Dirac structures and geometry of nonholonomic constraints

References

Hiroaki Yoshimura, Jerrold E. Marsden: "Dirac structures in Lagrangian Mechanics", *J. Geom. Phys.*, **57**, (2006) Part I 133–156, Part II 209-250;

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Introduction

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Contents

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- Nonholonomic constraints and Dirac structures according to Y & M
- My point of view
- Comments about the terminology (and example)

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Dirac structure

Definition

A *Dirac structure* L on a manifold M is a vector subbundle of $\mathsf{T}M \oplus_M \mathsf{T}^*M$ which is maximal isotropic with respect to the metric

$$\langle X_p + \varphi_p \,|\, Y_p + \psi_p \rangle = \frac{1}{2}(\langle \psi_p, X_p \rangle + \langle \varphi_p, Y_p \rangle)$$

and involutve with respect to Dorfman bracket

$$[X + \varphi, Y + \psi]_D = [X, Y] + (\mathcal{L}_X \psi - \imath_Y \mathrm{d}\varphi).$$

We call L an *almost Dirac structure* when it is maximal isotropic and not involutive.

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Almost Dirac structure in mechanics

- $M = \mathrm{T}^*Q$, symplectic form ω_Q, associated map β_Q:

$$\beta_Q : \mathrm{T}\mathrm{T}^*Q \longrightarrow \mathrm{T}^*\mathrm{T}^*Q, \qquad \beta(w) = \omega_Q(\cdot, w)$$

- $\Delta_Q \subset \mathrm{T}Q$ is a regular distribution not necessarily involutive, representing (nonholonomic) constraints,
- $\Delta_Q^\circ \subset \mathrm{T}^*Q$ is an anihilator of Δ_Q,

$$\pi_Q : \mathrm{T}^*Q \longrightarrow Q$$
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Almost Dirac structures in mechanics

Definition

The almost Dirac structure on T^*Q induced by a distribution Δ_Q is the subbundle

$$\Delta \subset \mathsf{T}\mathsf{T}^*Q \oplus_{\mathsf{T}^*Q} \mathsf{T}^*\mathsf{T}^*Q$$

$$\Delta = \{w + \varphi : \quad w \in \Delta_{\mathsf{T}^*Q}, \; \varphi - \beta_Q(w) \in \Delta^\circ_{\mathsf{T}^*Q}\}$$

- If $\Delta_Q = \mathsf{T}Q$ then $\Delta_{\mathsf{T}^*Q} = \mathsf{T}\mathsf{T}^*Q$ and Δ is the graph of β_Q.
- If Δ_Q is involutive then Δ is a true Dirac structure.

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Almost Dirac structures in mechanics

Local expressions: For $\mathcal{O}$ being an open subset of Q we identify

$$\begin{aligned}
&\mathrm{T}\mathcal{O} \simeq \mathcal{O} \times V \ni (q, \dot{q}),\\
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&\mathrm{T}\mathrm{T}^*\mathcal{O} \simeq \mathcal{O} \times V^* \times V \times V^* \ni (q, p, \dot{q}, \dot{p}) = w,\\
&\mathrm{T}^*\mathrm{T}^*\mathcal{O} \simeq \mathcal{O} \times V^* \times V^* \times V \ni (q, p, a, b) = \varphi.
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&\Delta_Q = \{(q, \dot{q}) : \ \dot{q} \in \Delta_Q(q)\}, \qquad \Delta_Q(q) \subset V,\\
&\Delta^\circ_Q = \{(q, \dot{q}) : \ \dot{q} \in \Delta^\circ_Q(q)\}, \qquad \Delta^\circ_Q(q) \subset V^*,\\
&\Delta_{\mathrm{T}^*Q} = \{(q, p, \dot{q}, \dot{p}) : \ \dot{q} \in \Delta_Q(q)\},\\
&\Delta^\circ_{\mathrm{T}^*Q} = \{(q, p, a, b) : \ a \in \Delta^\circ_Q(q), \ b = 0\}.
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Constrained Lagrangian system according to Y & M

Let $L : \mathsf{T}Q \to \mathbb{R}$ be a Lagrangian (possibly not hyperregular).

Definition

The Legendre map $\mathbb{F}L : \mathsf{T}Q \to \mathsf{T}^*Q$ is defined by the formula

$$\mathbb{F}L = \zeta \circ \mathrm{d}L.$$

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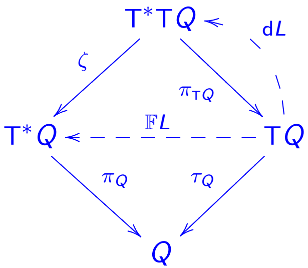

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There exists the canonical isomorphism

$$\gamma_Q : \mathsf{T}^*\mathsf{T}Q \longrightarrow \mathsf{T}^*\mathsf{T}^*Q$$

Locally for $\mathcal{O} \subset Q$ we have

$$\mathsf{T}^*\mathsf{T}\mathcal{O} \simeq \mathcal{O} \times V \times V^* \times V^*, \qquad \mathsf{T}^*\mathsf{T}^*\mathcal{O} \simeq \mathcal{O} \times V^* \times V^* \times V$$

and

$$\gamma_Q(q, \dot{q}, c, d) = (q, d, -c, \dot{q})$$

It exists for any vector bundle (not only tangent bundle), it is a double vector bundle isomorphism and antisymplectomorphism.

Constrained Lagrangian system according to Y & M

Definition

For $L : \mathsf{T}Q \to \mathbb{R}$ the composition

$$\gamma_Q \circ \mathrm{d}L : \mathsf{T}Q \longrightarrow \mathsf{T}^*\mathsf{T}^*Q$$

will be called the *Dirac differential* of L and denoted by $\mathcal{D}L$

Locally:

$$\mathcal{D}L(q, \dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}, -\frac{\partial L}{\partial q}, \dot{q})$$

Constrained Lagrangian system according to Y & M

Definition

A *partial vector field* on T^*Q is a map

$$X : \mathsf{T}Q \oplus_Q \mathsf{T}^*Q \longrightarrow \mathsf{T}\mathsf{T}^*Q$$

such that

$$X(v,p) \in \mathsf{T}_p\mathsf{T}^*Q.$$

Constrained Lagrangian system according to Y & M

A Lagrangian system with nonholonomic constraints Δ_Q can be described by a triple

$$(L, \Delta, X)$$

where

- L is a Lagrangian function, possibly not hyperregular,
- Δ is the Dirac structure associated to the constraint distribution Δ_Q,
- X is a partial vector field defined in points

$$(v, p) \in \mathsf{T}Q \oplus \mathsf{T}^*Q, \text{ such that } v \in \Delta_Q, \quad p = \mathbb{F}L(v)$$

- and such that

$$X(v, p) + \mathcal{D}L(v) \in \Delta$$

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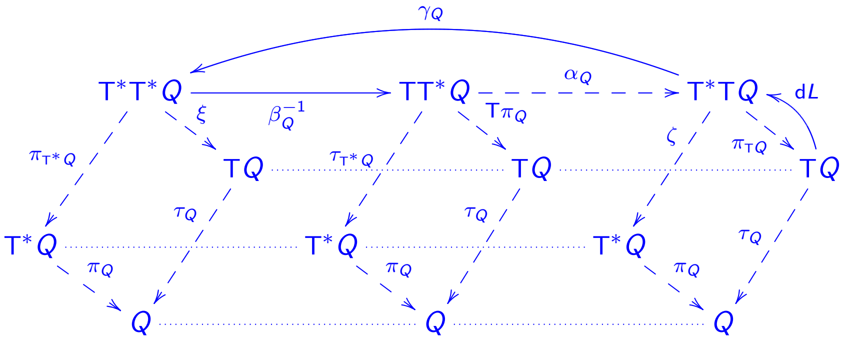
My point of view

Lagrangian dynamics without constraints according to Y & M (when we forget partial vector fields)



$$D_L = \beta_Q^{-1}(\gamma_Q(\mathrm{d}L(\mathsf{T}Q)))$$

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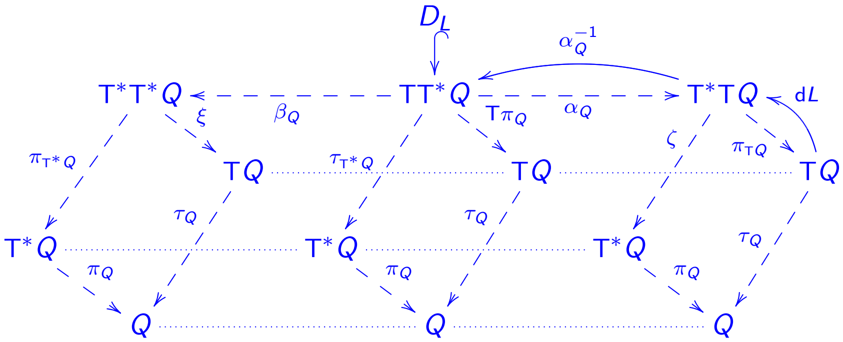

$$D_L = \alpha_Q^{-1}(\mathrm{d}L(\mathsf{T}Q))$$

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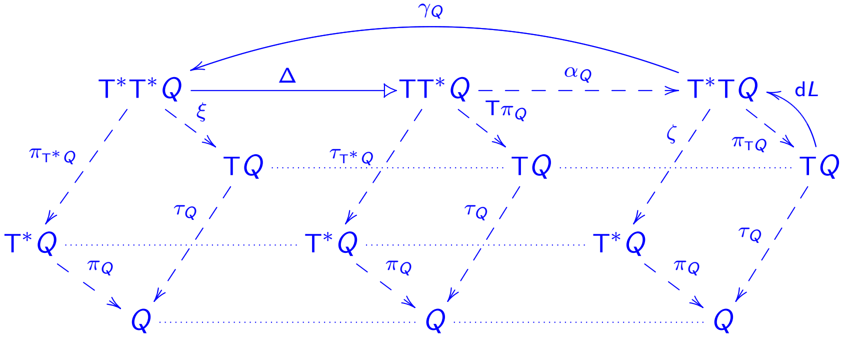
Lagrangian dynamics with constraints according to Y & M



$$D_L = \Delta(\gamma_Q(\mathrm{d}L(\mathrm{T}Q)))$$

My point of view

- We treat Δ as a relation $\mathsf{T}^*\mathsf{T}^*Q \dashrightarrow \mathsf{T}\mathsf{T}^*Q$
- We can take a "short-cut" identifying $\mathsf{T}^*\mathsf{T}^*Q$ with $\mathsf{T}^*\mathsf{T}Q$

- Local expressions:

$$\mathsf{T}^*\mathsf{T}\mathcal{O} \simeq \mathcal{O} \times V \times V^* \times V^* \ni (q, \dot{q}, c, d)$$
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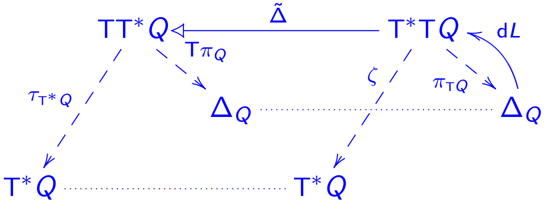

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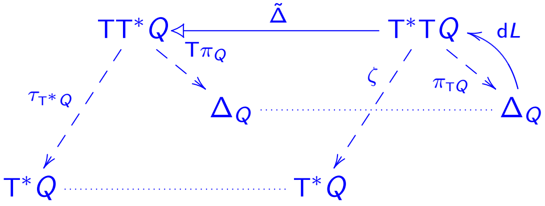

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How to get $\tilde{\Delta}$ not going to the "Hamiltonian" side of the diagram?

- Many of the ideas od Analytical Mechanics come from statics,
- In statics constraints are described by admissible configurations and admissible virtual displacements (WMT),
- We observed in "Variational ... in general algebroids" that different set of admissible virtual displacements produce different E-L equations (vaconomic, nonholonomic) for the same set of admissible configurations.
- How to put virtual displacements into the game?

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How to get $\tilde{\Delta}$ not going to the "Hamiltonian" side of the diagram?

- Many of the ideas od Analytical Mechanics come from statics,
- In statics constraints are described by admissible configurations and admissible virtual displacements (WMT),
- We observed in "Variational ... in general algebroids" that different set of admissible virtual displacements produce different E-L equations (vaconomic, nonholonomic) for the same set of admissible configurations.
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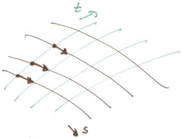
$$\mathcal{D}_L \subset TT^*Q \xrightarrow{\alpha_Q} T^*TQ \supset dL(TQ)$$

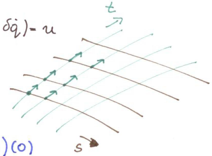
$$w=(q,p,\dot q,\dot p) \qquad\qquad (q,\dot q,c,d)$$

$$TTQ \xleftarrow{\kappa_Q} TTQ$$

$$(q,\delta q,\dot q,\delta\dot q) \qquad\qquad (q,\dot q,\delta q,\delta\dot q)=u$$



$\frac{d}{ds}\chi(t,\cdot)(0)$



$\frac{d}{dt}\chi(\cdot,s)(0)$

$$\langle\langle w, \kappa(u)\rangle\rangle = \langle \alpha(w), u\rangle$$

$$p\delta\dot q + \dot p\delta q = c\,\delta q + d\,\delta\dot q$$

$$\alpha_Q: \quad c=\dot p,\quad d=p$$

My point of view

The constrained case: we choose admissible displacements

$$U = \{u \in \mathsf{T}\mathsf{T}Q : \ \tau_{\mathsf{T}Q}(u) \in \Delta_Q, \ \mathsf{T}\tau_Q(u) \in \Delta_Q\}$$

Locally:

$$u = (q, \dot{q}, \delta q, \delta \dot{q}) : \quad \dot{q} \in \Delta_Q(q), \ \delta q \in \Delta_Q(q)$$

$$\kappa_Q(U) = U$$

We treat $\kappa_{Q|U}$ as a relation in $\mathsf{T}\mathsf{T}Q$ and we look for the dual relation with respect to the evaluations $\langle \cdot, \cdot \rangle$, $\langle\langle \cdot, \cdot \rangle\rangle$.

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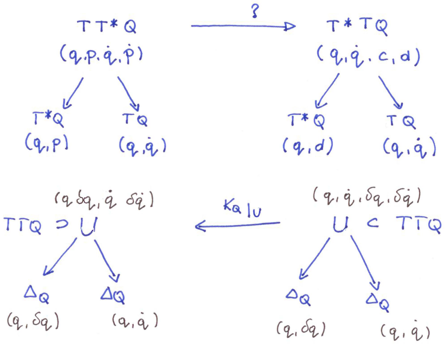

$$\langle \alpha(w), u \rangle = \langle\!\langle w, \kappa_Q(u) \rangle\!\rangle \quad \text{for } u \in U$$

$$c\,\delta q + d\,\delta\dot{q} = p\,\delta\dot{q} + \dot{p}\,\delta q \quad \Rightarrow \quad \begin{array}{l} d = p \\ c - \dot{p} \in \Delta_Q^{\circ}(q) \\ \dot{q} \in \Delta_Q(q) \end{array}$$

▸ $\tilde{\Delta}$

My point of view

Conclusion

The relation $\tilde{\Delta}$ is dual to the relation κ_Q restricted to the set of admissible virtual displacements.

Equations

The equations for a curve $t \longmapsto \wp(t)$ in the phase space T^*Q are

$$\mathsf{t}\wp(t) \in \tilde{\Delta} \circ \mathrm{d}L(\Delta_Q)$$

Local expressions:

$$t \longmapsto (q(t), p(t))$$

$$\dot{q} \in \Delta_Q(q(t)), \quad p(t) = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \quad \dot{p}(t) \in \frac{\partial L}{\partial q}(q, \dot{q}) + \Delta_Q^\circ(q(t))$$

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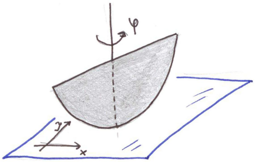
Example

Lagrangian system with constraints: a sleigh on a horizontal plane

$$Q = \mathbb{R}^2 \times S^1 \ni (x, y, \varphi),$$
$$\Delta_Q(x, y, \varphi) = \{(\dot{x}, \dot{y}, \dot{\varphi}) : \quad \dot{x}\sin(\varphi) = \dot{y}\cos(\varphi)\},$$
$$\Delta_Q = \langle \frac{\partial}{\partial\varphi}, \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y} \rangle$$

$$L(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\varphi}^2$$

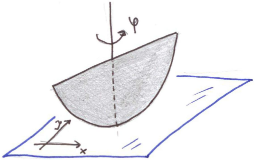


Example

Equations for a curve $t \longmapsto (x(t), y(t), \varphi(t), p_x(t), p_y(t), \pi(t))$

$$\dot{x} = p_x/m, \quad \dot{p}_x = \mu(t)\sin\varphi$$
$$\dot{y} = p_y/m, \quad \dot{p}_y = -\mu(t)\cos\varphi$$
$$\dot{\varphi} = \pi/m, \quad \dot{\pi} = 0$$
$$\dot{x}\sin\varphi = \dot{y}\cos\varphi$$

Solution (momentum)



$$\pi(t) = \pi_0$$
$$p_x(t) = p_0\cos(\frac{\pi_0}{I}t + \varphi_0)$$
$$p_y(t) = p_0\sin(\frac{\pi_0}{I}t + \varphi_0)$$

Example

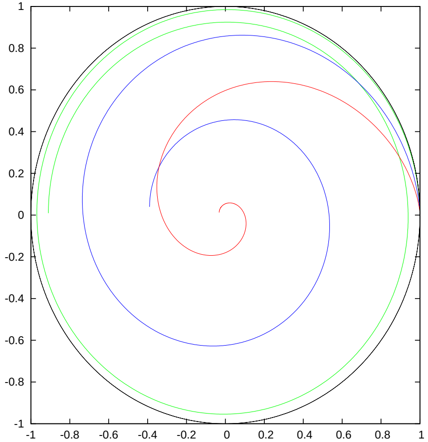
Modeling constraints: a sleigh subject to strong viscous force perpendicular to the sleigh

Equations for a curve $t \longmapsto (x(t), y(t), \varphi(t), p_x(t), p_y(t), \pi(t))$ $(\gamma < 0)$

$$\begin{aligned}
\dot{x} &= p_x/m, \quad \dot{p}_x = (\gamma/m)(p_y \cos\varphi - p_x \sin\varphi)(-\sin\varphi) \\
\dot{y} &= p_y/m, \quad \dot{p}_y = (\gamma/m)(p_y \cos\varphi - p_x \sin\varphi)(\cos\varphi) \\
\dot{\varphi} &= \pi/I, \quad \dot{\pi} = 0
\end{aligned}$$

Example

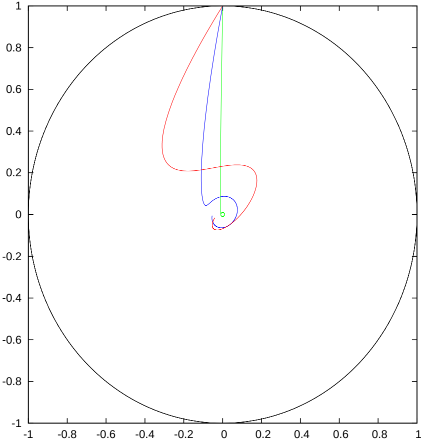
Initial momentum along the sleigh, $\pi_0/I = 1$, $\varphi_0 = 0$



black: solution with constraints
red: $\gamma/m = -3$
blue: $\gamma/m = -10$
green: $\gamma/m = -100$

Example

Initial momentum perpendicular to the sleigh, $\pi_0/I = 1$, $\varphi_0 = 0$



red: $\gamma/m = -3$
blue: $\gamma/m = -10$
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