding to \(N\). We call \(K_{21}\) the Legendre relation and \(\kappa_{21}\) the Legendre mapping of \(Q_1\) into \(Q_2\) corresponding to \(N\). The Legendre relation can be obtained from the generating functions \(F_1\) and \(E_{21}\) following Proposition 3.3 or Proposition 3.3'.

6. THE LEGENDRE TRANSFORMATION

OF PARTICLE DYNAMICS

Let \(\Phi_p\) denote the graded algebra of differential forms on a manifold \(P\) and let \(\Phi_{TP}\) be the graded algebra of forms on the tangent bundle \(TP\) of \(P\). A linear mapping \(a : \Phi_p \to \Phi_{TP}: \mu \mapsto a\mu\) is called a derivation of degree \(r\) of \(\Phi_p\) into \(\Phi_{TP}\) relative to \(\tau_p\) if

\[
\text{degree} (a\mu) = \text{degree} \mu + r \quad \text{and} \quad a(\mu \wedge \nu) = a\mu \wedge \tau_p^* \nu + (-1)^p \tau_p^* \mu \wedge a\nu,
\]

where \(p = \text{degree} \mu\).

An important property of derivations is that a derivation is completely characterized by its action on functions and 1-forms [3]. We define derivations \(i_T\) and \(d_T\) of \(\Phi_p\) into \(\Phi_{TP}\) of degrees \(-1\) and \(0\) respectively [7], [8].
If \( f \) is a function on \( P \) then \( i_f f = 0 \) and if \( \mu \) is a 1-form on \( P \) then \( i_\mu \) is a function on \( TP \) defined by \( (i_\mu)(u) = \langle u, \mu \rangle \) for each \( u \in TP \). The derivation \( d_T \) is defined by \( d_T \mu = i_T d\mu + d i_T \mu \) for each \( \mu \in \Phi_p \).

We summarize results derived in earlier publications [6], [8], [9], [10]. Let \( P \) be the cotangent bundle \( T^*Q \) of a differential manifold \( Q \). Let \( \pi \) denote the bundle projection \( \pi_Q : P \to Q \), let \( \theta \) be the canonical 1-form \( \theta_Q \) on \( P \) and \( \omega \) the canonical 2-form \( \omega_Q = d \theta_Q \) on \( P \). The tangent bundle \( TP \) together with the 2-form \( d_T \omega \) form a symplectic manifold \( (TP, d_T \omega) \). The symplectic manifold \( (TP, d_T \omega) \) is the underlying symplectic manifold of two special symplectic manifolds \( (TP, T^*_Q, \tau, \lambda) \) and \( (TP, T^*_Q, \tau, \lambda) \), where \( \tau \) is the tangent bundle projection \( \tau_p : TP \to P \), \( \chi \) is the 1-form \( i_T \omega \) and \( \lambda \) is the 1-form \( d_T \theta \).

Let \( Q \) be the configuration manifold of a particle system and let the dynamics of the system be represented by a Lagrangian submanifold \( D \) of \( (TP, d_T \omega) \) [8], [9], [10]. If \( D \) is generated by generating functions with respect to both special symplectic structures given above then the generating functions are related by Legendre transformations.

**Definition 6.1.** If the Lagrangian submanifold \( D \) representing the dynamics of a particle system is generated with respect to the special symplectic structure \( (TP, T^*_Q, \tau, \lambda) \) by a generating function \( L \) on a submanifold \( J \) of \( T^*_Q \) then \( L \) is called a Lagrangian of the particle system and \( J \) is called the Lagrangian constraint.

**Definition 6.2.** If the Lagrangian submanifold \( D \) is generated with respect to the special symplectic structure \( (TP, P, \tau, \chi) \) by a function \( F \) on a submanifold \( K \) of \( P \) then the function \( H = - F \) is called a Hamiltonian of the particle system and \( K \) is called the Hamiltonian constraint.

**Definition 6.3.** The Legendre transformation from \( (TP, T^*_Q, \tau, \lambda) \) to \( (TP, P, \tau, \chi) \) is called the Legendre transformation of particle dynamics and the Legendre transformation from \( (TP, P, \tau, \chi) \) to \( (TP, T^*_Q, \tau, \lambda) \) is called the inverse Legendre transformation of particle dynamics.

**Proposition 6.1.** The Legendre transformation of particle dynamics is generated by the function \( E \) defined on the Whitney sum

\[
I = T^*Q \times_Q T^*Q \subset P \times T^*Q
\]

by \( E(p, v) = - \langle v, p \rangle \).

**Proof.** Let \( \rho \) be the mapping defined by the commutative diagram

\[
\begin{array}{ccc}
TP & \xrightarrow{\Lambda} & TP \times TP \\
\downarrow \rho & & \downarrow \tau \times T\pi \\
I & \xrightarrow{\text{injection}} & P \times T^*Q,
\end{array}
\]

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where $\Delta$ is the diagonal mapping. Then
\[
(\mathbf{E} \circ \rho)(w) = \mathbf{E}(\tau(w), T\tau(w)) = - \langle T\tau(w), \tau(w) \rangle = - \langle w, \delta \rangle.
\]
Hence $\mathbf{E} \circ \rho = - i_\tau \delta$. Further
\[
\Delta^*(\chi \otimes \lambda) = \chi - \lambda = i_\tau d\delta - d_\tau \delta = - di_\tau \delta = d(\mathbf{E} \circ \rho).
\]
It follows that the diagonal of $TP \times TP$ is contained in the Lagrangian submanifold generated by $\mathbf{E}$. The diagonal of $TP \times TP$ and the Lagrangian submanifold generated by $\mathbf{E}$ are closed submanifolds of $TP \times TP$ of the same dimension. If $Q$ is connected then the Lagrangian submanifold generated by $\mathbf{E}$ is connected and hence equal to the diagonal of $TP \times TP$. If $Q$ is not connected then the same argument applies to each connected component of $Q$.

The proof of the following proposition is similar.

**Proposition 6.2.** — The inverse Legendre transformation of particle dynamics is generated by the function $\mathbf{E}'$ on $I' = TQ \times QT^*Q$ defined by
\[
\mathbf{E}'(v, p) = \langle v, p \rangle.
\]

### 7. Local Expressions and Examples

Let $(x^i, 1 \leq i \leq n)$ be local coordinates of $Q$ and $(x^i, y_j, 1 \leq i, j \leq n)$ local coordinates of $P = T^*Q$ such that $\theta_Q = \sum_i y_i dx^i$. We use coordinates $(x^i, x_j), 1 \leq i, j \leq n$ for $TQ$ and coordinates $(x^i, y_j, x^k, y^l), 1 \leq i, j, k, l \leq n$ for $TP$. Functions $\dot{x}^i$ and $\dot{y}_j$ are defined by $\dot{x}^i = d_\tau x^i$ and $\dot{y}_j = d_\tau y_j$. Local expressions of the forms $d_\tau \omega, \lambda$ and $\chi$ are $d_\tau \omega = \Sigma_i (dy^i dx^i + dy^i dx^i), \lambda = \Sigma_i (y_i dx^i + y_i dx^i)$ and $\chi = \Sigma_i (y_i dx^i - x^i dy_i)$. Let $(x^i, \dot{x}^i, y_k, 1 \leq i, j, k \leq n$ be coordinates of $I$ and also of $I'$. Then $E(x^i, \dot{x}^i, y_k) = - \Sigma_i y_i x^i$ and $E'(x^i, \dot{x}^i, x_k) = \Sigma_i y_i x^i$ are local expressions of functions $E$ and $E'$.

**Example 7.1.** — Let $Q$ be the configuration manifold of a non-relativistic particle of mass $m$ and let $V(x^i)$ be the local expression of the potential energy of the particle. The dynamics of the particle is represented by the Lagrangian submanifold $D$ of $(TP, d_\tau \omega)$ defined locally by $y_i = m\dot{x}^i$ and $\dot{y}_j = - \frac{\partial V}{\partial x^j}$. The submanifold $D$ can also be described by equations
\[
\Sigma_i (y_i dx^i + y_i dx^i) = d \left( \frac{1}{2} m\dot{x}^i - V(x^i) \right)
\]
or
\[
\Sigma_i (y^i dx^i - \dot{x}^i dy_i) = - d \left( \frac{1}{2m} \dot{x}^i - V(x^i) \right).
\]
Hence
\[ L(x^i, \dot{x}^j) = \frac{1}{2} m \sum_j (\dot{x}^j)^2 - V(x^i) \quad \text{and} \quad H(x^i, y_j) = \frac{1}{2m} \sum_j (y_j)^2 + V(x^i) \]
are local expressions of a Lagrangian \( L \) and a Hamiltonian \( H \). Relations
\[ \text{Stat}_{(x^i)}[\sum_j y_j \dot{x}^j - L(x^i, \dot{x}^j)] \quad \text{and} \quad \text{Stat}_{(y_k)}[\sum_j y_j \dot{x}^j - H(x^i, y_k)] \]
are local expressions of the Legendre transformation and the inverse Legendre transformation.

The following example illustrates a situation slightly more general than that described in Section 6.

**Example 7.2.** — Let \( Q \) be the flat space-time of special relativity, let \((x^i), 0 \leq i \leq 3)\) be affine coordinates of \( Q \) and let \( g_{ij}, 0 \leq i, j \leq 3 \) be components of the constant indefinite metric tensor on \( Q \). The dynamics of a free particle of mass \( m \) is represented by the Lagrangian submanifold \( D \) defined locally by \( y_3 = 0 \) and \( y_1^3 > 0 \). The definition is equivalent to: \( \sum_i (g_{ij} y_j + \dot{x}^i) > 0 \).

Hence \( D \) is generated by a Lagrangian \( L(x^i, \dot{x}^j) = \sum_i (g_{ij} y_j + \dot{x}^i) \) defined on the open submanifold \( J \) of \( TQ \) satisfying \( \dot{x}^i > 0 \). The submanifold \( D \) is not generated by a generating function with respect to \((P, P, \tau, \chi)\).

The definition of \( D \) is equivalent to: there is a number \( \lambda > 0 \) such that \( \sum_i (g_{ij} y_j + \dot{x}^i) = -d[\lambda (\sum_i g_{ij} y_j y_j)^{1/2} - m] \), where \( g^{ij}, 0 \leq i, j \leq 3 \) are components of the contravariant metric tensor. We call the function \( H \) defined locally on \( P \times R \) by \( H(x^i, y_j, \lambda) = \lambda (\sum_i g^{ij} y_j y_j)^{1/2} - m \) the generalized Hamiltonian of the relativistic particle. We call the submanifold \( K \) of \( P \) defined by \( \sum_i g^{ij} y_j = m \) the Hamiltonian constraint. The relation
\[ \sum_i (g_{ij} y_j + \dot{x}^i)^{1/2} = \text{Stat}_{(y_i, \lambda > 0)}[\sum_i y_i \dot{x}^i - \lambda (\sum_k g^{kj} y_k y_i)^{1/2} - m] \]
is the local expression of a generalized version of the inverse Legendre transformation.

8. **LEGENDRE TRANSFORMATIONS IN THERMOSTATICS OF IDEAL GASES**

Let \( P \) be a manifold with coordinates \((V, S, p, T)\) interpreted as the volume, the metrical entropy, the pressure and the absolute temperature respectively of one mole of an ideal gas. The manifold \( P \) together with the form
\[ \omega = dV \wedge dp + dT \wedge dS \]

is the local expression of a generalized version of the inverse Legendre transformation.
define a symplectic manifold \((P, \omega)\). The behaviour of the gas is governed by the two equations of state: \(pV = RT\) and \(pV' = K \exp \frac{S}{c_v}\), where \(R\), \(\gamma\) and \(K\) are constants and \(c_v = \frac{R}{\gamma - 1}\). It is easy to see that the equations of state define a Lagrangian submanifold \(N\) of \((P, \omega)\).

Let \(Q_1, Q_2, Q_3\) and \(Q_4\) be manifolds with coordinate systems \((V, S)\), \((V, T)\), \((p, T)\) and \((S, p)\) respectively. The mappings

\[
\begin{align*}
\pi_1 : P &\rightarrow Q_1 : (V, S, p, T) \mapsto (V, S), \\
\pi_2 : P &\rightarrow Q_2 : (V, S, p, T) \mapsto (V, T), \\
\pi_3 : P &\rightarrow Q_3 : (V, S, p, T) \mapsto (p, T), \\
\pi_4 : P &\rightarrow Q_4 : (V, S, p, T) \mapsto (S, p)
\end{align*}
\]

and forms

\[
\begin{align*}
\vartheta_1 &= -pdV + TdS, \\
\vartheta_2 &= -pdV - SdT, \\
\vartheta_3 &= Vdp - SdT, \\
\vartheta_4 &= Vdp + TdS
\end{align*}
\]

define special symplectic manifolds \((P, Q_1, \vartheta_1)\), \((P, Q_2, \vartheta_2)\), \((P, Q_3, \vartheta_3)\) and \((P, Q_4, \vartheta_4)\). The Lagrangian submanifold \(N\) is generated by generating functions \(F_1 = U\), \(F_2 = F\), \(F_3 = G\) and \(F_4 = H\) with respect to the above special symplectic structures. The generating functions are given by formulæ:

\[
\begin{align*}
U(V, S) &= \frac{K}{\gamma - 1} V^{(1-\gamma)} \exp \frac{S}{c_v}, \\
F(V, T) &= c_v T(1 - \ln T + \ln K - \ln R) - RT \ln V, \\
G(p, T) &= c_p T(1 - \ln T - \ln R) + c_v T \ln K + RT \ln p,
\end{align*}
\]

and

\[
H(S, p) = \frac{\gamma}{\gamma - 1} K^\frac{\gamma - 1}{p \gamma} \exp \frac{S}{c_p},
\]

where \(c_p = R + c_v\). The generating functions \(U\), \(F\), \(G\) and \(H\) are known as thermodynamic potentials and are called the internal energy, the Helmholtz function, the Gibbs function and the enthalpy respectively.

Three examples of the twelve Legendre transformations relating the four special symplectic structures are given below. The mapping \(\pi_2 \times \pi_1\) maps the diagonal of \(P \times P\) onto a submanifold \(I_{21}\) of \(Q_2 \times Q_1\) with coordinates \((V, S, T)\) related to the coordinates \((V, S, p, T)\) in an obvious way. The Legendre transformation from \((P, Q_1, \vartheta_1, \vartheta_1)\) to \((P, Q_2, \vartheta_2, \vartheta_2)\) is generated by the function \(E_{21}\) defined on \(I_{21}\) by \(E_{21}(V, S, T) = -TS\). The Legendre transformation from \((P, Q_1, \vartheta_1, \vartheta_1)\) to \((P, Q_3, \vartheta_3, \vartheta_3)\) is generated by the function \(E_{31}\) defined on \(I_{31}\) by \(E_{31}(V, S, p, T) = pV - TS\).

and the Legendre transformation from \((P, Q_1, \pi_1, \vartheta_1)\) to \((P, Q_4, \pi_4, \vartheta_4)\) is generated by the function \(E_{41}\) on a submanifold \(I_{41}\) of \(Q_4 \times Q_1\) with coordinates \((V, S, p)\) defined by \(E_{41}(V, S, p) = pV\). Relations
\[
\begin{align*}
F(V, T) &= \text{Stat}_S (U(V, S) - TS), \\
G(p, T) &= \text{Stat}_{\{V,S\}} (U(V, S) + pV - TS), \\
H(S, p) &= \text{Stat}_V (U(V, S) + pV)
\end{align*}
\]
are easily verified.

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