Liouville structures

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the fundamental geometric structure used in variational formulations of physical theories are related to the goemetry of cotangent bundles.

There are rarely directly cotangent bundles, but only structurally isomorphic to a cotangent bundle.

For reasons of interpretation this structure can not be replaced by the corresponding cotangent fibration.

Preliminary definition

A Liouville structure is a vector fibration isomorphism

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \mathsf{T}^*Q \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Q} \\ Q & = & Q \end{array}$$

This is a preliminary definition.

Example 1

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$$\begin{array}{c} E \\ \downarrow{\scriptstyle \eta} \\ Q \end{array},$$

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(it is also a symplectomorphism).
There are TWO different Liouville structures on T^*E.

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We have a vector bundle isomorphism, which is also a symplectomorphism

$$\begin{array}{ccc}
\mathsf{T}P & \xrightarrow{\beta_{(P,\omega)}} & \mathsf{T}^*P \\
{\scriptstyle \tau_P}\downarrow & & \downarrow{\scriptstyle \pi_P} \\
P & = & P
\end{array}$$

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be a vector fibration.
A form μ on E is linear if $\mathrm{d}_Z\mu = \mu$, where Z is the Euler vector field on E.
A 1-form ϑ is linear if it is a vector fibration morphism

$$\begin{array}{ccc} E & \xrightarrow{\vartheta} & \mathsf{T}^*E \\ \eta \downarrow & & \downarrow \mathsf{T}^*\eta \\ Q & \longrightarrow & E^* \end{array}$$

A 2-form μ is linear if the associated linear map $\widetilde{\mu}$

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\beta_{(E,\mu)}} & \mathsf{T}^*E \\ {\scriptstyle \tau_E}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ E & = & E \end{array}$$

is also a vector fibration morphism

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\beta_{(E,\mu)}} & \mathsf{T}^*E \\ {\scriptstyle \mathsf{T}\eta}\downarrow & & \downarrow{\scriptstyle \mathsf{T}^*\eta} \\ \mathsf{T}Q & \longrightarrow & E^* \end{array}$$

Alternative definitions of a Liouville structure

- A bilinear non degenerate pairing

$$\langle \ , \ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \to \mathbb{R}.$$

$$\langle \alpha(p), v\rangle_Q = \langle p, v\rangle.$$

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- Linear symplectic form ω on P.

$$\langle\ ,\ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \to : (p,v) \mapsto \omega(\chi_\pi(O_\pi(\pi(p)), p), \mathsf{T}O_{\tau_Q}(v)).$$

Here $\alpha^* \omega_Q = \omega$.

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- Linear and vertical 1-form ϑ (a Liouville form) with non degenerate

$$\omega = \mathrm{d}\vartheta.$$

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$$\vartheta - \alpha^*\vartheta_Q = \mathrm{d}f.$$

f is linear, $\mathrm{d}f$ is vertical, hence $f = 0$.

Relations

A vector fibration relation is differential relation of fibrations

$$\begin{array}{ccc} P & \xrightarrow{\rho} & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Q & \xrightarrow{\sigma} & Q' \end{array}$$

such that for each $(q', q) \in \mathrm{graph}(\sigma)$ the set $\mathrm{graph}(\rho) \cap (P'_{q'} \times P_q)$ is a vector subspace of $P'_{q'} \times P_q$.

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A Liouville structure morphism is a vector fibration relation such that one of the following equivalent conditions is satisfied:

- If the Liouville structures are characterized by symplectic forms ω and ω', then the relation ρ is a symplectic relation from (P, ω) to (P', ω').

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- If the Liouville structures are characterized by .....

Propositions

Proposition. *Let* $\pi : P \to Q$ *be a vector fibration. If* K *is a closed submanifold of* P *such that for each* $q \in C = \pi(K)$ *the intersection* $K_q = K \cap P_q$ *of* K *with* $P_q = \pi^{-1}(q)$ *is a vector subspace of* P_q*, then* C *is a submanifold of* Q *and the dimension of* K_q *is locally constant and the mapping*

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Proposition. *If* K *is a submanifold of the cotangent bundle* T^*Q *with the properties*

1. *the dimension of* K *is equal to the dimension of* Q*,*
2. *for each* $q \in C = \pi_Q(K)$ *the intersection* $K_q = K \cap \mathsf{T}^*_q Q$ *of* K *with the fibre* $\mathsf{T}^*_q Q = \pi_Q^{-1}(q)$ *is a vector subspace of the fibre,*
3. *the Liouville form* ϑ_Q *vanishes on* $\mathsf{T}K$*,*

then $C \subset Q$ *is a submanifold and* $K = \mathsf{T}^\circ C$*.*

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A morphism of Liouville structures is the ’phase lift’ of a differential relation between base manifolds.

The tangent functor

The *tangent functor* T associates the structure

$$\begin{array}{c} (\mathsf{T}P, \mathrm{d}_T\vartheta) \\ \mathsf{T}\pi \Big\downarrow \\ \mathsf{T}Q \end{array}$$

with a Liouville structure

$$\begin{array}{c} (P, \vartheta) \\ \pi \Big\downarrow \\ Q \end{array}$$

and the morphism

$$\begin{array}{ccc}
(\mathsf{T}P, \mathrm{d}_T\vartheta) & \xrightarrow{\mathsf{T}\rho} & (\mathsf{T}P', \mathrm{d}_T\vartheta') \\
\downarrow{\scriptstyle \mathsf{T}\pi} & & \downarrow{\scriptstyle \mathsf{T}\pi'} \\
\mathsf{T}Q & \xrightarrow{\mathsf{T}\sigma} & \mathsf{T}Q'
\end{array}$$

with a Liouville structure morphism

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(P, \vartheta) & \xrightarrow{\rho} & (P', \vartheta') \\
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The Hamilton functor

The *Hamilton functor* H is a covariant functor from the category of symplectic manifolds to the category of Liouville structures.

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It associates the Liouville structure

$$\begin{array}{c}(\mathsf{T}P, \mathrm{i}_T\omega) \\ \Big\downarrow{\scriptstyle \tau_P} \\ P\end{array}$$

with a symplectic manifold (P, ω).

and the morphism

$$
\begin{array}{ccc}
(\mathsf{T}P, \mathrm{i}_T\omega) & \xrightarrow{\mathsf{T}\varphi} & (\mathsf{T}P', \mathrm{i}_T\omega') \\
\downarrow{\scriptstyle \mathsf{T}\pi} & & \downarrow{\scriptstyle \tau'_P} \\
P & \xrightarrow{\varphi} & P'
\end{array}
\qquad (237)
$$

with a symplectomorphism $\varphi : (P, \omega) \longrightarrow (P', \omega')$.

Generating functions

A Liouville structure offers the possibility of generating from generating objects subsets of a symplectic manifold (P, ω) for which the Liouville structure is established. Such subsets are usually Lagrangian submanifolds.

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An example of a generating object is a constrained generating function

$$U : C \to \mathbb{R},$$

defined on a submanifold of $C \subset Q$. The set

$$S = \Big\{ f \in P \; ; \; \pi_Q(f) \in C \;\text{ and }\; \forall_{\delta q \in \mathsf{T}C} \;\text{ if }\; \tau_Q(\delta q) = \pi(f),$$

$$\text{then } \; \langle f, \delta q \rangle = \langle \mathrm{d}U, \delta q \rangle_C \Big\}$$

is the Lagrangian submanifold of (P, ω) generated by the constrained function U. This submanifold is an affine bundle over C, modelled on the vector bundle $\mathsf{T}^\circ C$.

Example 3

Let

$$\begin{array}{c} (\mathsf{T}P, \mathrm{d}_T\vartheta) \\ \downarrow \mathsf{T}\pi \\ \mathsf{T}Q \end{array} \qquad \begin{array}{c} (\mathsf{T}P, \mathrm{i}_T\mathrm{d}\vartheta) \\ \downarrow \tau_P \\ P \end{array}$$

be Liouville structures derived from a Liouville structure

$$\begin{array}{c} (P, \vartheta) \\ \downarrow \pi \\ Q \end{array}$$

The pairing

$$\langle\ ,\ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \to \mathbb{R}$$

is a differentiable function defined on the submanifold $P \times_{(\pi,\tau_Q)} \mathsf{T}Q \subset P \times \mathsf{T}Q$ and the diagonal Δ of $\mathsf{T}P \times \mathsf{T}P$ is the graph of the identity symplectomorphism in $(\mathsf{T}P, \mathrm{d}_T\omega)$.

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The diagonal Δ is generated by the function $-\langle\ ,\ \rangle$.