Differential forms on differential spaces

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The presented approach to differential forms on differential spaces is motivated by the use of differential forms in the theory of static systems ([10, 11, 3]). The concept of a covector of higher order is based on duality between forms and multivectors rather then on the idea of the Grassman algebra.

1 Differential spaces

Let M be a set and let $\mathcal{C}$ be a family of real functions on M. The topology induced on M by the family $\mathcal{C}$, i.e., the weakest topology such that all functions from the family $\mathcal{C}$ are continuous, will be denoted by $\tau_{\mathcal{C}}$. If the family $\mathcal{C}$ separates points of M, then $(M, \tau_{\mathcal{C}})$ is a Hausdorff space.

Let $\mathrm{sc}(\mathcal{C})$ denote the set of all functions of the form

$$\alpha \circ (f_1, \ldots, f_k)$$

where $f_1, \ldots, f_k \in \mathcal{C}$ and α is a smooth real-valued function defined on an open neighborhood of $f_1(M) \times \ldots \times f_k(M)$ in $\mathbf{R}^k$.

It is easy to show ([9]) that:

$$\mathcal{C} \subset \mathrm{sc}(\mathcal{C})$$

$$\mathrm{sc}(\mathcal{C}) = \mathrm{sc}(\mathrm{sc}(\mathcal{C}))$$

$$\tau_{\mathrm{sc}(\mathcal{C})} = \tau_{\mathcal{C}}$$

Definition 1.1 *A function f on $A \subset M$ is a* local $\mathcal{C}$-function *if at each point $p \in A$ there exist $g \in \mathcal{C}$ and an open neighborhood U of p such that $f|_U = g|_U$.*

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The set of all local $\mathcal{C}$-functions on A will be denoted by $\mathcal{C}_A$. We list the following properties of $\mathcal{C}_A$ (see [9]):

$$\mathcal{C}|_A \subset \mathcal{C}_A \text{ (in particular } \mathcal{C} \subset \mathcal{C}_M)$$

$$(\mathcal{C}_A)_A = \mathcal{C}_A$$

$$\mathcal{C} \subset \mathrm{sc}(\mathcal{C}_M) \subset (\mathrm{sc}(\mathcal{C}))_M$$

Definition 1.2 *A* differential space *is a pair* $(M, \mathcal{C})$ *where* M *is a set and* $\mathcal{C}$ *is a family of real functions on* M *such that* $\mathcal{C} = (\mathrm{sc}(\mathcal{C}))_M$. *The family* $\mathcal{C}$ *will be called the* differential structure *on* M *and functions in* $\mathcal{C}$ *will be called* smooth functions *on* M.

We use also $C^\infty(M)$ to denote the family af all smooth functions.

For each family $\mathcal{C}_0$ of functions on M there exists the smallest differential structure $\mathcal{C}$ which includes $\mathcal{C}_0$. We observe that $\mathcal{C} = (\mathrm{sc}(\mathcal{C}_0))_M$.

Definition 1.3 *We say that a family* $\mathcal{C}_0$ generates *a differential structure* $\mathcal{C}$ *if* $\mathcal{C} = (\mathrm{sc}(\mathcal{C}_0))_M$.

Definition 1.4 *Let* $(M, \mathcal{C})$ *and* $(N, \mathcal{D})$ *be differential spaces. A mapping* $\alpha: M \to N$ *is a* smooth mapping *between differential spaces if for each* $f \in \mathcal{D}$ *we have* $f \circ \alpha \in \mathcal{C}$.

Proposition 1.1 *Let* $(M, \mathcal{C})$ *and* $(N, \mathcal{D})$ *be differential spaces and let* $\mathcal{D}_0$ *generate* $\mathcal{D}$. *A mapping* $\alpha: M \to N$ *is smooth if for each* $f \in \mathcal{D}_0$ *we have* $f \circ \alpha \in \mathcal{C}$.

2 Tangent spaces and the module of smooth 1-forms

Let the interval $[0, 1[\in \mathbf{R}$ be endowed with the natural differential structure (see [9]) and let $(M, C^\infty(M))$ be a differential space.

Definition 2.1 *A* smooth half-curve *on* M *is a smooth mapping* $\gamma: [0, 1[\to M$.

Let Σ^1 denote the set of all smooth half-curves on M. We define a fibration $\tau: \Sigma^1 \to M$ by

$$\tau(\gamma) = \gamma(0).$$

By Π^1 we denote a trivial fibration $M \times C^\infty(M)$ with the canonical projection

$$\pi: \Pi^1 \to M: (q, f) \mapsto q.$$

We define a mapping $<,>: \Sigma^1 \times_M \Pi^1 \to \mathbf{R}$ by the formula:

$$< \gamma, (q, f) >= \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma)|_{t=0}$$

where $q = \gamma(0)$. We refer to this mapping as a pairing between fibrations Σ^1 and Π^1.

Let be $\gamma_1, \gamma_2 \in \Sigma^1$. We say that γ_1 and γ_2 are equivalent if $\pi(\gamma_1) = \pi(\gamma_2) = q$ and

$$< \gamma_1, (q, f) >=< \gamma_2, (q, f) > \text{ for each } f \in C^\infty(M).$$

An equivalence class $[\gamma]$ is a *tangent vector* at the point q. A *tangent space* $\mathrm{T}_q(M)$ is the set of all tangent vectors at the point q.

Definition 2.2 *The* tangent bundle *of a differential space M is a triple $(\mathrm{T}M, M, \tau)$ where $\mathrm{T}M = \bigcup_{q \in M} \mathrm{T}_q M$ is the disjoint sum of tangent spaces and τ is the canonical projection $\tau: \mathrm{T}M \to M: v_q \mapsto q$.*

Now, we introduce an equivalence relation in Π^1: two pairs (q_1, f_1) and (q_2, f_2) are equivalent if $q_1 = q_2$ and $< \gamma, (q_1, f_1) >=< \gamma, (q_2, f_2) >$ for each $\gamma \in \Sigma$ such that $\gamma(0) = q_1 = q_2$.

An equivalence class $[(q, f)]$ is called the differential of the function f at the point q and will be denoted by $\mathrm{d}_q f$. The *cotangent space* at q is the set of all differentials of functions at the point q and will be denoted $\mathrm{T}^*_q M$.

Definition 2.3 *The* cotangent bundle $(\mathrm{T}^* M, M, \pi)$ *of a differential space M is the disjoint sum of cotangent spaces $\mathrm{T}^* M = \bigcup_{q \in M} \mathrm{T}^*_q M$ with the canonical projection*

$$\pi: \mathrm{T}^* M \to M: \mathrm{d}_q f \mapsto q.$$

The pairing between Σ^1 and Π^1 defines a pairing between $\mathrm{T}M$ and $\mathrm{T}^* M$ by the formula

$$< v, \mathrm{d}_q f >= \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma)|_{t=0}$$

where γ represents v.

Remark. $C^\infty(M)$ has a natural structure of a linear space. This structure defines a linear space structure in $\mathrm{T}^* M$. Let be $f, h \in C^\infty(M)$ and $a \in \mathbf{R}$ then:

$$\mathrm{d}_q f + \mathrm{d}_q h = \mathrm{d}_q (f + h)$$

$$a \mathrm{d}_q f = \mathrm{d}_q (a f).$$

In general, the dimension of a cotangent space is not the same at each point of M and a tangent space is not a linear space.

Definition 2.4 *The* differential *of a function* $f \in C^\infty(M)$ *is a mapping* $\mathrm{d}f\colon \mathrm{T}M \to \mathbf{R}$ *defined by the formula* $\mathrm{d}f(v) =< v, \mathrm{d}f >$.

A differential structure $C^\infty(\mathrm{T}M)$ on a tangent bundle $\mathrm{T}M$ we define in the following way:

$$C^\infty(\mathrm{T}M) = (\mathrm{sc}(\{\tau^* f\colon f \in C^\infty(M)\} \cup \{\mathrm{d}g\colon g \in C^\infty(M)\}))_{\mathrm{T}M}.$$

It follows that that the projection $\tau\colon \mathrm{T}M \to M$ is smooth and that if $C^\infty(M)$ separates points of M then $C^\infty(\mathrm{T}M)$ separates points of $\mathrm{T}M$.

We may introduce a differential structure on T^*M generated by functions of the form $\pi^* f$ where $f \in C^\infty(M)$ and of the form $\mathrm{d}_q f \mapsto \langle X(q), \mathrm{d}_q f\rangle$ where X is a smooth vector field on M ($X\colon M \to \mathrm{T}M$). However this differential structure does not, in general, separate points of T^*M and, consequently, is not very usefull.

Definition 2.5 *A* smooth 1-form *on* M *is a smooth function* $\phi\colon \mathrm{T}M \to \mathbf{R}$ *such that for each point* $q \in M$ *there exists* $f \in C^\infty(M)$ *such that* $\phi|_{\mathrm{T}_q M} = \mathrm{d}_q f$.

The space $\Lambda^1 M$ of all smooth 1-forms has natural structure of a module over the ring of smooth functions on M. The structure of a linear space and of a module in $\Lambda^1 M$ is given by:

$$(\varphi_1 + \varphi_2)(v_q) = \varphi_1(v_q) + \varphi_2(v_q)$$
$$(f\varphi)(v_q) = f(q)\varphi(v_q).$$

It is evident that the differential of a smooth function is a smooth 1-form.

3 The fibration of 2-vectors and the module of smooth 2-forms

A 2-cube is a smooth mapping $\xi\colon [0,1[\times[0,1[\to M$. Let Σ^2 be the set of all 2-cubes on M. We define a fibration $\tau\colon \Sigma^2 \to M$ by $\Sigma^2 \ni \xi \mapsto \xi(0,0)$. Let $\pi\colon \Pi^2 = M \times \Lambda^1 M \to M$ be the trivial fibration defined by $\pi(q,\varphi) = q$.

For each $t \in [0,1[$ half-curves

$$t_1 \to \xi^2_t(t_1) = \xi(t_1, t)$$

and

$$t_2 \to \xi^1_t(t_2) = \xi(t, t_2)$$

are smooth and, consequently, represent tangent vectors denoted by $[\xi^2_t]$ and $[\xi^1_t]$, respectively. Mappings $[0,1[\ni t \mapsto [\xi^1_t] \in \mathrm{T}M$ and $[0,1[\ni t \mapsto [\xi^2_t] \in \mathrm{T}M$ are smooth half-curves. Thus we can define a pairing between fibrations τ and π by the formula:

$$< \xi, (q,\varphi) >= \frac{\mathrm{d}}{\mathrm{d}t}(\varphi([\xi_t^1]) - \varphi([\xi_t^2]))|_{t=0}$$

We say that two 2-cubes ξ and ξ' are equivalent if

$$\tau(\xi) = \tau(\xi') = q \text{ and } < \xi, (q,\varphi) >=< \xi', (q,\varphi) >$$

for each $\varphi \in \Lambda^1 M$.

An equivalence class $[\xi]$ of 2-cubes is called a *2-vector* at point q. By $\mathrm{T}_q^{(2)} M$ we denote the space of all 2-vectors at point q.

Definition 3.1 *The* fibre bundle of 2-vectors *on the differential space* M *is the triple* $(\mathrm{T}^{(2)}M, M, \tau_2)$ *where* $\mathrm{T}^{(2)}M = \bigcup_{q\in M} \mathrm{T}_q^{(2)} M$ *and* $\tau_2 \colon \mathrm{T}^{(2)}M \to M \colon [\xi] \mapsto \xi(0,0)$

Now, we introduce an equivalent relation in Π^2. Two pairs (q,φ) and (q,φ') are equivalent if $< \xi, (q,\varphi) >=< \xi, (q,\varphi) >$ for each $\xi \in \Sigma^2$ such that $\tau(\xi) = q$. An equivalence class $[(q,\varphi)]$ is called the *differential of 1-form* φ at the point q and it will be denoted by $\mathrm{d}_q\varphi$.

Definition 3.2 *The* differential *of a smooth 1-form* φ *is a map* $\mathrm{d}\varphi \colon \mathrm{T}^{(2)}M \to \mathbf{R}$ *defined by the formula:*

$$\mathrm{d}\varphi([\xi_q]) =< \xi_q, \mathrm{d}_q\varphi >$$

.

We introduce a differential structure $C^\infty(\mathrm{T}^{(2)}M)$ on $\mathrm{T}^{(2)}M$ by

$$C^\infty(\mathrm{T}^{(2)}M) = (\mathrm{sc}(\{d\varphi \colon \varphi \in \Lambda^1 M\} \cup \{\tau^* f \colon f \in C^\infty(M)\}))_{\mathrm{T}^{(2)}M}.$$

We observe that the projection $\tau_2 \colon \mathrm{T}^{(2)}M \to M$ is projection and that $C^\infty(\mathrm{T}^{(2)}M)$ separates points of $\mathrm{T}^{(2)}M$ if $C^\infty(M)$ separates points of M.

Definition 3.3 *A* smooth 2-form *on* M *is a smooth function* $\omega \colon \mathrm{T}^2 M \to \mathbf{R}$ *such that for each point* $q \in M$ *there exists* $\varphi \in \Lambda^1 M$ *such that* $\omega|_{\mathrm{T}_q^2 M} = \mathrm{d}_q\varphi$.

The set of all smooth 2-forms will be denoted by $\Lambda^2 M$. We introduce in $\Lambda^2 M$ a structure of the module over $C^\infty(M)$ in the following way:

$$(\omega_1 + \omega_2)(\xi_q) = \omega_1(\xi_q) + \omega_2(\xi_q),$$
$$(f\omega)(\xi_q) = f(q)\omega(\xi_q).$$

We notice that for $\varphi \in \Lambda^1 M$ the 2-form $\mathrm{d}\varphi$ is smooth.

Lemma 3.1 *Let be* $f \in C^\infty(M)$ *then* $\mathrm{d}(\mathrm{d}f) = 0$

Proof: Let ξ be a 2-cube representing a 2-vector $[\xi]$. We have

$$\begin{aligned} < [\xi], \mathrm{d}_q(\mathrm{d}f) > &= (\partial_t(\mathrm{d}f([\xi_t^1])) - \partial_s(\mathrm{d}f([\xi_s^2])))|_{t=s=0} \\ &= (\partial_t\partial_s(f\circ\xi) - \partial_s\partial_t(f\circ\xi))|_{t=s=0} = 0 \quad \square \end{aligned}$$

4 Differential forms on a differential space.

An l-cell ξ on M is a smooth mapping

$$\xi\colon [0,1[^l \to M.$$

Let ξ be an $l+r$-cell on M and let $(i_1,\dots,i_r)$ be a sequence of integers such that $i_p \neq i_q$ and $1 \geq i_p \leq l+r$, $p=1,\dots,r$. For each $(t_1,t_2,\dots,t_r) \in [0,1[^r$ we define in the obvious way an l-cell $\xi^{i_1\dots i_r}_{t_1\dots t_r}$.

Let us assume that we have already defined k-vectors as equivalence classes of k-cells and differentials of $k-1$-forms as equivalence classes of pairs (q,φ) where $q \in M$ and φ is a $k-1$-form. We denote by $\mathrm{T}^{(k)}M$ the bundle of k-vectors and by $\Lambda^k M$ the module of k-forms defined in the obvious way. We also assume that for each $k+r$-cell on M mappings defined by

$$(t_1,\dots,t_r) \mapsto [\xi^{i_1\dots i_r}_{t_1\dots t_r}]$$

are smooth r-cells on $\mathrm{T}^{(k)}M$.

Now, we define $k+1$-vectors and forms. Let Σ^{k+1} be the set of all smooth (k+1)-cells on M. We define a fibration $\tau^{k+1}\colon \Sigma^{k+1} \to M\colon \xi \mapsto \xi(0,\dots,0)$. Let $\Pi^{k+1} = M \times \Lambda^k M$. We define a fibration

$$\pi^{k+1}\colon \Pi^{k+1} \to M\colon (q,\omega) \mapsto q.$$

We introduce a pairing between these fibrations by the formula:

$$<\xi,(q,\omega)>= \sum_{i=1}^{k+1} (-1)^i \frac{\mathrm{d}}{\mathrm{d}t}\omega([\xi^i_t])|_{t=0}$$

($\tau(\xi)=q$). With this pairing we define equivalence relations in Σ^{k+1} and Π^{k+1}. We say that two $k+1$-cells ξ and ξ' are equivalent if $\tau^{k+1}(\xi)=\tau^{k+1}(\xi')$ and for each $(q,\omega) \in \Pi^{k+1}$ $<\xi,(q,\omega)>=<\xi',(q,\omega)>$, where $q=\tau^{k+1}(\xi)$. An equivalence class of (k+1)-cells is, by the definition, a (k+1)-vector at the point q. $\mathrm{T}^{(k+1)}_q M$ will denote the set of all (k+1)-vectors at q. In the standard way we define the bundle $(\mathrm{T}^{(k+1)}M, M, \tau_{k+1})$ of (k+1)-vectors on M.

In a similar way we introduce an equivalence relation in Π^{k+1}. An equivalence class $[(q,\omega)]$ is called the *differential* of a form ω at the point q. It will be denoted by $\mathrm{d}_q\omega$.

Definition 4.1 *The* differential *of a k-form ω is a mapping* $\mathrm{d}\omega\colon \mathrm{T}^{(k+1)}M \to \mathbf{R}^1$ *defined by the formula:*

$$\mathrm{d}\omega([\xi]) =<\xi,(q,\omega)> \quad \text{where } q=\xi(0,\dots,0).$$

We introduce a differential structure $C^\infty(\mathrm{T}^{(k+1)}M)$ on $\mathrm{T}^{(k+1)}M$ generated by the family of functions:

$$\{\tau_{k+1}^* f\colon f \in C^\infty(M)\} \cup \{\mathrm{d}\omega\colon \omega \in \Lambda^k M\}.$$

With this differential structure the projection $\tau_{k+1}\colon \mathrm{T}^{(k+1)}M \to M$ is a smooth. Moreover, if $C^\infty(M)$ separates points of M, then $C^\infty(\mathrm{T}^{(k+1)}M)$ separates points of $\mathrm{T}^{(k+1)}M$. It is easy to check that for a given $k+r+1$-cell ξ mappings defined by

$$(t_1, \ldots, t_r) \mapsto [\xi^{i_1 \ldots i_r}_{t_1 \ldots t_r}]$$

are smooth r-cells on $\mathrm{T}^{(k+1)}M$.

Definition 4.2 *A* smooth $(k+1)$-form *on M is a smooth function $\eta\colon \mathrm{T}^{(k+1)}M \to \mathbf{R}$ such that for each point $q \in M$ there exists $\omega \in \Lambda^k M$ such that $\eta|_{\mathrm{T}^{(k+1)}_q M} = \mathrm{d}_q\omega$.*

Thus we have defined the fibration of k-vectors $\tau_k\colon \mathrm{T}^{(k)}M \to M$ and the module of smooth k-forms $\Lambda^k M$ for each integer k.

Lemma 4.1 *For each $(k-1)$-form ω we have* $\mathrm{d}(\mathrm{d}\omega) = 0$.

Proof: Let ξ be a (k+1)-cell, $\xi(0, \ldots, 0) = q$ and let be $\epsilon(i,j) = 0$ for $j < i$, $\epsilon(i,j) = 1$ for $j > i$. We have

$$< [\xi], \mathrm{d}_q(\mathrm{d}\omega) >=< \xi, (q, \mathrm{d}\omega) >= \sum_{i=1}^{k+1} (-1)^i \partial_t \omega(\xi^i_t)|_{t=0} =$$

$$= \sum_{i=1}^{k+1} (-1)^i \sum_{i \neq j=1}^{k+1} (-1)^{j-\epsilon(i,j)} \partial_s \partial_t \omega(\xi^{ij}_{ts})|_{t=s=0}$$

$$= \sum_{i=1}^{k+1} \sum_{i \neq j=1}^{k+1} (-1)^{j+i-\epsilon} \partial_t \partial_s \xi^{ij}_{ts} = 0. \quad \square$$

Let α be a smooth mapping of differential spaces $\alpha\colon (M, \mathcal{C}) \to (N, \mathcal{D})$.

Definition 4.3 *The* k-tangent *of α is a mapping $\mathrm{T}^{(k)}\alpha\colon \mathrm{T}^{(k)}M \to \mathrm{T}^{(k)}N$ defined by the formula:* $\mathrm{T}^{(k)}\alpha([\xi]) = [\alpha \circ \xi]$.

Lemma 4.2 $\mathrm{T}^{(k)}\alpha\colon (\mathrm{T}^{(k)}M, C^\infty(\mathrm{T}^{(k)}M)) \to (\mathrm{T}^{(k)}N, C^\infty(\mathrm{T}^{(k)}N))$ *is a smooth mapping of differential spaces.*

Proof. We have to show that $(\mathrm{T}^{(k)}\alpha)^* g \in C^\infty(\mathrm{T}^{(k)}M)$ for each $g \in C^\infty(\mathrm{T}^{(k)}N)$. The differential structure on $\mathrm{T}^k N$ is generated by

$$\{\tau_k^* f \colon f \in C^\infty(N)\} \cup \{\mathrm{d}\omega \colon \omega \in \Lambda^{k-1}N\}.$$

Let be $f \in C^\infty$. We have

$$(\mathrm{T}^{(k)}\alpha)^*\tau_k^* f = (\tau_k \mathrm{T}^{(k)}\alpha)^* f = (\alpha \circ \tau_k)^* f = \tau_k^* \alpha^* f \in C^\infty(\mathrm{T}^{(k)}M).$$

Now, let ξ be a k-cell in M and let be $\varphi \in \Lambda^{k-1}M$. We have

$$(\mathrm{T}^{(k)}\alpha)^*(\mathrm{d}\varphi)([\xi]) = \mathrm{d}\varphi([\alpha \circ \eta])]) =$$

$$= \sum_{i=1}^{k} (-1)^i \frac{\mathrm{d}}{\mathrm{d}t} \varphi([\alpha \circ \xi]_t^i)|_{t=0} = \mathrm{d}((\mathrm{T}^{(k-1)}\alpha)^*(\varphi)([\xi]).$$

It follows that $(\mathrm{T}^{(k)}\alpha)^*(\mathrm{d}\varphi) = \mathrm{d}(\mathrm{T}^{(k-1)}\alpha)^*\varphi$.

The induction with respect to k shows that $(\mathrm{T}^{(k)}\alpha)^*(\mathrm{d}\varphi) \in C^\infty(\mathrm{T}^{(k)}M)$. Indeed, for $k = 1$ we have $(\mathrm{T}\alpha)(\mathrm{d}f) = \mathrm{d}(f \circ \alpha) \in C^\infty(\mathrm{T}M)$. Now, let be $(\mathrm{T}^{(k-1)}\alpha)^*\varphi \in C^\infty(\mathrm{T}^{(k-1)}M)$ for each $\varphi \in \Lambda^{k-1}N$. We have

$$(\mathrm{T}^{(k)}\alpha)^*(\mathrm{d}\varphi) = \mathrm{d}((\mathrm{T}^{(k-1)}\alpha)\varphi \in \Lambda^k M$$

for $\varphi \in \Lambda^{k-1}N$. It folows that $(\mathrm{T}^{(k)}\alpha)^* g \in C^\infty(\mathrm{T}^{(k)}M)$ for g from the set of generators of $C^\infty(\mathrm{T}^{(k)}N)$. Thus $\mathrm{T}^{(k)}\alpha$ is smooth. $\square$

Definition 4.4 *The* pull-back *of a smooth k-form ω on N is a smooth k-form $\alpha^*\omega$ on M defined by the formula*

$$\alpha^*\omega = \omega \circ \mathrm{T}^{(k)}\alpha.$$

Lemma 4.3 *Let be $\varphi, \psi \in \Lambda^k M$ then:*

1. $\alpha^*(\varphi + \psi) = \alpha^*(\varphi) + \alpha^*(\psi)$.
2. $\mathrm{d}(\alpha^*(\varphi)) = \alpha^*(\mathrm{d}\varphi)$.

Proof:

1. Obvious.
2. Let ξ be a $k+1$-cell on M. We have $\alpha^*(\mathrm{d}\varphi)([\xi]) = \mathrm{d}\varphi([\alpha \circ \xi]) =$

$$\sum_i (-1)^i \frac{\mathrm{d}}{\mathrm{d}t} \varphi([\alpha \circ \xi)]_t^i) = \sum_i (-1)^i \frac{\mathrm{d}}{\mathrm{d}t} (\alpha^*\varphi)([\xi]_t^i) = \mathrm{d}(\alpha^*\varphi)([\xi]). \quad \square$$

5 Integration of differential forms. Stokes' theorem.

Let us denote by I^k the standard closed k-dimensional cube in $\mathbf{R}^k$ ($I = [0,1] \in \mathbf{R}^1$).

Definition 5.1 *A closed k-*cell *in M is a smooth mapping $\xi \colon I^k \to M$.*

For a given k-cell on M and for each $(t_1, \ldots, t_k) \in [0,1[^k$ we define in the obvious way a k-vector at $\eta(t_1, \ldots, t_k)$ denoted by $[\xi](t_1, \ldots, t_k)$.

Definition 5.2 *The integral of a k-form φ over a k-cell η is a number $\int_\eta \varphi$ defined by*

$$\int_\eta \varphi = \int_{I^k} \xi^* \varphi = \int_0^1 \ldots \int_0^1 \varphi([\eta](t_1, \ldots, t_k)) \mathrm{d}t_1 \ldots \mathrm{d}t_k.$$

Let us denote $\int_\eta \varphi$ by $< \eta, \varphi >$. The integral of φ over a k-chain $c = \sum m_i \eta_i$ is defined by

$$< c, \varphi >= \sum m_i < \eta_i, \varphi > .$$

The boundary of a k-cell η is a $(k-1)$-chain

$$\partial \eta = \sum_{m=1}^{k} (-1)^m (w_+^m - w_-^m)$$

where $w_+^m = \eta|_{t_m = 0}$ and $w_-^m = \eta | t_m = 1$.

Theorem 5.1 (The Stokes' identity)

$$< \eta, \mathrm{d}\varphi >=< \partial\eta, \varphi > .$$

Proof: Let φ be a smooth k-form and let η be a $(k+1)$-cell. Then

$$< \eta, \mathrm{d}\varphi >= \int_0^1 \ldots \int_0^1 \mathrm{d}\varphi([\eta](t_1, \ldots, t_{k+1})) \mathrm{d}t_1 \ldots \mathrm{d}t_{k+1} =$$

$$= \int_0^1 \ldots \int_0^1 \sum_{i=1}^{k+1} (-1)^i \partial_i \varphi([\eta^i_{t_i}](t_1, \overset{t_i}{\overset{\vee}{\ldots}}, t_{k+1})) \mathrm{d}t_1 \ldots \mathrm{d}t_{k+1}$$

$$= \int_0^1 \ldots \int_0^1 \sum_{i=1}^{k+1} (-1)^i (\varphi([\eta](\ldots, 1, \ldots)) - \varphi([\eta^i_1](t_1, \overset{t_i}{\overset{\vee}{\ldots}}, t_{k+1}))) \mathrm{d}t_1, \overset{i}{\overset{\vee}{\ldots}}, \mathrm{d}t_{k+1}$$

$$= \sum_{i=1}^{k+1} (-1)^i (< w_+^i, \varphi > - < w_-^i, \varphi >) =< \partial\eta, \varphi > . \quad \square$$

6 Poincare Lemma for differential spaces.

In $M \times I$ we introduce the product differential structure ([9]). For a $k-1$-cell η on M and the identity 1-cell on I we define a k-cell $\eta \times e$ on $M \times I$ by

$$\eta \times e(t_1, \ldots, t_{k-1}, t) = (\eta(t_1, \ldots, t_{k-1}), t).$$

For a k-form ω on $M \times I$ we define a (k-1)-form $D\omega$ on M by

$$(D\omega)([\eta]) = \int_0^1 \omega([\eta \times e](0, \ldots, 0, t)\mathrm{d}t.$$

Let η be a k-cell on M. We have

$$\mathrm{d}\omega([\eta \times e](0, \ldots, 0, t)) = \sum_{i=1}^{k}(-1)^i \frac{\mathrm{d}}{\mathrm{d}s}\omega([\eta_s^i \times e](0, \ldots, 0, t)|_{s=0} + (-1)^{k+1}\frac{\mathrm{d}}{\mathrm{d}s}\omega([\eta \times e_s^1]|_{s=t}).$$

Hence

$$(D\mathrm{d}\omega)([\eta]) = \int_0^1 \mathrm{d}\omega([\eta \times e](0, \ldots, 0, t)\mathrm{d}t$$

$$= \int_0^1 \sum_{i=1}^{k}(-1)^i \frac{\mathrm{d}}{\mathrm{d}s}\omega([\eta_s^i \times e](0, \ldots, 0, t)|_{s=0} + (-1)^{k+1}(\omega([\eta \times e_1^1]) - \omega([\eta \times e_0^1])).$$

On the other side

$$(\mathrm{d}D\omega)(\eta) = \sum_{i=1}^{k}(-1)^i \frac{\mathrm{d}}{\mathrm{d}s}\omega([\eta_s^i])|_{s=0}$$

$$= \sum_{i=1}^{k}(-1)^i \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\omega([\eta_s^i \times e](0, \ldots, 0, t)|s = 0\mathrm{d}t\ .$$

It follows that

$$(\mathrm{d}D\omega)(\eta) + (-1)^{k+1}(Dd\omega)(\eta) = \omega([\eta \times e_1^1]) - \omega([\eta \times e_0^1]).$$

Let $F\colon M \times I \to M$ be a smooth homotopy between the identity and the constant mapping and let be $\omega = F^*\varphi$, where φ is a k-form on M. We have then the homotopy formula

$$\mathrm{d}D(F^*\varphi) + (-1)^{k+1}D\mathrm{d}(F^*\varphi) = \varphi$$

Theorem 6.1 (Poincare Lemma.) *If for each point of M there exists a local, smooth homotopy between the constant and the identity mappings then each closed form φ is locally exact:*

$$\varphi = \mathrm{d}(D(F^*\varphi)).$$

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