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Some geometric aspects of the calculus of variations in several independent variables

D. J. Saunders

Banach Center Warszawa

28 April 2010

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Abstract

In this talk I shall describe some recent research on parametric problems in the calculus of variations (of which the minimal surfaces problem is perhaps the most basic example).

I shall also explain the relationship between these problems and the type of problem more usual in physics, where there is a given space of independent variables.

Aspects to be covered will include an interpretation of the first variation formula in terms of cohomology.

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What are 'parametric' variational systems?

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Compare two problems:

- 1. Find the trajectory of a free unit-mass particle in 3-dimensional space;
- 2. Find the shortest curve between two points in three-dimensional space.

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- 1. Find the trajectory of a free unit-mass particle in 3-dimensional space;
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Solutions:

- 1. A map $[0,T] \rightarrow \mathbb{R}^3$, $t \mapsto (a^i t + b^i)$
- 2. A straight line segment $[(p^i), (q^i)] \subset \mathbb{R}^3$.

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Lagrangians:

1.
$$\frac{1}{2} \left((\dot{u}^1)^2 + (\dot{u}^2)^2 + (\dot{u}^3)^2 \right)$$

2. $\sqrt{(\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2}$ positively homogeneous

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What are 'parametric' variational systems? (2)

In physics, variational problems are commonly defined on fibred manifolds $\pi: E \to M$.

(For the free particle, this is $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$.)

Intro

Extremals are local sections of π , and the Lagrangian is defined on a jet bundle $J^1\pi$ (or $J^k\pi$) of jets of local sections of π .

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Extremals are submanifolds of *E*, defined 'parametrically'.

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So where is the Lagrangian defined?



There are different types of jet bundle.

To understand the difference, think of the relationships between a vector space, an affine space and a projective space.

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Take a vector space V, with $\dim V = n + 1$, basis (e_0, e_1, \ldots, e_n) and corresponding coordinate functions $(\dot{y}^0, \dot{y}^1, \ldots, \dot{y}^n)$.

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The set $A = \{v \in V : \dot{y}^0(v) = 1\}$ is an *n*-dimensional affine space.

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The set $P = (V - \{0\})/(\mathbb{R} - \{0\})$ is an *n*-dimensional projective space.



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The set $P = (V - \{0\})/(\mathbb{R} - \{0\})$ is an *n*-dimensional projective space. There is a natural injection $A \rightarrow P$.

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Let $\pi : E \to \mathbb{R}$ be a fibred manifold, with dim E = n + 1 and coordinates $(y^0 = t, y^1, \dots, y^n)$.

Jet manifolds: $J^1\pi$ contains jets of local sections of π , and $J^1(E, 1)$ contains jets of immersed submanifolds in E.

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We identify $J^1\pi$ with an open submanifold of $J^1(E, 1)$ by mapping the jet of a local section to the jet of its image.

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The bundle $\mathring{T}E \to J^1(E, 1)$ is a principal bundle with structure group $\mathbb{R} - \{0\}$.

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An example

Finsler geometry (special case: Riemannian geometry) Manifold *E*, coordinates y^a ($0 \le a \le n$) Lagrangian *L* defined on $\mathring{T}E$

Positive homogeneity:

$$\dot{y}^a \frac{\partial L}{\partial \dot{y}^a} = L$$

Variational problem: find extremals γ of $\int j^1 \gamma^*(L) dt$

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Positive homogeneity:

$$\dot{y}^a \frac{\partial L}{\partial \dot{y}^a} = L$$

Variational problem: find extremals γ of $\int j^1 \gamma^*(L) dt$

If γ is an extremal then so is $\gamma \circ \phi$ where

$$\phi: \mathbb{R} \to \mathbb{R}$$
 diffeomorphism, $\phi' > 0$

The problem may also be formulated on $PTE^+ = \mathring{T}E/\mathbb{R}^+$ (double cover of *PTE*).

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Regular velocities on a manifold

Finsler geometry is defined on the slit tangent bundle $T^{\circ}E$.

First-order multiple integral problems are defined on a sub-bundle of the Whitney sum $\bigoplus^m TE$.

The bundle of regular velocities on *E* is $\mathring{T}_{(m)}E = \{(\xi_1, \dots, \xi_m) \in \bigoplus^m TE : (\xi_i) \text{ linearly independent}\}.$

Equivalently:

 $\mathring{T}_{(m)}E$ is the bundle of non-degenerate velocities — 1-jets (at the origin) of non-singular maps $\mathbb{R}^m \to E$.

Coordinates: (y^a) on E, (y^a, y^a_i) on $\mathring{T}_{(m)}E$ $(1 \le i \le m)$.

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Regular velocities on a manifold (2)

Contact forms on $\mathring{T}_{(m)}E$:

 $\omega \in \Omega(\mathring{T}_{(m)}E)$ where the pull-back $(j^1\sigma)^*\omega$ by a prolongation of $\sigma : \mathbb{R}^m \to E$ always vanishes.

In coordinates, contact 1-forms are sums of $(m + 1) \times (m + 1)$ determinants:

(Compare the contact 1-forms $du^{\alpha} - u_i^{\alpha} dx^i$ on a jet bundle.)

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Regular velocities on a manifold (3) For each function $f : E \to \mathbb{R}$, define the functions $d_i f : \mathring{T}_{(m)} E \to \mathbb{R}$ by

$$d_i f(j_0^1 \sigma) = \frac{\partial (f \circ \sigma)}{\partial t^i}$$
 where $\sigma : \mathbb{R}^m \to E$.

 d_i is a vector field along $au_m : \mathring{T}_{(m)} E o E$, called a

total derivative

A 1-form $heta \in \Omega^1 \mathring{T}_{(m)} E$ is a contact form exactly when

$$\langle d_i, \theta \rangle = 0, \qquad 1 \le i \le m.$$

In coordinates

$$d_i = y_i^a \frac{\partial}{\partial y^a}.$$

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Regular velocities on a manifold (4) The Whitney sum $\bigoplus^m TE \rightarrow E$ is a vector bundle. Denote its vertical lift operator to (η_i) by

$$\bigoplus^m T_{\tau_m(\eta_i)} E \to T_{(\eta_i)} \left(\bigoplus^m TE \right) , \qquad (\xi_i) \mapsto (\xi_i)^{\dagger(\eta_i)} .$$

For each vector $\zeta \in T_{(\eta_i)} \mathring{T}_{(m)} E$ define the vector $S^i \zeta \in T_{(\eta_i)} \mathring{T}_{(m)} E$ by

$$S^{i}\boldsymbol{\zeta} = (0,\ldots,0,T\boldsymbol{\tau}_{m}(\boldsymbol{\zeta}),0,\ldots,0)^{\dagger(\eta_{i})}$$

 S^i is a type (1,1) tensor field on $\mathring{T}_{(m)}E$, called a

vertical endomorphism

In coordinates

$$S^i = dy^a \otimes \frac{\partial}{\partial y^a_i}.$$

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Grassmannians

Regular velocities $\mathring{T}_{(m)}E$: equivalence classes of maps $\mathbb{R}^m \to E$

Grassmannian bundle $J^1(E, m)$: equivalence classes of images of maps $\mathbb{R}^m \to E$ (*m*-dimensional subspaces of *TE*)

Two regular velocities $j_0^1\sigma$, $j_0^1\hat{\sigma}$ represent the same subspace when

$$j_0^1\hat{\sigma}=j_0^1(\sigma\circ\phi)$$

for some diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ with $\phi(0) = 0$.

Oriented Grassmannians $J^1(E, m)^+$: diffeomorphism ϕ preserves orientation on \mathbb{R}^m .

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Grassmannians (2)

The projections give principal bundles

$$\begin{split} \rho &: \mathring{T}_{(m)} E \to J^1(E,m) \qquad (\text{group } GL(m,\mathbb{R})) \\ \rho^+ &: \mathring{T}_{(m)} E \to J^1(E,m)^+ \qquad (\text{group } GL(m,\mathbb{R})^+) \end{split}$$

Fundamental vector fields are $\Delta_j^i = S^i(d_j)$

In coordinates

$$\Delta_j^i = \gamma_j^a \frac{\partial}{\partial \gamma_i^a}$$

Any fibration $\pi : E \to M$ defines open submanifolds $J^1\pi \subset J^1(E,M)$ and $J^1\pi \subset J^1(E,M)^+$.

Special case: $J^{1}(E, 1) = PTE$ and $J^{1}(E, 1)^{+} = PTE^{+}$.

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Higher-order regular velocities

k-th order regular velocities $\mathring{T}^{k}_{(m)}E$: *k*-jets (at the origin) of non-singular maps $\mathbb{R}^{m} \to E$ Coordinates \mathscr{Y}^{a}_{I} on $\mathring{T}^{k}_{(m)}E$ (*I* multi-index, $0 \leq |I| \leq k$) Total derivatives d_{i} and vertical endomorphisms S^{i} :

$$d_i = \sum_{|I|=0}^{k-1} y_{I+1_i}^a \frac{\partial}{\partial y_I^a}, \qquad S^i = \sum_{|I|=0}^{k-1} (I(i)+1) dy_I^a \otimes \frac{\partial}{\partial y_{I+1_i}^a}.$$

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Higher-order regular velocities (2)

Principal bundles:

 $\rho^k: \mathring{T}^k_{(m)} E \to J^k(E,m)\,, \qquad \rho^{k+}: \mathring{T}^k_{(m)} E \to J^k(E,m)^+\,.$

Groups are the jet groups L_m^k , L_m^{k+} :

 $L_m^k = \{ j_0^k \phi : \phi : \mathbb{R}^m \to \mathbb{R}^m \text{ diffeomorphism} \}$ $L_m^{k+} = \{ j_0^k \phi \in L_m^k : |\mathcal{J}(\phi)| > 0 \}$

Fundamental vector fields $\Delta_j^I = S^I(d_j) \ (0 \le |I| \le k)$.

Put i_j^I for contraction with Δ_j^I and d_j^I for Lie derivative by Δ_j^I .

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Vector forms

We often use vectors of operators, tensors, forms, ... $d_i, S^i, \theta^i, \ldots$

These fit into a framework of vector forms.

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Vector forms

We often use vectors of operators, tensors, forms, ... $d_i, S^i, \theta^i, \ldots$

These fit into a framework of vector forms.

We consider forms on $\mathring{T}^k_{(m)}E$ taking values in the vector space \mathbb{R}^{m*} and its exterior powers.

Put

$$\Omega_k^{r,s} = \left(\Omega^r \mathring{T}^k_{(m)} E\right) \otimes \left(\bigwedge^s \mathbb{R}^{m*}\right) \,.$$

Let the standard basis for \mathbb{R}^{m*} be denoted by (dt^i) . Then

$$\Phi = \phi_{i_1 \cdots i_s} \otimes dt^{i_1} \wedge \ldots \wedge dt^{i_s} \in \Omega_k^{r,s};$$

the scalar forms $\phi_{i_1 \cdots i_s}$ are skew-symmetric in their indices.

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Vector forms (2)

Operators on vector forms:

$$\begin{split} d: \Omega_k^{r,s} &\to \Omega_k^{r+1,s}, \qquad d_{\mathrm{T}}: \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1} \\ d(\phi \otimes \omega) &= d\phi \otimes \omega, \\ d_{\mathrm{T}}(\phi \otimes \omega) &= d_i \phi \otimes dt^i \wedge \omega. \end{split}$$

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Properties: $dd_{\rm T} = d_{\rm T}d$, $d_{\rm T}^2 = 0$.

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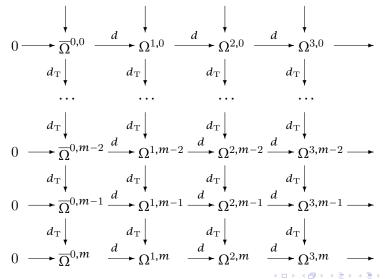
 $i_{\mathrm{T}}:\Omega_k^{r,s} o \Omega_{k+1}^{r-1,s+1}, \qquad i_{\mathrm{T}}(\phi \otimes \omega) = (d_i \,\lrcorner\, \phi) \otimes dt^i \wedge \omega$

where

$$d_{\mathrm{T}} = di_{\mathrm{T}} + i_{\mathrm{T}}d.$$

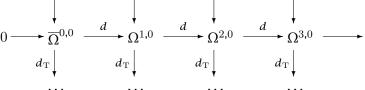
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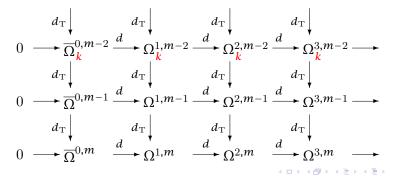




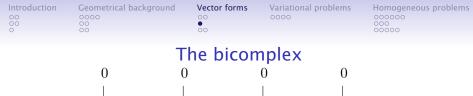
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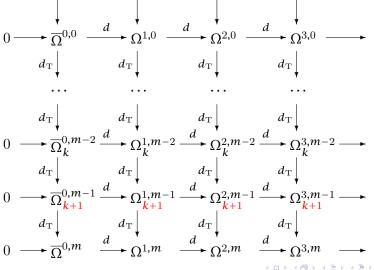






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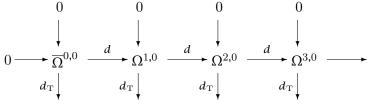


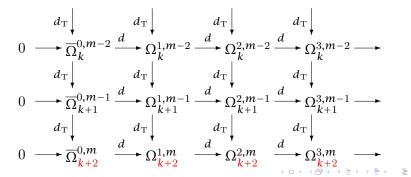


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The homotopy operators for d_{T}

The map $d_{\mathrm{T}}: \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1}$ is not exact (even locally).



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The homotopy operators for d_{T}

Vector forms

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The map $d_{\mathrm{T}}: \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1}$ is not exact (even locally). But it is globally exact modulo pull-backs (for $r \ge 1$).

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The homotopy operators for $d_{ m T}$

The map $d_{\mathrm{T}}: \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1}$ is not exact (even locally). But it is globally exact modulo pull-backs (for $r \ge 1$). The homotopy operators are $P, \widetilde{P}: \Omega_k^{r,s} \to \Omega_{(r+1)k-1}^{r,s-1}$:

$$P(\Phi) = P_{(s)}^{j}(\phi_{i_{1}\cdots i_{s}}) \otimes \left\{ \frac{\partial}{\partial t^{j}} \, \, \, \, \left(dt^{i_{1}} \wedge \dots \wedge dt^{i_{s}} \right) \right\}$$
$$\widetilde{P}(\Phi) = \widetilde{P}_{(s)}^{j}(\phi_{i_{1}}\cdots i_{s}) \otimes \left\{ \frac{\partial}{\partial t^{j}} \, \, \, \, \, \left(dt^{i_{1}} \wedge \dots \wedge dt^{i_{s}} \right) \right\}$$

where $P = \widetilde{P}$ when acting on vector 1-forms, or on first-order forms.

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The homotopy operators for $d_{\rm T}$ (2) The scalar operators $P^j_{(s)}$ and $\widetilde{P}^j_{(s)}$ are given by

$$\begin{split} P_{(s)}^{j} &= \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|} (m-s)! |J|!}{r^{|J|+1} (m-s+|J|+1)! J!} d_{J} S^{J+1_{j}},\\ \widetilde{P}_{(s)}^{j} &= \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|} (m-s)! |J|!}{r (m-s+|J|+1)! J!} d_{J} \widetilde{S}^{J+1_{j}} \end{split}$$

where, for a scalar form θ ,

$$S^{1_{j_1}1_{j_2}\cdots 1_{j_r}}\theta = i_{S^{j_1}}i_{S^{j_2}}\cdots i_{S^{j_r}}\theta$$
$$\widetilde{S}^{1_{j_1}1_{j_2}\cdots 1_{j_r}}\theta = i_{S^{j_1}S^{j_2}\cdots S^{j_r}}\theta.$$

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Homogeneous problems

The homotopy operators for $d_{\rm T}$ (2) The scalar operators $P^j_{(s)}$ and $\widetilde{P}^j_{(s)}$ are given by

$$P_{(s)}^{j} = \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s)!|J|!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J}S^{J+1j}$$

$$\widetilde{P}_{(s)}^{j} = \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s)!|J|!}{r(m-s+|J|+1)!J!} d_{J}\widetilde{S}^{J+1j}$$

where, for a scalar form θ ,

$$S^{1_{j_1}1_{j_2}\cdots 1_{j_r}}\theta = i_{S^{j_1}}i_{S^{j_2}}\cdots i_{S^{j_r}}\theta$$
$$\widetilde{S}^{1_{j_1}1_{j_2}\cdots 1_{j_r}}\theta = i_{S^{j_1}S^{j_2}\cdots S^{j_r}}\theta.$$

Note that $P^2 = 0$ but $\widetilde{P}^2 \neq 0$.

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Variational problems on fibred manifolds

Let $\pi : E \to M$ be a fibred manifold, with $\dim M = m$ and $\dim E = m + n$, where M is orientable.

A Lagrangian is an *m*-form $\lambda = L d^m x$ on the jet bundle $J^k \pi$, horizontal over *M*.

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Homogeneous problems

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for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$.

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Lepage equivalents

Let $\lambda = L d^m x$ be a Lagrangian *m*-form on $J^k \pi$.

Another *m*-form θ on $J^l \pi$ (where $l \ge k$) is a *Lepage form* if: $i_Y d\theta$ is a contact form whenever the vector field *Y* is vertical over *E*.

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It is a *Lepage equivalent* of λ if: it is a Lepage form, and $\pi_{lk}^* \lambda - \theta$ is a contact form.

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Every Lagrangian m-form has a Lepage equivalent, defined on $J^{2k-1}\pi$.

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Examples of Lepage equivalents

For the case m = 1 (single integral variational problems): For a first-order Lagrangian $\lambda = L dx$ on $J^{1}\pi$:

$$\theta = L \, dx + \frac{\partial L}{\partial \dot{y}^{\alpha}} (dy^{\alpha} - \dot{y}^{\alpha} dx)$$

is the unique Lepage equivalent, the *Poincaré-Cartan form*, also defined on $J^1\pi$.

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is the unique Lepage equivalent, the *Poincaré-Cartan form*, also defined on $J^1\pi$.

For a higher-order Lagrangian $\lambda = L dx$ on $J^k \pi$:

$$\theta = L dx + \sum_{p=0}^{k-1} \left(\sum_{q=0}^{k-p-1} (-1)^q \frac{d^q}{dx^q} \frac{\partial L}{\partial y^{\alpha}_{(p+q+1)}} \right) (dy^{\alpha}_{(p)} - y^{\alpha}_{(p+1)} dx)$$

is the unique Lepage equivalent, defined on $J^{2k-1}\pi$.

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Examples of Lepage equivalents (2)

For the case $m \ge 2$ (multiple integral variational problems) and a first-order Lagrangian $\lambda = L d^m x$ on $J^1 \pi$:

$$\theta_{1} = L d^{m}x + \frac{\partial L}{\partial y_{i}^{\alpha}} \omega^{\alpha} \wedge d^{m-1}x_{i}$$

$$\theta_{2} = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(L dx^{i} + \frac{\partial L}{\partial y_{i}^{\alpha}} \omega^{\alpha} \right)$$

$$\theta_{3} = \sum_{r=0}^{\min\{m,n\}} \frac{1}{(r!)^{2}} \frac{\partial^{r}L}{\partial y_{i_{1}}^{\alpha_{1}} \cdots \partial y_{i_{r}}^{\alpha_{r}}} \omega^{\alpha_{1}} \wedge \cdots \wedge \omega^{\alpha_{r}} \wedge d^{m-r}x_{i_{1}\cdots i_{r}}$$

(where $\omega^{\alpha} = dy^{\alpha} - y_{j}^{\alpha} dx^{j}$) are globally-defined Lepage equivalents.

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Homogeneous variational problems

We now consider m-dimensional variational problems on E, with fixed boundary conditions.

It is sufficient to consider submanifolds of the form $\sigma(C)$ where $\sigma : \mathbb{R}^m \to E$ and $C \subset \mathbb{R}^m$ is a compact *m*-dimensional submanifold.

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Homogeneous problems

Homogeneous variational problems

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This is because variational problems are local.

'An m-dimensional submanifold of E is extremal with fixed boundary conditions if, and only if, every small piece of it is extremal with fixed boundary conditions.'

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Homogeneous Lagrangians

A vector function $\Lambda = L d^m t \in \Omega^{0,m}$ is called a *Lagrangian* for a parametric variational problem. It is called *homogeneous* if it is equivariant with respect to the action of the jet group L_m^{k+} , where k is the order of the Lagrangian.

If Λ is homogeneous then the scalar function L satisfies

$$d^i_j L = \delta^i_j L$$
, $d^I_j L = 0$ for $|I| \ge 2$.

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$$\int_C ((j\sigma)^* X^k L) d^m t = 0$$

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Equivalents of Lagrangians

Let $\Lambda \in \Omega^{0,m}$ be a homogeneous Lagrangian.

Any scalar *m*-form $\Theta_m \in \Omega^{m,0}$ is called an *integral equivalent* of Λ if $\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i_{\mathrm{T}}^m \Theta_m.$

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Any vector r-form $\Theta_r \in \Omega^{r,m-r}$ is called an *intermediate* equivalent if

$$\Lambda = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!} i_{\rm T}^r \Theta_r \qquad 0 \le r \le m-1.$$

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If Θ_{r+1} is an equivalent of Λ then

$$\Theta_r = \frac{(-1)^r}{m-r} \, i_{\rm T} \Theta_{r+1}$$

is also an equivalent.

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Integral equivalents

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i_{\mathrm{T}}^{m} \Theta_{m}$$

Why 'integral equivalent'?

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Integral equivalents

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i_{\mathrm{T}}^{m} \Theta_{m}$$

Why 'integral equivalent'?

If
$$\sigma : \mathbb{R}^m \to E$$
 then $(j\sigma)^*\Lambda = (j\sigma)^*\Theta_m$

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Integral equivalents

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i_{\mathrm{T}}^{m} \Theta_{m}$$

Why 'integral equivalent'?

If
$$\sigma : \mathbb{R}^m \to E$$
 then $(j\sigma)^*\Lambda = (j\sigma)^*\Theta_m$

so that

$$\int_{\mathcal{C}} (j\sigma)^* \Lambda = \int_{\mathcal{C}} (j\sigma)^* \Theta_m.$$

Thus $\Lambda = \Theta_0$ and Θ_m have the same extremals.

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Euler forms

Let Θ_m be an integral equivalent of Λ .

Define the scalar (m + 1)-form $\mathcal{E}_m \in \Omega^{m+1,0}$ by

 $\mathcal{E}_m = d\Theta_m$

and the vector forms $\mathcal{F}_r \in \Omega^{r+1,m-r}$ by

 $\mathcal{E}_{r} = d\Theta_{r} - (-1)^{r} d_{\mathrm{T}} \Theta_{r+1} \qquad 0 \leq r \leq m-1.$

The forms \mathcal{L}_r are called the *Euler forms* of Θ_m .

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Euler forms

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$$\mathcal{E}_{\mathcal{F}} = d\Theta_{\mathcal{F}} - (-1)^{\mathcal{F}} d_{\mathrm{T}} \Theta_{\mathcal{F}+1} \qquad 0 \leq \mathcal{F} \leq m-1.$$

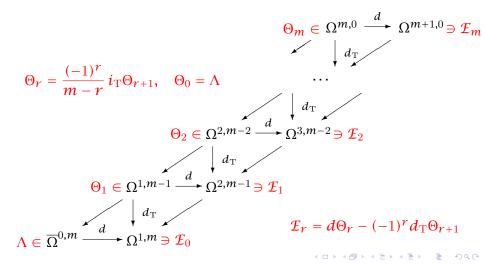
The forms \mathcal{L}_{r} are called the *Euler forms* of Θ_{m} .

By a straightforward calculation

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r} i_{\mathrm{T}} \mathcal{E}_{r+1} \qquad 0 \le r \le m-1.$$

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Equivalents and Euler forms



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Lepagian forms

Let Λ be a homogeneous Lagrangian.

Let Θ_r be an equivalent of Λ ($1 \le r \le m$).

Say that Θ_r is *Lepagian* if the corresponding Euler form $\mathcal{E}_0 \in \Omega^{1,m}$ satisfies

$$S\mathcal{E}_0=0,$$

so that \mathcal{E}_0 is horizontal over *E*.

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so that \mathcal{E}_0 is horizontal over *E*.

Theorem The vector 1-form

$$\Theta_1 = Pd\Lambda$$

is an integral equivalent of Λ (m = 1) or an intermediate equivalent ($m \ge 2$), and is Lepagian. It is called the *Hilbert* equivalent of Λ .

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The Hilbert equivalent

 $\Theta_1 = P d\Lambda$ is a Lepagian equivalent of $\Lambda = L d^m t$.

Outline of proof: if $\Phi = \phi \otimes d^m t \in \Omega^{1,m}$ use

$$P\Phi = P^{j}\phi \otimes d^{m-1}t_{j}, \qquad P^{j} = \sum_{J} \frac{(-1)^{|J|}}{(|J|+1)J!} d_{J}S^{J+1_{j}}.$$

To show $i_T P d\Lambda = \Lambda$, use: commutators $[i_k, d_j] = 0$, $[i_k^I, S^j] = i_k^{I+1_j}$, homogeneity $i_k^I dL = d_k^I L = 0$ ($|I| \ge 1$), vanishing of S^i on functions $i_k dL$ and $i_k^j dL$, homogeneity again $i_k^j dL = d_k^j dL = \delta_k^j dL$.

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To show $S\mathcal{E}_0 = S(d\Lambda - d_T P d\Lambda) = 0$ use: commutators $[S^j, d_i] = \delta_i^j$ to give a collapsing sum so that $Sd_T P(d\Lambda) = S(d\Lambda)$.

If $\widetilde{\Theta}_1$ is another Lepagian vector 1-form equivalent to Λ , with corresponding Euler form $\widetilde{\mathcal{I}}_0$, then

$$\widetilde{\mathcal{E}}_0 = \mathcal{E}_0, \qquad \widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}} \Phi \qquad (\Phi \in \Omega^{r, m-2})$$

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The First Variation Formula

Given a variation field X on E with $X|_C = 0$, and its prolongation \hat{X} ,

$$\int_{C} (j\sigma)^{*} d_{\hat{X}} \Lambda = \int_{C} (j\sigma)^{*} di_{\hat{X}} \Lambda + \int_{C} (j\sigma)^{*} i_{\hat{X}} d\Lambda$$
$$= \int_{\partial C} (j\sigma)^{*} i_{\hat{X}} \Lambda + \int_{C} (j\sigma)^{*} i_{\hat{X}} d\Lambda$$
$$= \int_{C} (j\sigma)^{*} i_{\hat{X}} \mathcal{E}_{0} + \int_{C} (j\sigma)^{*} i_{\hat{X}} d_{T} \Theta$$

But

$$\int_{C} (j\sigma)^* i_{\hat{X}} d_{\mathrm{T}} \Theta_1 = \int_{C} (j\sigma)^* d_{\mathrm{T}} i_{\hat{X}} \Theta_1 = \int_{C} d(j\sigma)^* i_{\hat{X}} \Theta_1 = 0$$

because prolongations commute with total derivatives. Thus

$$\int_C (j\sigma)^* d_{\hat{X}} \Lambda = \int_C (j\sigma)^* i_{\hat{X}} \mathcal{E}_0 = \int_C (j\sigma)^* i_X \mathcal{E}_0$$

because \mathcal{E}_0 is horizontal over E.

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Integral equivalents for $m \ge 2$

Let $\Lambda = L d^m t$ be a homogeneous Lagrangian with $m \ge 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1}t_i;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ .

If Λ never vanishes, define the Carathéodory equivalent $\widetilde{\Theta}_m \in \Omega^{m,0}$ by

$$\widetilde{\Theta}_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^m \mathfrak{P}^i.$$

Theorem The Carathéodory equivalent $\widetilde{\Theta}_m$ is an integral equivalent of Λ .

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The Carathéodory equivalent

Given $\widetilde{\Theta}_m = (1/L^{m-1}) \mathfrak{P}^1 \wedge \cdots \wedge \mathfrak{P}^m$, we must show that

$$i_{\mathrm{T}}^{m}\widetilde{\Theta}_{m} = (-1)^{m(m-1)/2} m! \Lambda.$$

Outline of proof, using $i_k \vartheta^i = \delta^i_k L$: suppose

$$i_{\mathrm{T}}^{s} \widetilde{\Theta}_{m} = \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}} \left\{ \sum_{\sigma \in S_{m}} (-1)^{\sigma} \mathfrak{g}^{\sigma(1)} \wedge \dots \wedge \mathfrak{g}^{\sigma(m-s)} \otimes dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \right\}$$

(where S_m is the permutation group) and use induction.



The variation formula for $\widetilde{\Theta}_m$

From the induction formula

$$i_{\mathrm{T}}^{m-1}\widetilde{\Theta}_m = (-1)^{m(m-1)/2} (m-1)! \Theta_1$$

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where Θ_1 is the Hilbert equivalent.



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From the induction formula

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The variation formula for $\tilde{\Theta}_m$

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From the induction formula

$$i_{\mathrm{T}}^{m-1}\widetilde{\Theta}_m = (-1)^{m(m-1)/2} (m-1)!\Theta_1$$

where Θ_1 is the Hilbert equivalent. Thus $\widetilde{\Theta}_m$ is Lepagian.

Then, as $d\Theta_m = \mathcal{E}_m$, $\int_C (j\sigma)^* d_Y \Theta_M = \int_C (j\sigma)^* i_Y \mathcal{E}_m$ $= \int_C (j\sigma)^* i_Y \mathcal{E}_0$

for any vector field Y on $\mathring{T}_{(m)}^k E$ vanishing on $j\sigma(\partial C)$, because contractions by vector fields anticommute, so that $i_T^m i_Y \mathcal{E}_m = (-1)^m i_Y i_T^m \mathcal{E}_m$.

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Another integral equivalent

Let Λ be a *first-order* Lagrangian, and put

 $\Theta_{r+1} = (-1)^r P d\Theta_r \qquad (1 \le r < m) \,.$



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Another integral equivalent

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Each Θ_r is a first-order vector form.

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$$\Theta_{r+1} = (-1)^r P d\Theta_r \qquad (1 \le r < m) \,.$$

Each Θ_r is a first-order vector form.

Using commutator relations, we obtain

$$\Theta_r = \frac{(-1)^r}{m-r} \, i_{\rm T} \Theta_{r+1}$$

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so that Θ_m is a Lepagian integral equivalent of Λ , the *fundamental equivalent* of Λ .

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Another integral equivalent

Let Λ be a *first-order* Lagrangian, and put

$$\Theta_{r+1} = (-1)^r P d\Theta_r \qquad (1 \le r < m) \,.$$

Each Θ_r is a first-order vector form.

Using commutator relations, we obtain

$$\Theta_r = \frac{(-1)^r}{m-r} \, i_{\rm T} \Theta_{r+1}$$

so that Θ_m is a Lepagian integral equivalent of Λ , the *fundamental equivalent* of Λ .

Thus $d\Theta_m = \mathcal{E}_m = 0$ if, and only if, $\mathcal{E}_0 = 0$.

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Other matters

- Regularity
- Symmetry
- Helmholtz equations