An affine framework for the dynamics of charged particles

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1 Introduction

Gauge independence of the Lagrangian formulation of dynamics of charged particles can be achieved by increasing the dimension of the configuration space of the particle. The four dimensional space-time of general relativity is replaced by the five dimensional space-time-phase of Kaluza. The phase space of the particle is the cotangent bundle of the Kaluza space and the gauge independent Lagrangian is a function on the tangent bundle of the Kaluza space [1]. An alternate approach is proposed in the present note. The four dimensional space-time is used as the configuration space of the charged particle. The phase space is no longer a cotangent bundle and not even a vector bundle. It is an affine bundle modelled on the cotangent bundle of the space-time manifold. The Lagrangian is a section of an affine line bundle over the tangent bundle of the space-time manifold.

2 Affine fibrations

Let $\xi: E \to M$ be a vector fibration. An *affine fibration modelled on* ξ is a differential fibration $\eta: A \to M$ and a differentiable mapping $\rho: A \times_M A \to E$ such that

1. $\xi \circ \rho = \eta \times_M \eta$,
2. $\rho(a_3, a_2) + \rho(a_2, a_1) = \rho(a_3, a_1)$ for each triple $(a_3, a_2, a_1) \in A \times_M A \times_M A$,
3. for each local section $\sigma: U \to A$ of η, the mapping $\rho_\sigma: \eta^{-1}(U) \to \xi^{-1}(U)$ defined by

   $$\rho_\sigma(a) = \rho(a, \sigma(\eta(a)))$$

   is a diffeomorphism.

For each pair $(a_2, a_1) \in A \times_M A$ we will use the symbol $a_2 - a_1$ to denote the element $\rho(a_2, a_1) \in E$. We will write $a_2 = a_1 + e$ if $a_2 - a_1 = e$. These notational conventions are extended to local sections of A and E. If α_2 and α_1 are local sections of A over $U \subset M$ then $\alpha_2 - \alpha_1$ denotes the local section of E defined by $(\alpha_2 - \alpha_1)(m) = \alpha_2(m) - \alpha_1(m)$. We will write $\alpha_2 = \alpha_1 + \varepsilon$ if $\alpha_2 - \alpha_1 = \varepsilon$.

An affine fibration modelled on the trivial vector fibration $pr_M\colon M \times \mathbf{R} \to M$ is usually interpreted as a principal fibration with structure group $\mathbf{R}$.

3 The phase fibration and the contact fibration of a principal fibration

Let $\mathbf{Z} = (\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$. We define an equivalence relation in the set of all pairs (m, σ), where m is a point in M and σ is a section of ζ. Two pairs (m, σ) and (m', σ') are equivalent if $m' = m$ and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We have identified the section $\sigma' - \sigma$ of pr_M with a function on M for the purpose of evaluating the differential $\mathrm{d}(\sigma' - \sigma)(m)$. We denote by $\mathrm{P}Z$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathrm{d}\sigma(m)$ and will be called the *differential* of σ at m. We define a mapping $\mathrm{P}\zeta\colon \mathrm{P}Z \to M$ by $\mathrm{P}\zeta(\mathrm{d}\sigma(m)) = m$. We define a mapping $\mathrm{P}\rho\colon \mathrm{P}Z \times_M \mathrm{P}Z \to \mathrm{T}^*M$ by

$$\mathrm{P}\rho(\mathrm{d}\sigma_2(m), \mathrm{d}\sigma_1(m)) = \mathrm{d}(\sigma_2 - \sigma_1)(m).$$

The pair $\mathrm{P}\mathbf{Z} = (\mathrm{P}\zeta, \mathrm{P}\rho)$ is an affine fibration modelled on the fibration $\pi_M\colon \mathrm{T}^*M \to M$. This fibration is called the *phase fibration* of $\mathbf{Z} = (\zeta, \rho)$. Let φ be a section of $\mathrm{P}\zeta$ and let σ be a section of ζ. We define the *differential* $\mathrm{d}\varphi$ of φ by $\mathrm{d}\varphi = \mathrm{d}(\varphi - \mathrm{d}\sigma)$. Since for two sections σ, σ' of ζ we have $\mathrm{d}(\mathrm{d}\sigma - \mathrm{d}\sigma') = \mathrm{dd}(\sigma - \sigma') = 0$ it follows that the definition of the differential does not depend on the choice of σ.

Let (ζ, ρ) be again an affine fibration modelled on the trivial fibration $pr_M : M \times \mathbf{R} \to M$. We define an equivalence relation in the set of all pairs (m, σ) , where m is a point in M and σ is a section of ζ. Two pairs (m, σ) and $(m, , \sigma')$ are equivalent if $m = m'$, $(\sigma' - \sigma)(m) = 0$ and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We denote by $\mathrm{C}Z$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathrm{C}\sigma(m)$ and will be called the *contact element* of σ at m. We define a mapping $\mathrm{C}\zeta\colon \mathrm{C}Z \to M$ by $\mathrm{C}\zeta(\mathrm{C}\sigma(m) = m$. We define a mapping $\mathrm{C}\rho\colon \mathrm{C}Z \times_M \mathrm{C}Z \to \mathrm{T}^*M \times \mathbf{R}$ by

$$\mathrm{C}\rho(\mathrm{C}\sigma_2(m), \mathrm{C}\sigma_1(m)) = ((\sigma_2 - \sigma_1)(m), \mathrm{d}(\sigma_2 - \sigma_1)(m).$$

The pair $(\mathrm{C}\zeta, \mathrm{C}\rho)$ is an affine fibration modelled on the vector fibration $\mathrm{T}^*M \times \mathbf{R} \to M$.

There is a natural morphism $\gamma_{\mathbf{Z}}$ between $\mathrm{C}Z$ and $\mathrm{P}Z$ defined by

$$\gamma_{\mathbf{Z}}(\mathrm{C}\sigma(m)) = \mathrm{d}\sigma(m).$$

The pair $(\gamma_{\mathbf{Z}}, \tilde{\mathrm{C}}\rho)$ where $\tilde{\mathrm{C}}\rho$ denotes the mapping $\mathrm{C}\rho$ restricted to $\mathrm{C}Z \times_{\mathrm{P}Z} \mathrm{C}Z$ is an affine fibration modelled on the trivial fibration $\mathrm{P}Z \times \mathbf{R} \to \mathrm{P}Z$. This fibration will be called the *contact fibration* of (ζ, ρ) and will be denoted $\mathrm{C}\mathbf{Z}$. The fibration $\mathrm{C}\mathbf{Z}$ is a pull-back of the fibration (ζ, ρ) with respect to the mapping $\mathrm{d}\zeta$.

4 The symplectic structure of the phase fibration

Let $(\eta\colon A \to M, \rho\colon A \times_M A \to \mathrm{T}^*M)$ be an affine fibration modelled on the cotangent fibration $\pi_M\colon \mathrm{T}^*M \to M$. We define an equivalence relation between triples (a, φ, ψ) where a is a point in A, φ is a section of η and ψ is a section of the cotangent fibration $\pi_A\colon \mathrm{T}^*A \to A$. Two triples (a, φ, ψ) and (a', φ', ψ') are equivalent if $a = a'$ and $(\eta^*(\varphi - \varphi'))(a) = (\psi' - \psi)(a)$. We denote by η^*A the set of equivalence classes. We define a mapping $\rho^*\colon \eta^*A \times \eta^*A \to \mathrm{T}^*A$ by

$$\rho^*([(a, \varphi, \psi)], [(a, \varphi', \psi')]) = \eta^*(\varphi - \varphi)(a) + (\psi - \psi')(a).$$

The pair (η^*, ρ^*) is an affine fibration modelled on the vector fibration π_A. We define a section $\vartheta_{\mathbf{A}}$ of η^* by $\vartheta_{\mathbf{A}}(a) = [(a, \varphi, 0)]$ where φ is such that $\varphi(\eta(a)) = a$. The section $\vartheta_{\mathbf{A}}$ will be called the *Liouville section* of $\mathbf{A}$. For a section φ of η we define a section $\eta^*\varphi$ of the fibration (η^*, ρ^*) by $\eta^*\varphi(a) = [(a, \varphi, 0)]$. We observe that for two sections φ and φ' of η we have $\eta^*\varphi - \eta^*\varphi' = \eta^*(\varphi - \varphi')$.

Let $(\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$. Let φ be a section of the affine fibration $((\mathrm{P}\zeta)^*, (\mathrm{P}\rho)^*)$. We define a 2-form $\mathrm{d}\varphi$ on $\mathrm{P}Z$ by $\mathrm{d}\varphi = \mathrm{d}(\varphi - (\mathrm{P}\zeta)^*\mathrm{d}\sigma)$ where σ is a section of ζ. The definition does not depend on the choice of a section σ because for two sections σ and σ' of ζ we have $\mathrm{d}(\mathrm{P}\zeta^*\mathrm{d}\sigma - \mathrm{P}\zeta^*\mathrm{d}\sigma') = \mathrm{d}(\mathrm{P}\zeta^*(\mathrm{d}\sigma - \mathrm{d}\sigma') = (\mathrm{P}\zeta^*\mathrm{dd}(\sigma - \sigma') = 0$.

PROPOSITION 4.1 *The differential $\omega_{\mathrm{P}\mathbf{Z}}$ of the Liouville section $\vartheta_{\mathrm{P}\mathbf{Z}}$ is a symplectic form on $\mathrm{P}Z$.*

Proof. It is enough to observe that for a section σ of ζ the 1-form $\vartheta_{\mathrm{P}\mathbf{Z}} - \mathrm{P}\zeta^*\mathrm{d}\sigma$ on $\mathrm{P}Z$ is ϑ on T^*M is the pull-back of the canonical 1-form on T^*M with respect to a mapping $\mathrm{P}\rho_\sigma\colon \mathrm{P}Z \to \mathrm{T}^*M$ defined by $\mathrm{P}\rho_\sigma(a) = a - \mathrm{d}\sigma(a)$. $\square$

Thus we have shown that $\mathrm{P}Z$ has the canonical structure of a symplectic manifold.

5 The structure of the contact fibration

Let $\mathbf{Z} = (\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$.

PROPOSITION 5.1 *There is a canonical isomorphism of affine fibrations* $\mathrm{PC}\mathbf{Z}$ *and* $((\mathrm{P}\zeta)^*, (\mathrm{P}\rho)^*)$.

Proof. Let us choose a section σ of ζ. We define a section $\tilde{\sigma}$ of $\gamma_{\mathbf{Z}}$ by

$$\tilde{\sigma}(d) = [(\mathrm{P}\zeta(d), \sigma)] + (d - \mathrm{d}\sigma(\mathrm{P}\zeta(d), 0).$$

A mapping $\Phi_\sigma\colon \mathrm{PC}Z \to \mathrm{T}^*\mathrm{P}Z$ defined by

$$\Phi_\sigma(a) = a - \mathrm{d}\tilde{\sigma}(\mathrm{P}\gamma_{\mathbf{Z}}(a)$$

is an isomorphism of affine fibrations. A mapping $\Psi_\sigma\colon (\mathrm{P}\zeta)^* \to \mathrm{T}^*\mathrm{P}Z$ defined by $\Psi_\sigma(b) = \mathrm{d}\psi(a)$ where $b = [(a, \mathrm{d}\sigma, \psi)]$ is also an isomorphism of affine fibrations. We observe, that for two sections σ and σ' of ζ we have $\Phi_\sigma(d) = \Phi_{\sigma'}(d) + (\mathrm{P}\zeta)^*\mathrm{d}(\sigma - \sigma')(\gamma_{\mathbf{Z}}(b))$ and $\Psi_\sigma(b) =$

$\Psi'_\sigma(b) + (\mathrm{P}\zeta)^*\mathrm{d}(\sigma - \sigma')(\gamma_{\mathbf{Z}}(b))$. It follows that $(\Phi_\sigma)^{-1} \circ \Psi_\sigma$ does not depend on the choice of σ and defines a canonical isomorphism of $\mathrm{P}\tau(\zeta)$ and $(\mathrm{P}\zeta)^*$. $\square$

We define a canonical section $\vartheta_{\mathbf{Z}}$ of $\mathrm{P}\gamma_{\mathbf{Z}}$ as $\vartheta_{\mathrm{P}\mathbf{Z}}$ transported by the isomorphism introduced in the proposition. We observe that $\mathrm{d}\vartheta_{\mathbf{Z}} = \mathrm{d}\vartheta_{\mathrm{P}\mathbf{Z}}$. We refer to $\vartheta_{\mathbf{Z}}$ as a connection in the fibration $\gamma_{\mathbf{Z}}$ and to the symplectic form $\mathrm{d}\vartheta_{\mathbf{Z}} = \omega_{\mathrm{P}\mathbf{Z}}$ as the curvature form of this connection.

6 The reduced tangent fibration

Let $\mathbf{Z} = (\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$. We introduce an equivalence relation in the set of triples (v, a, r) where v is an element of the tangent fibration $\tau_M\colon \mathrm{T}M \to M$, a is an element of the phase fibration $\mathrm{P}\zeta$ such that $\tau_M(v) = \mathrm{P}\zeta(a)$ and r is a real number. Two such triples (v, a, r) and (v', a', r') are equivalent if and only if $v = v'$ and $\langle v, a - a'\rangle = r' - r$. We denote by $\widetilde{\mathrm{T}}Z$ the set of equivalence classes. The class of $(v, a, 0)$ will be denoted by $\langle v, a\rangle$ and will be called the *evaluation* of a on v. We define a mapping $\widetilde{\tau}\colon \widetilde{\mathrm{T}}Z \to \mathrm{T}M$ by $\widetilde{\mathrm{T}}\zeta([(v, a, r)] = v$ and a mapping $\widetilde{\mathrm{T}}\rho\colon \widetilde{\mathrm{T}}Z \times_{\mathrm{T}M} \widetilde{\mathrm{T}}Z \to \mathrm{T}M \times \mathbf{R}$ by

$$\widetilde{\mathrm{T}}\rho([(v, a, r)], [(v, a', r')]) = (v, r - r' + \langle v, a - a'\rangle).$$

The affine fibration $\widetilde{\mathrm{T}}\mathbf{Z} = (\widetilde{\mathrm{T}}\zeta, \widetilde{\mathrm{T}}\rho)$ will be called the *reduced tangent fibration* of $\mathbf{Z}$.

For two triples (v, a, r) and ($v', a', r')$ such that $\tau_M(v) = \tau_M(v')$ we introduce the sum

$$[(v, a, r)] + [(v', a', r')] = [(v + v', a, r + r' + \langle v', a' - a\rangle)].$$

For a number s and a triple (v, a, r) we introduce the product

$$s[(v, a, r)] = [(sv, a, sr)].$$

With this operations the fibration $\tau_Z\colon \widetilde{\mathrm{T}}Z \to M$ is a vector fibration.

Let σ be a section of ζ. We define a section $\widetilde{\mathrm{T}}\sigma$ of $\widetilde{\mathrm{T}}\zeta$ by

$$\widetilde{\mathrm{T}}\sigma(v) = [(v, \mathrm{d}\sigma(\tau_M v), 0)] = \langle v, \mathrm{d}\sigma\rangle.$$

7 The definition of $\alpha_{\mathbf{Z}}$

Let $\mathbf{A} = (\eta\colon A \to M, \rho\colon A \times_M A \to E)$ be an affine fibration modelled on the vector fibration $\xi\colon E \to M$. The tangent fibration $(\mathrm{T}\eta, \mathrm{T}\rho)$ is an affine fibration modelled on the vector fibration $\mathrm{T}\xi\colon \mathrm{T}E \to \mathrm{T}M$. The tangent fibration will be denoted $\mathrm{T}\mathbf{A}$

Let $\mathbf{Z} = (\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$.

PROPOSITION 7.1 *There is a canonical isomorphism of affine fibrations* $\mathrm{T}\mathrm{P}\mathbf{Z}$ *and* $\mathrm{P}\widetilde{\mathrm{T}}\mathbf{Z}$.

Proof. The affine fibration $\mathrm{TP}\mathbf{Z}$ is modelled on the vector fibration $\mathrm{T}\pi_M\colon \mathrm{TT}^* \to \mathrm{T}M$. The affine fibration $\mathrm{P}\widetilde{\mathrm{T}}\mathbf{Z}$ is modelled on the vector fibration $\pi_{\mathrm{T}M}\colon \mathrm{T}^*\mathrm{T}M \to \mathrm{T}M$. There is a canonical isomorphism of vector fibrations $\alpha_M\colon \mathrm{T}\pi_M \to \pi_{\mathrm{T}M}$ with the property that for a function f on M $\alpha_M(\mathrm{Td}f) = \mathrm{d}\widetilde{\mathrm{T}}f$. Let σ be a section of ζ, then $\mathrm{Td}\sigma$ is a section of $\mathrm{T}\pi$ and $\mathrm{d}\widetilde{\mathrm{T}}\sigma$ is a section of $\pi_{\mathrm{T}M}$. We define a mapping $\alpha_{\mathbf{Z},\sigma}\colon \mathrm{TP}Z \to \mathrm{P}\widetilde{\mathrm{T}}Z$ by

$$\alpha_{\mathbf{Z},\sigma}(w) = \mathrm{d}\widetilde{\mathrm{T}}\sigma + \alpha_M(w - \mathrm{Td}\sigma(\mathrm{TP}\zeta(w)).$$

For two sections σ and σ' of ζ we have

$$\alpha_M(\mathrm{Td}\sigma) - \alpha_M(\mathrm{Td}\sigma') = \alpha_M(\mathrm{Td}(\sigma - \sigma')) = \mathrm{d}\widetilde{T}(\sigma - \sigma') = \mathrm{d}\widetilde{T}\sigma - \mathrm{d}\widetilde{T}\sigma'.$$

It follows that the mapping $\alpha_{\mathbf{Z},\sigma}$ does not depend on the choice of σ. It follows also that it is an isomorphism of affine fibrations. □

Let φ be a section of the fibration $\mathrm{P}\zeta$. We define a section $\mathrm{i}_{\mathrm{T}}\varphi$ of $\widetilde{T}\zeta$ by $\mathrm{i}_{\mathrm{T}}\varphi(v) = \langle v, \varphi(\tau_M(v))\rangle$. The section $\alpha_{\mathbf{Z}} \circ \mathrm{T}\varphi$ of $\mathrm{P}\widetilde{\mathrm{T}}\zeta$ will be denoted $\mathrm{d}_{\mathrm{T}}\varphi$.

PROPOSITION 7.2 *Let φ be a section of $\mathrm{P}\zeta$. Then $\mathrm{d}_{\mathrm{T}}\varphi = \mathrm{di}_{\mathrm{T}}\varphi + \mathrm{i}_{\mathrm{T}}\mathrm{d}\varphi$.*

Proof. We know ([**?**, **?**]) that for a section σ of ζ we have

$$\mathrm{d}_{\mathrm{T}}(\varphi - \mathrm{d}\sigma) = \mathrm{di}_{\mathrm{T}}(\varphi - \mathrm{d}\sigma) + \mathrm{i}_{\mathrm{T}}\mathrm{d}(\varphi - \mathrm{d}\sigma).$$

From the definition of $\alpha_{\mathbf{Z}}$ it follows that $\mathrm{d}_{\mathrm{T}}\mathrm{d}\sigma = \mathrm{di}_{\mathrm{T}}\mathrm{d}\sigma + \mathrm{i}_{\mathrm{T}}\mathrm{dd}\sigma$. Thus $\mathrm{d}_{\mathrm{T}}\varphi = \mathrm{di}_{\mathrm{T}}\varphi + \mathrm{i}_{\mathrm{T}}\mathrm{d}\varphi$ for every section φ. □

8 The definition of $\widehat{\alpha}_{\mathbf{Z}}$

We already know from Section **??** that the fibration $\mathbf{CZ}$ is a pull-back of the fibration (ζ, ρ) with respect to the mapping $\mathrm{P}\zeta$. It follows that the fibration $\gamma_{\widetilde{\mathrm{T}}Z}$ is a pull-back of the fibration $\widetilde{\mathrm{T}}\zeta$ with respect to the mapping $\mathrm{P}\widetilde{\mathrm{T}}\zeta\colon \mathrm{P}\widetilde{\mathrm{T}}Z \to \mathrm{T}Z$.

PROPOSITION 8.1 *The fibration $\widetilde{\mathrm{T}}\mathrm{C}\zeta$ is a pull-back of the fibration $\widetilde{\mathrm{T}}\zeta$ with respect to the mapping $\mathrm{TP}\zeta : \mathrm{TP}Z \to \mathrm{T}M$.*

Proof. Let σ be a section of ζ and let $\widetilde{\sigma}$ be its pull-back to a section of $\gamma_{\mathbf{Z}}$. We define a mapping from $\widetilde{\mathrm{T}}\mathrm{C}Z$ to $\widetilde{\mathrm{T}}Z$ by $[(w, a, r)] \mapsto [(v, \mathrm{d}\sigma(\tau_M v), r + \langle w, a - \mathrm{d}\widetilde{\sigma}\rangle)]$ where $w \in \mathrm{TP}Z$, $v = \mathrm{TP}\zeta w$ and $a \in \mathrm{PC}Z$. For two sections σ and σ' of ζ we have $\langle w, \mathrm{d}\widetilde{\sigma} - \mathrm{d}\widetilde{\sigma'}\rangle = \langle v, \mathrm{d}(\sigma - \sigma')\rangle$. It follows that the triples $(v, \mathrm{d}\sigma(\tau_M v), r + \langle w, a - \mathrm{d}\widetilde{\sigma}\rangle)$ and $(v, \mathrm{d}\sigma'(\tau_M v), r + \langle w, a - \mathrm{d}\widetilde{\sigma'}\rangle)$ define the same element of $\widetilde{\mathrm{T}}Z$. Thus the introduced mapping does not depend on the choice of σ and, consequently, the fibration $\widetilde{\mathrm{T}}\mathrm{C}\zeta$ is a pull-back of the fibration $\widetilde{\mathrm{T}}\zeta$ with respect to the mapping $\mathrm{TP}\zeta : \mathrm{TP}Z \to \mathrm{T}M$. □

Since both fibrations $\widetilde{\mathrm{T}}\mathrm{C}\zeta$ and $\mathrm{C}\widetilde{\mathrm{T}}\zeta$ are pull-backs of the same fibration it follows that $\alpha_{\mathbf{Z}}$ has a natural extension $\widetilde{\alpha}_{\mathbf{Z}}$ to the isomorphism of fibrations $\widetilde{\mathrm{T}}\gamma_{\mathbf{Z}}$ and $\gamma_{\widetilde{\mathrm{T}}\mathbf{Z}}$. It is also an isomorphism of affine fibrations $\widetilde{\mathrm{T}}\mathrm{C}\mathbf{Z}$ and $\mathrm{C}\widetilde{\mathrm{T}}\mathbf{Z}$.

Let us apply the isomorphism of affine fibrations defined in Proposition **??** to the contact fibration $\mathrm{C}\mathbf{Z}$. We obtain an isomorphism $\alpha_{\mathbf{Z}}$ of $\mathrm{TP}\zeta$ and $\mathrm{P}\widetilde{\mathrm{T}}\zeta$. There is the Liouville section $\vartheta_{\mathbf{Z}}$ of $\mathrm{PC}\mathbf{Z}$. It follows from the construction of $\alpha_{\mathbf{Z}}$ that the differential $\mathrm{dd_T}\vartheta_{\mathbf{Z}}$ is a symplectic form on $\mathrm{TP}Z$. Thus $\widetilde{\mathrm{T}}$ has the canonical structure of a contact fibration defined by $\mathrm{d_T}\vartheta_{\mathbf{Z}}$.

Since $\widehat{\alpha}_{\mathbf{Z}}$ is an isomorphism of $\widetilde{\mathrm{T}}\mathrm{C}\mathbf{Z}$ and $\mathrm{C}\widetilde{\mathrm{T}}\mathbf{Z}$ it defines an isomorphism of $\mathrm{P}\widetilde{\mathrm{T}}\mathrm{C}\mathbf{Z}$ and $\mathrm{PC}\widetilde{\mathrm{T}}\mathbf{Z}$. It follows from the construction of $\widehat{\alpha}_{\mathbf{Z}}$ that $\widehat{\alpha}_{\mathbf{Z}} \circ \mathrm{d_T}\vartheta_{\mathbf{Z}} = \vartheta_{\widetilde{\mathrm{T}}\mathbf{Z}}$. We say that $\widehat{\alpha}_{\mathbf{Z}}$ is an isomorphism of contact fibrations.

9 The definition of $\widehat{\beta}_{\mathbf{Z}}$

Let $\mathbf{Z} = (\zeta\colon Z \to M, \rho\colon Z \times_M Z \to M \times \mathbf{R})$ be an affine fibration modelled on the trivial fibration $pr_M\colon M \times \mathbf{R} \to M$. The symplectic form $\omega_{\mathrm{P}\mathbf{Z}}$ on $\mathrm{P}Z$ defines the canonical symplectomorphism $\beta_{\mathrm{P}Z}\colon \mathrm{TP}Z \to \mathrm{T}^*\mathrm{P}Z$. The trivial fibration $pr_{\mathrm{T}^*\mathrm{P}Z}\colon \mathrm{T}^*\mathrm{P}Z \times \mathbf{R} \to \mathrm{T}^*\mathrm{P}Z$ is the contact fibration of the trivial fibration $pr_{\mathrm{P}Z}\colon \mathrm{P}Z \times \mathbf{R} \to \mathrm{P}Z$ with the connection form $\vartheta_{\mathrm{P}Z}$. The trivial fibration $pr_{\mathrm{TP}Z}\colon \mathrm{TP}Z \times \mathbf{R} \to \mathrm{TP}Z$ is also a contact fibration with the connection form $\mathrm{i_T}\omega_{\mathrm{P}Z}$. The trivial lift of $\beta_{\mathrm{P}Z}$ to the fibration isomorphism $\widehat{\beta}_{\mathrm{P}Z}\colon \mathrm{T}^*\mathrm{P}Z \times \mathbf{R} \to \mathrm{TP}Z \times \mathbf{R}$ is the isomorphism of contact bundles.

The Liouville section $\vartheta_{\mathbf{Z}}$ of $\mathrm{P}\gamma_{\mathbf{Z}}$ defines a section $\mathrm{i_T}\vartheta$ of $\widetilde{\mathrm{T}}\gamma_{\mathbf{Z}}$. Since

$$\mathrm{i_T}\omega_{\mathrm{P}Z} = \mathrm{i_T}\mathrm{d}\vartheta_{\mathbf{Z}} = \mathrm{d_T}\vartheta_{\mathbf{Z}} - \mathrm{d}\mathrm{i_T}\vartheta_{\mathbf{Z}}$$

it follows that the mapping $\varepsilon_{\mathbf{Z}}\colon \widetilde{\mathrm{T}}\mathrm{C}Z \to \mathrm{TP}Z \times \mathbf{R}$ defined by

$$\varepsilon_{\mathbf{Z}}(\widetilde{w}) = \widetilde{w} - \mathrm{i_T}\vartheta_{\mathbf{Z}}(\widetilde{\mathrm{T}}\gamma_{\mathbf{Z}}(\widetilde{w}))$$

is an isomorphism of contact bundles. We define $\widehat{\beta}_{\mathbf{Z}}\colon \widetilde{\mathrm{T}}\mathrm{C}\mathbf{Z} \to \mathrm{T}^*\mathrm{P}Z \times \mathbf{R}$ by $\widehat{\beta}_{\mathbf{Z}} = \widehat{\beta}_{\mathrm{P}Z} \circ \varepsilon_{\mathbf{Z}}$.

10 The dynamics of a charged particle

Let M be the space-time with the metric tensor g. Let

$$\mathbf{Y} = (\xi\colon Y \to M, \eta\colon Y \times_M Y \to M \times \mathbf{R}$$

be the Kaluza-Klein fibration. An electromagnetic potential A is a section of $\mathrm{P}\xi$ and the electromagnetic field is its differential $\mathrm{d}A$. Let e be the charge of a particle with the mass m. We define an equivalence relation between pairs (y, r) where $y \in Y$ and $r \in \mathbf{R}$. Two pairs (y, r) and (y', r') are equivalent if $\xi(y) = \xi(y')$ and $(\xi(y), r - r') = \eta(y - y')$. We denote by Z the set of equivalence classes. We define $\zeta\colon Z \to M$ by $\zeta([(y, r)]) = \xi(y)$ and $\rho\colon Z \times_M Z \to M \times \mathbf{R}$ by $\rho([(y, r)], [(y, r')]) = (\xi(y), r - r')$. Let σ be a section of ξ. We define a section σ_e of ζ by $\sigma_e(m) = [(\sigma(m), 0)]$. The correspondence between sections of ξ and sections of ζ defines an isomorphism of fibrations $\Phi\colon \mathrm{P}\xi \to \mathrm{P}\zeta$ by the correspondence $(m, \sigma) \mapsto (m, \sigma_e)$ of representants of elements of the phase fibrations. We denote by A_e the section $\Phi \circ A$ of $\mathrm{P}\zeta$. The Lagrangian of a charged particle is a section L of the reduced tangent fibration $\widetilde{\mathrm{T}}\zeta$ defined by $L(v) = \langle v, A_e \rangle + m\sqrt{g(v, v)}$ defined on the submanifold of positive vectors. Let D_l and W_l denote sets of all elements of $\mathrm{P}\widetilde{\mathrm{T}}Z$ and $\mathrm{C}\widetilde{\mathrm{T}}Z$ respectively which have representatives of the form (m, L). A submanifold D of the phase manifold $\mathrm{P}Z$ of the system defined by $\alpha_{\mathbf{Z}}(D) = D_l$ is the dynamics of the system.

Let us choose a section (a gauge) σ of ζ. The mapping $\Psi_\sigma\colon \mathrm{P}Z \to \mathrm{T}^*M$ defined by $\Psi_\sigma(a) = a - \mathrm{d}\sigma(\mathrm{P}\zeta(a))$ is a symplectomorphism. We introduce symbols $A_{e,\sigma} = \Psi_\sigma \circ A_e$ and

$$C_{A_{e,\sigma}} = \{p \in \mathrm{T}^*M : g(p - A_{e,\sigma}, p - A_{e,\sigma}) = \mathrm{m}^2\}.$$

We then have

$$\mathrm{T}\Psi_\sigma(D) = \{w \in \mathrm{TT}^*M : w \in \mathrm{T}C_{A_{e,\sigma}}, w \,\lrcorner\, (\omega_M | C_{A_{e,\sigma}}) = 0,$$

$$mv = \sqrt{g(v,v)}g(p - A_{e,\sigma}(\pi_M(p)), \cdot),$$
$$v = \mathrm{T}\pi_M(w)$$

and

$$p = \tau_{\mathrm{T}^*(M)}(w)\}.$$

Another representation of $\mathrm{P}Z$ as TT^*M is obtained by choosing A_e as the zero section. We define a mapping $\Psi_{A_e}\colon \mathrm{P}Z \to \mathrm{T}^*M$ by $\Psi_{A_e}(a) = a - A_e(\mathrm{P}\zeta(a))$. This mapping is not symplectomorphic and the canonical symplectic form $\omega_{\mathrm{P}Z}$ is transported by this mapping to the 2-form $\omega_M - \pi_M^* F = \omega_F$ where $F = \mathrm{d}A_e$. Let be $C_0 = \{p \in \mathrm{T}^*M : g(p,p) = m^2$. We have $\Psi_{A_e}(D) = \{w \in \mathrm{TT}^*M : w \in \mathrm{T}C_0, w \,\lrcorner\, (\pi_F|C_0) = 0, mv = \sqrt{g(v,v)}g(p,\cdot), v = \mathrm{T}\pi_M(w), p = \tau_{\mathrm{T}^*M}(w)\}$.

In order to obtain the Hamilton description of the system we project $\widehat{\beta}_{\mathbf{Z}}$ to $\mathrm{P}Z \times \mathbf{R}$. We obtain the zero function on the constraint submanifold

$$C_{A_e} = \{p \in \mathrm{T}^*M : g(p - A_{e,\sigma}, p - A_{e,\sigma}) = m^2\}.$$

References

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