Some geometric aspects of the calculus of variations in several independent variables

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Abstract

In this talk I shall describe some recent research on parametric problems in the calculus of variations (of which the minimal surfaces problem is perhaps the most basic example).

I shall also explain the relationship between these problems and the type of problem more usual in physics, where there is a given space of independent variables.

Aspects to be covered will include an interpretation of the first variation formula in terms of cohomology.

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Lagrangians:

1. $\frac{1}{2}\left((\dot{u}^1)^2 + (\dot{u}^2)^2 + (\dot{u}^3)^2\right)$
2. $\sqrt{(\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2}$ positively homogeneous

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In physics, variational problems are commonly defined on fibred manifolds $\pi : E \to M$.

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Extremals are local sections of π, and the Lagrangian is defined on a jet bundle $J^1\pi$ (or $J^k\pi$) of jets of local sections of π.

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So where is the Lagrangian defined?

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The set $P = (V - \{0\})/(\mathbb{R} - \{0\})$ is an n-dimensional projective space. There is a natural injection $A \to P$.

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The bundle $\mathring{T}E \to J^1(E, 1)$ is a principal bundle with structure group $\mathbb{R} - \{0\}$.

An example

Finsler geometry (special case: Riemannian geometry)
Manifold E, coordinates y^a $(0 \leq a \leq n)$
Lagrangian L defined on $\mathring{T}E$

Positive homogeneity: $$\dot{y}^a \frac{\partial L}{\partial \dot{y}^a} = L$$

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Variational problem: find extremals γ of $$\int j^1\gamma^*(L)\,dt$$

If γ is an extremal then so is $\gamma \circ \phi$ where

$$\phi : \mathbb{R} \to \mathbb{R} \text{ diffeomorphism}, \qquad \phi' > 0$$

The problem may also be formulated on $PTE^+ = \mathring{T}E/\mathbb{R}^+$ (double cover of PTE).

Regular velocities on a manifold

Finsler geometry is defined on the slit tangent bundle $T^\circ E$.

First-order multiple integral problems are defined on a sub-bundle of the Whitney sum $\bigoplus^m TE$.

The bundle of regular velocities on E is
$\mathring{T}_{(m)}E = \{(\xi_1, \ldots, \xi_m) \in \bigoplus^m TE : (\xi_i) \text{ linearly independent}\}$.

Equivalently:

$\mathring{T}_{(m)}E$ is the bundle of non-degenerate velocities —
1-jets (at the origin) of non-singular maps $\mathbb{R}^m \to E$.

Coordinates: (y^a) on E, (y^a, y^a_i) on $\mathring{T}_{(m)}E$ $(1 \leq i \leq m)$.

Regular velocities on a manifold (2)

Contact forms on $\mathring{T}_{(m)}E$:

$\omega \in \Omega(\mathring{T}_{(m)}E)$ where the pull-back $(j^1\sigma)^*\omega$ by a prolongation of $\sigma : \mathbb{R}^m \to E$ always vanishes.

In coordinates, contact 1-forms are sums of $(m+1) \times (m+1)$ determinants:

$$\begin{vmatrix} y_1^{a_1} & y_1^{a_2} & \cdots & y_1^{a_{m+1}} \\ y_2^{a_1} & y_2^{a_2} & \cdots & y_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ y_m^{a_1} & y_m^{a_2} & \cdots & y_m^{a_{m+1}} \\ dy^{a_1} & dy^{a_2} & \cdots & dy^{a_{m+1}} \end{vmatrix}$$

(Compare the contact 1-forms $du^\alpha - u_i^\alpha dx^i$ on a jet bundle.)

Regular velocities on a manifold (3)

For each function $f : E \to \mathbb{R}$, define the functions $d_i f : \mathring{T}_{(m)} E \to \mathbb{R}$ by

$$d_i f(j^1_0 \sigma) = \frac{\partial (f \circ \sigma)}{\partial t^i} \qquad \text{where } \sigma : \mathbb{R}^m \to E .$$

d_i is a vector field along $\tau_m : \mathring{T}_{(m)} E \to E$, called a

total derivative

A 1-form $\theta \in \Omega^1 \mathring{T}_{(m)} E$ is a contact form exactly when

$$\langle d_i, \theta \rangle = 0 , \qquad 1 \leq i \leq m .$$

In coordinates

$$d_i = y^a_i \frac{\partial}{\partial y^a} .$$

Regular velocities on a manifold (4)

The Whitney sum $\bigoplus^m TE \to E$ is a vector bundle. Denote its vertical lift operator to (η_i) by

$$\bigoplus^m T_{\tau_m(\eta_i)}E \to T_{(\eta_i)}\left(\bigoplus^m TE\right), \qquad (\xi_i) \mapsto (\xi_i)^{\uparrow(\eta_i)}.$$

For each vector $\zeta \in T_{(\eta_i)}\mathring{T}_{(m)}E$ define the vector $S^i\zeta \in T_{(\eta_i)}\mathring{T}_{(m)}E$ by

$$S^i\zeta = (0, \ldots, 0, T\tau_m(\zeta), 0, \ldots, 0)^{\uparrow(\eta_i)}.$$

S^i is a type $(1,1)$ tensor field on $\mathring{T}_{(m)}E$, called a

vertical endomorphism

In coordinates

$$S^i = dy^a \otimes \frac{\partial}{\partial y_i^a}.$$

Grassmannians

Regular velocities $\mathring{T}_{(m)}E$: equivalence classes of maps $\mathbb{R}^m \to E$

Grassmannian bundle $J^1(E, m)$: equivalence classes of images of maps $\mathbb{R}^m \to E$ (m-dimensional subspaces of TE)

Two regular velocities $j^1_0\sigma$, $j^1_0\hat{\sigma}$ represent the same subspace when

$$j^1_0\hat{\sigma} = j^1_0(\sigma \circ \phi)$$

for some diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ with $\phi(0) = 0$.

Oriented Grassmannians $J^1(E, m)^+$: diffeomorphism ϕ preserves orientation on $\mathbb{R}^m$.

Grassmannians (2)

The projections give principal bundles

$$\rho : \mathring{T}_{(m)}E \to J^1(E, m) \qquad \text{(group } GL(m, \mathbb{R})\text{)}$$
$$\rho^+ : \mathring{T}_{(m)}E \to J^1(E, m)^+ \qquad \text{(group } GL(m, \mathbb{R})^+\text{)}$$

Fundamental vector fields are $\Delta^i_j = S^i(d_j)$

In coordinates

$$\Delta^i_j = y^a_j \frac{\partial}{\partial y^a_i}$$

Any fibration $\pi : E \to M$ defines open submanifolds $J^1\pi \subset J^1(E, M)$ and $J^1\pi \subset J^1(E, M)^+$.

Special case: $J^1(E, 1) = PTE$ and $J^1(E, 1)^+ = PTE^+$.

Higher-order regular velocities

k-th order regular velocities $\mathring{T}^k_{(m)}E$:
k-jets (at the origin) of non-singular maps $\mathbb{R}^m \to E$

Coordinates y^a_I on $\mathring{T}^k_{(m)}E$ (I multi-index, $0 \le |I| \le k$)

Total derivatives d_i and vertical endomorphisms S^i:

$$d_i = \sum_{|I|=0}^{k-1} y^a_{I+1_i} \frac{\partial}{\partial y^a_I}\,, \qquad S^i = \sum_{|I|=0}^{k-1} (I(i)+1)\, dy^a_I \otimes \frac{\partial}{\partial y^a_{I+1_i}}\,.$$

Higher-order regular velocities (2)

Principal bundles:

$$\rho^k : \mathring{T}^k_{(m)}E \to J^k(E,m)\,, \qquad \rho^{k+} : \mathring{T}^k_{(m)}E \to J^k(E,m)^+\,.$$

Groups are the jet groups L^k_m, L^{k+}_m:

$$\begin{aligned} L^k_m &= \{j^k_0\phi : \phi : \mathbb{R}^m \to \mathbb{R}^m \text{ diffeomorphism}\} \\ L^{k+}_m &= \{j^k_0\phi \in L^k_m : |\mathcal{J}(\phi)| > 0\} \end{aligned}$$

Fundamental vector fields $\Delta^I_j = S^I(d_j)$ $(0 \le |I| \le k)$.

Put i^I_j for contraction with Δ^I_j and d^I_j for Lie derivative by Δ^I_j.

Vector forms

We often use vectors of operators, tensors, forms, ...
d_i, S^i, ϑ^i, $\ldots$

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We consider forms on $\mathring{T}^k_{(m)}E$ taking values in the vector space $\mathbb{R}^{m*}$ and its exterior powers.

Put

$$\Omega^{r,s}_k = \left(\Omega^r \mathring{T}^k_{(m)}E\right) \otimes \left(\bigwedge^s \mathbb{R}^{m*}\right) .$$

Let the standard basis for $\mathbb{R}^{m*}$ be denoted by (dt^i). Then

$$\Phi = \phi_{i_1 \ldots i_s} \otimes dt^{i_1} \wedge \ldots \wedge dt^{i_s} \in \Omega^{r,s}_k ;$$

the scalar forms $\phi_{i_1 \ldots i_s}$ are skew-symmetric in their indices.

Vector forms (2)

Operators on vector forms:

$$d : \Omega_k^{r,s} \to \Omega_k^{r+1,s}\,, \qquad d_{\mathrm{T}} : \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1}$$
$$d(\phi \otimes \omega) = d\phi \otimes \omega\,,$$
$$d_{\mathrm{T}}(\phi \otimes \omega) = d_i\phi \otimes dt^i \wedge \omega\,.$$

Properties: $dd_{\mathrm{T}} = d_{\mathrm{T}}d, \quad d_{\mathrm{T}}^2 = 0.$

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$$d(\phi \otimes \omega) = d\phi \otimes \omega\,,$$

$$d_{\mathrm{T}}(\phi \otimes \omega) = d_i\phi \otimes dt^i \wedge \omega\,.$$

Properties: $dd_{\mathrm{T}} = d_{\mathrm{T}}d, \quad d_{\mathrm{T}}^2 = 0.$

Also

$$i_{\mathrm{T}} : \Omega_k^{r,s} \to \Omega_{k+1}^{r-1,s+1}\,, \qquad i_{\mathrm{T}}(\phi \otimes \omega) = (d_i \lrcorner \phi) \otimes dt^i \wedge \omega$$

where

$$d_{\mathrm{T}} = di_{\mathrm{T}} + i_{\mathrm{T}}d\,.$$

The bicomplex

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
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The homotopy operators are $P, \tilde{P} : \Omega^{r,s}_{k} \to \Omega^{r,s-1}_{(r+1)k-1}$:

$$
\begin{aligned}
P(\Phi) &= P^{j}_{(s)}(\phi_{i_1 \ldots i_s}) \otimes \left\{ \frac{\partial}{\partial t^{j}} \lrcorner \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\} \\
\tilde{P}(\Phi) &= \tilde{P}^{j}_{(s)}(\phi_{i_1 \ldots i_s}) \otimes \left\{ \frac{\partial}{\partial t^{j}} \lrcorner \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\}
\end{aligned}
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where $P = \tilde{P}$ when acting on vector 1-forms, or on first-order forms.

The homotopy operators for d_{T} (2)

The scalar operators $P^j_{(s)}$ and $\widetilde{P}^j_{(s)}$ are given by

$$P^j_{(s)} = \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s)!|J|!}{r^{|J|+1}(m-s+|J|+1)!J!} d_J S^{J+1_j} ,$$

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where, for a scalar form θ,

$$S^{1_{j_1}1_{j_2}\cdots 1_{j_r}}\theta = i_{S^{j_1}} i_{S^{j_2}} \cdots i_{S^{j_r}}\theta$$

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Note that $P^2 = 0$ but $\tilde{P}^2 \neq 0$.

Variational problems on fibred manifolds

Let $\pi : E \to M$ be a fibred manifold, with $\dim M = m$ and $\dim E = m + n$, where M is orientable.

A Lagrangian is an m-form $\lambda = L\, d^m x$ on the jet bundle $J^k\pi$, horizontal over M.

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The fixed-boundary variational problem defined by λ is the search for submanifolds $\sigma(C) \subset E$ satisfying

$$\int_C ((j\sigma)^* X^k \lambda) = 0$$

for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$.

Lepage equivalents

Let $\lambda = L\, d^m x$ be a Lagrangian m-form on $J^k\pi$.

Another m-form θ on $J^l\pi$ (where $l \geq k$) is a *Lepage form* if:
$i_Y d\theta$ is a contact form whenever the vector field Y is vertical over E.

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Every Lagrangian m-form has a Lepage equivalent, defined on $J^{2k-1}\pi$.

Examples of Lepage equivalents

For the case $m = 1$ (single integral variational problems):
For a first-order Lagrangian $\lambda = L\,dx$ on $J^1\pi$:

$$\theta = L\,dx + \frac{\partial L}{\partial \dot{y}^\alpha}(dy^\alpha - \dot{y}^\alpha dx)$$

is the unique Lepage equivalent, the *Poincaré-Cartan form*, also defined on $J^1\pi$.

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For a higher-order Lagrangian $\lambda = L\,dx$ on $J^k\pi$:

$$\theta = L\,dx + \sum_{p=0}^{k-1}\left(\sum_{q=0}^{k-p-1}(-1)^q \frac{d^q}{dx^q}\frac{\partial L}{\partial y^\alpha_{(p+q+1)}}\right)(dy^\alpha_{(p)} - y^\alpha_{(p+1)}dx)$$

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Examples of Lepage equivalents (2)

For the case $m \geq 2$ (multiple integral variational problems) and a first-order Lagrangian $\lambda = L\, d^m x$ on $J^1\pi$:

$$\theta_1 = L\, d^m x + \frac{\partial L}{\partial y_i^\alpha} \omega^\alpha \wedge d^{m-1} x_i$$

$$\theta_2 = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(L\, dx^i + \frac{\partial L}{\partial y_i^\alpha} \omega^\alpha \right)$$

$$\theta_3 = \sum_{r=0}^{\min\{m,n\}} \frac{1}{(r!)^2} \frac{\partial^r L}{\partial y_{i_1}^{\alpha_1} \cdots \partial y_{i_r}^{\alpha_r}} \omega^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_r} \wedge d^{m-r} x_{i_1 \cdots i_r}$$

(where $\omega^\alpha = dy^\alpha - y_j^\alpha dx^j$) are globally-defined Lepage equivalents.

Homogeneous variational problems

We now consider m-dimensional variational problems on E, with fixed boundary conditions.

It is sufficient to consider submanifolds of the form $\sigma(C)$ where $\sigma : \mathbb{R}^m \to E$ and $C \subset \mathbb{R}^m$ is a compact m-dimensional submanifold.

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It is sufficient to consider submanifolds of the form $\sigma(C)$ where $\sigma : \mathbb{R}^m \to E$ and $C \subset \mathbb{R}^m$ is a compact m-dimensional submanifold.

This is because variational problems are local.

'An m-dimensional submanifold of E is extremal with fixed boundary conditions if, and only if, every small piece of it is extremal with fixed boundary conditions.'

Homogeneous Lagrangians

A vector function $\Lambda = L\, d^m t \in \Omega^{0,m}$ is called a *Lagrangian* for a parametric variational problem. It is called *homogeneous* if it is equivariant with respect to the action of the jet group L^{k+}_m, where k is the order of the Lagrangian.

If Λ is homogeneous then the scalar function L satisfies

$$d^i_j L = \delta^i_j L\,, \qquad d^I_j L = 0 \text{ for } |I| \geq 2\,.$$

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Equivalents of Lagrangians

Let $\Lambda \in \Omega^{0,m}$ be a homogeneous Lagrangian.

Any scalar m-form $\Theta_m \in \Omega^{m,0}$ is called an *integral equivalent* of Λ if

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!} \right) i_{\mathrm{T}}^{m} \Theta_m \, .$$

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If Θ_{r+1} is an equivalent of Λ then

$$\Theta_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}} \Theta_{r+1}$$

is also an equivalent.

Integral equivalents

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If $\sigma : \mathbb{R}^m \to E$ then $(j\sigma)^* \Lambda = (j\sigma)^* \Theta_m$

so that

$$\int_C (j\sigma)^* \Lambda = \int_C (j\sigma)^* \Theta_m \,.$$

Thus $\Lambda = \Theta_0$ and Θ_m have the same extremals.

Euler forms

Let Θ_m be an integral equivalent of Λ.

Define the scalar $(m+1)$-form $\mathcal{E}_m \in \Omega^{m+1,0}$ by

$$\mathcal{E}_m = d\Theta_m$$

and the vector forms $\mathcal{E}_r \in \Omega^{r+1,m-r}$ by

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1} \qquad 0 \le r \le m-1\,.$$

The forms $\mathcal{E}_r$ are called the *Euler forms* of Θ_m.

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By a straightforward calculation

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r}\, i_{\mathrm{T}}\mathcal{E}_{r+1} \qquad 0 \le r \le m-1\,.$$

Equivalents and Euler forms

$$
\begin{tikzcd}
 & & & & & \Theta_m \in \Omega^{m,0} \arrow[r, "d"] \arrow[dl, "d_T"] & \Omega^{m+1,0} \ni \mathcal{E}_m \arrow[dll] \\
 & & & & \cdots \arrow[dl, "d_T"] \arrow[dll] & & \\
 & & \Theta_2 \in \Omega^{2,m-2} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_T"] & \Omega^{3,m-2} \ni \mathcal{E}_2 \arrow[dl] & & & \\
 & \Theta_1 \in \Omega^{1,m-1} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_T"] & \Omega^{2,m-1} \ni \mathcal{E}_1 \arrow[dl] & & & & \\
\Lambda \in \overline{\Omega}^{0,m} \arrow[r, "d"] & \Omega^{1,m} \ni \mathcal{E}_0 & & & & &
\end{tikzcd}
$$

$$\Theta_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}}\Theta_{r+1}, \quad \Theta_0 = \Lambda$$

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1}$$

Lepagian forms

Let Λ be a homogeneous Lagrangian.

Let Θ_r be an equivalent of Λ $(1 \leq r \leq m)$.

Say that Θ_r is *Lepagian* if the corresponding Euler form $\mathcal{E}_0 \in \Omega^{1,m}$ satisfies

$$S\mathcal{E}_0 = 0\,,$$

so that $\mathcal{E}_0$ is horizontal over E.

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$$S\mathcal{E}_0 = 0\,,$$

so that $\mathcal{E}_0$ is horizontal over E.

Theorem The vector 1-form

$$\Theta_1 = Pd\Lambda$$

is an integral equivalent of Λ $(m = 1)$ or an intermediate equivalent $(m \geq 2)$, and is Lepagian. It is called the *Hilbert equivalent* of Λ.

The Hilbert equivalent

$\Theta_1 = P d\Lambda$ is a Lepagian equivalent of $\Lambda = L\, d^m t$.

Outline of proof: if $\Phi = \phi \otimes d^m t \in \Omega^{1,m}$ use

$$P\Phi = P^j \phi \otimes d^{m-1} t_j \,, \qquad P^j = \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J S^{J+1_j} \,.$$

To show $i_{\mathrm{T}} P d\Lambda = \Lambda$, use:

commutators $[i_k, d_j] = 0$, $[i^I_k, S^j] = i^{I+1_j}_k$,

homogeneity $i^I_k dL = d^I_k L = 0 \; (|I| \geq 1)$,

vanishing of S^i on functions $i_k dL$ and $i^j_k dL$,

homogeneity again $i^j_k dL = d^j_k dL = \delta^j_k dL$.

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To show $S\mathcal{E}_0 = S(d\Lambda - d_{\mathrm{T}} P d\Lambda) = 0$ use:

commutators $[S^j, d_i] = \delta^j_i$ to give a collapsing sum

so that $S d_{\mathrm{T}} P(d\Lambda) = S(d\Lambda)$.

If $\tilde{\Theta}_1$ is another Lepagian vector 1-form equivalent to Λ, with corresponding Euler form $\tilde{\mathcal{E}}_0$, then

$$\tilde{\mathcal{E}}_0 = \mathcal{E}_0\,, \qquad \tilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}\Phi \qquad (\Phi \in \Omega^{r,m-2})$$

The First Variation Formula

Given a variation field X on E with $X|_C = 0$, and its prolongation $\hat{X}$,

$$\begin{aligned}\int_C (j\sigma)^* d_{\hat{X}}\Lambda &= \int_C (j\sigma)^* d i_{\hat{X}}\Lambda + \int_C (j\sigma)^* i_{\hat{X}} d\Lambda \\ &= \int_{\partial C} (j\sigma)^* i_{\hat{X}}\Lambda + \int_C (j\sigma)^* i_{\hat{X}} d\Lambda \\ &= \int_C (j\sigma)^* i_{\hat{X}}\mathcal{E}_0 + \int_C (j\sigma)^* i_{\hat{X}} d_{\mathrm{T}}\Theta_1\end{aligned}$$

But

$$\int_C (j\sigma)^* i_{\hat{X}} d_{\mathrm{T}}\Theta_1 = \int_C (j\sigma)^* d_{\mathrm{T}} i_{\hat{X}}\Theta_1 = \int_C d(j\sigma)^* i_{\hat{X}}\Theta_1 = 0$$

because prolongations commute with total derivatives. Thus

$$\int_C (j\sigma)^* d_{\hat{X}}\Lambda = \int_C (j\sigma)^* i_{\hat{X}}\mathcal{E}_0 = \int_C (j\sigma)^* i_X\mathcal{E}_0$$

because $\mathcal{E}_0$ is horizontal over E.

Integral equivalents for $m \geq 2$

Let $\Lambda = L\, d^m t$ be a homogeneous Lagrangian with $m \geq 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1} t_i \,;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ.

If Λ never vanishes, define the *Carathéodory equivalent* $\tilde{\Theta}_m \in \Omega^{m,0}$ by

$$\tilde{\Theta}_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \vartheta^i \,.$$

Theorem The Carathéodory equivalent $\tilde{\Theta}_m$ is an integral equivalent of Λ.

The Carathéodory equivalent

Given $\widetilde{\Theta}_m = (1/L^{m-1})\vartheta^1 \wedge \cdots \wedge \vartheta^m$, we must show that

$$i_{\mathrm{T}}^m \widetilde{\Theta}_m = (-1)^{m(m-1)/2} m!\Lambda\,.$$

Outline of proof, using $i_k \vartheta^i = \delta^i_k L$: suppose

$$i_{\mathrm{T}}^s \widetilde{\Theta}_m = \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}} \Big\{ \sum_{\sigma \in S_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m-s)} \otimes \otimes\, dt^{\sigma(m-s+1)} \wedge \cdots \wedge dt^{\sigma(m)} \Big\}$$

(where S_m is the permutation group) and use induction.

The variation formula for $\widetilde{\Theta}_m$

From the induction formula

$$i_{\mathrm{T}}^{m-1}\widetilde{\Theta}_m = (-1)^{m(m-1)/2}\,(m-1)!\Theta_1$$

where Θ_1 is the Hilbert equivalent.

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Then, as $d\Theta_m = \mathcal{E}_m$,

$$\begin{aligned}\int_C (j\sigma)^* d_Y \Theta_M &= \int_C (j\sigma)^* i_Y \mathcal{E}_m \\ &= \int_C (j\sigma)^* i_Y \mathcal{E}_0\end{aligned}$$

for any vector field Y on $\mathring{T}^k_{(m)}E$ vanishing on $j\sigma(\partial C)$, because contractions by vector fields anticommute, so that $i_{\mathrm{T}}^m i_Y \mathcal{E}_m = (-1)^m i_Y i_{\mathrm{T}}^m \mathcal{E}_m$.

Another integral equivalent

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$$\Theta_{r+1} = (-1)^r P d\Theta_r \qquad (1 \leq r < m) \,.$$

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Using commutator relations, we obtain

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Thus $d\Theta_m = \mathcal{E}_m = 0$ if, and only if, $\mathcal{E}_0 = 0$.

Other matters

- Regularity
- Symmetry
- Helmholtz equations