Dirac Algebroids

Janusz Grabowski[*], **Katarzyna Grabowska**[†]

[*]Polish Academy of Sciences

[†] University of Warsaw

March 23, 2011

Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

Introduction

- Dirac structures
- Double vector bundles
- Dirac algebroids
- Examples and applications
- References

Dirac structures

- There is a symmetric pairing on the bundle $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$:

$$(X_1 + \alpha_1 \,|\, X_2 + \alpha_2) = \frac{1}{2}\left(\alpha_1(X_2) + \alpha_2(X_1)\right) .$$

- Courant-Dorfman bracket on the space of $Sec(\mathcal{T}N)$:

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \imath_{X_2}\mathrm{d}\alpha_1 .$$

Definition

An almost Dirac structure on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If additionally the space of sections of D is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

Dirac structures

- There is a symmetric pairing on the bundle $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$:

$$(X_1 + \alpha_1 \,|\, X_2 + \alpha_2) = \frac{1}{2}\left(\alpha_1(X_2) + \alpha_2(X_1)\right) .$$

- Courant-Dorfman bracket on the space of $Sec(\mathcal{T}N)$:

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \imath_{X_2}\mathrm{d}\alpha_1 .$$

Definition

An almost Dirac structure on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If additionally the space of sections of D is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

Dirac structures

- There is a symmetric pairing on the bundle $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^* N$:

$$(X_1 + \alpha_1 \,|\, X_2 + \alpha_2) = \frac{1}{2}\left(\alpha_1(X_2) + \alpha_2(X_1)\right) .$$

- Courant-Dorfman bracket on the space of $Sec(\mathcal{T}N)$:

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \imath_{X_2}\mathrm{d}\alpha_1 .$$

Definition

An almost Dirac structure on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If additionally the space of sections of D is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

Dirac structures

- There is a symmetric pairing on the bundle $\mathcal{T}N = \mathsf{T}N \oplus_N \mathsf{T}^*N$:

$$(X_1 + \alpha_1 \,|\, X_2 + \alpha_2) = \frac{1}{2}\left(\alpha_1(X_2) + \alpha_2(X_1)\right) .$$

- Courant-Dorfman bracket on the space of $Sec(\mathcal{T}N)$:

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \imath_{X_2}\mathrm{d}\alpha_1 .$$

Definition

An almost Dirac structure on the smooth manifold N is a subbundle D of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot|\cdot)$. If additionally the space of sections of D is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle D may be supported on a submanifold $N_0 \subset N$.

- The *first integrability condition* for the almost Dirac structure says that

$$pr_{\mathsf{T}N}(D) \subset \mathsf{T}N_0\,,$$

so the Courant-Dorfman bracket reduces to a well-defined bracket $[\![\cdot,\cdot]\!]_D$ on sections of D.

- The *second integrability* condition says that $[\![\cdot,\cdot]\!]_D$ takes values in $Sec(D)$:

$$[\![\cdot,\cdot]\!]_D : Sec(D) \times Sec(D) \to Sec(D) \subset Sec((\mathcal{T}N)_{|N_0})\,.$$

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions.

- The *first integrability condition* for the almost Dirac structure says that

$$pr_{\mathsf{T}N}(D) \subset \mathsf{T}N_0\,,$$

so the Courant-Dorfman bracket reduces to a well-defined bracket $[\![\cdot,\cdot]\!]_D$ on sections of D.

- The *second integrability* condition says that $[\![\cdot,\cdot]\!]_D$ takes values in $Sec(D)$:

$$[\![\cdot,\cdot]\!]_D : Sec(D) \times Sec(D) \to Sec(D) \subset Sec((\mathcal{T}N)_{|N_0})\,.$$

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions.

- The *first integrability condition* for the almost Dirac structure says that

$$pr_{\mathsf{T}N}(D) \subset \mathsf{T}N_0\,,$$

so the Courant-Dorfman bracket reduces to a well-defined bracket $[\![\cdot,\cdot]\!]_D$ on sections of D.

- The *second integrability* condition says that $[\![\cdot,\cdot]\!]_D$ takes values in $Sec(D)$:

$$[\![\cdot,\cdot]\!]_D : Sec(D) \times Sec(D) \to Sec(D) \subset Sec((\mathcal{T}N)_{|N_0})\,.$$

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions.

- For $\Pi \in Sec(\bigwedge^2 \mathsf{T}N)$, $\widetilde{\Pi} : \mathsf{T}^*N \ni \alpha \longmapsto \imath_\alpha \Pi \in \mathsf{T}N$,

$$graph(\widetilde{\Pi}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

  If Π is a Poisson tensor, then it is a Dirac structure.

- For $\omega \in Sec(\bigwedge^2 \mathsf{T}^*N)$, $\widetilde{\omega} : \mathsf{T}N \ni X \longmapsto \imath_X \omega \in \mathsf{T}^*N$,

$$graph(\widetilde{\omega}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

  If ω is a closed 2-form, then it is a Dirac structure.

- For a distribution Δ on N,

$$\Delta \oplus \Delta^\circ \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

  If Δ is integrable, then it is a Dirac structure.

- For $\Pi \in Sec(\bigwedge^2 \mathsf{T}N)$, $\widetilde{\Pi} : \mathsf{T}^*N \ni \alpha \longmapsto \imath_\alpha \Pi \in \mathsf{T}N$,

$$graph(\widetilde{\Pi}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}\,.$$

  If Π is a Poisson tensor, then it is a Dirac structure.

- For $\omega \in Sec(\bigwedge^2 \mathsf{T}^*N)$, $\widetilde{\omega} : \mathsf{T}N \ni X \longmapsto \imath_X \omega \in \mathsf{T}^*N$,

$$graph(\widetilde{\omega}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}\,.$$

  If ω is a closed 2-form, then it is a Dirac structure.

- For a distribution Δ on N,

$$\Delta \oplus \Delta^\circ \subset \mathcal{T}N \quad \text{is an almost Dirac structure}\,.$$

  If Δ is integrable, then it is a Dirac structure.

- For $\Pi \in Sec(\bigwedge^2 \mathsf{T}N)$, $\widetilde{\Pi} : \mathsf{T}^*N \ni \alpha \longmapsto \imath_\alpha \Pi \in \mathsf{T}N$,

$$graph(\widetilde{\Pi}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

If Π is a Poisson tensor, then it is a Dirac structure.

- For $\omega \in Sec(\bigwedge^2 \mathsf{T}^*N)$, $\widetilde{\omega} : \mathsf{T}N \ni X \longmapsto \imath_X \omega \in \mathsf{T}^*N$,

$$graph(\widetilde{\omega}) \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

If ω is a closed 2-form, then it is a Dirac structure.

- For a distribution Δ on N,

$$\Delta \oplus \Delta^\circ \subset \mathcal{T}N \quad \text{is an almost Dirac structure}.$$

If Δ is integrable, then it is a Dirac structure.

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.

- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \tau_1(k) = 0, \tau_2(k) = 0\},$$

τ_0 is a v.b.

- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

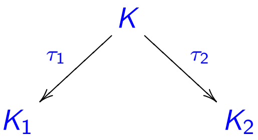
modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.



- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \tau_1(k) = 0, \tau_2(k) = 0\},$$

τ_0 is a v.b.

- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

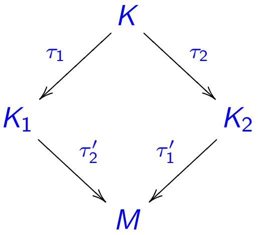
modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.



- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \ \tau_1(k) = 0, \tau_2(k) = 0\},$$

τ_0 is a v.b.
- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

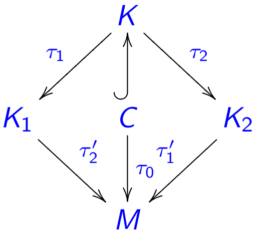
modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.



- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

  $$C = \{k \in K : \ \tau_1(k) = 0, \tau_2(k) = 0\},$$

  τ_0 is a v.b.
- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

  $$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

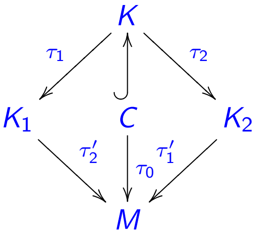
  modeled on the pull-back of the core

  $$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.



- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \tau_1(k) = 0, \tau_2(k) = 0\},$$

  τ_0 is a v.b.
- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

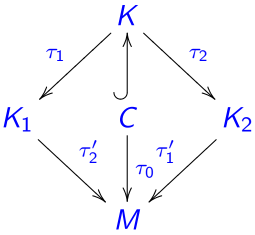
  modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

Double vector bundles

Definition

A double vector bundle is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.



- τ_1, τ_2, τ_1', τ_2' are v.b.
- The core

$$C = \{k \in K : \ \tau_1(k) = 0, \tau_2(k) = 0\},$$

  τ_0 is a v.b.
- $(\tau_1.\tau_1')$, $(\tau_2.\tau_2')$ are v.b. morphisms
- There is one more (affine) bundle

$$\tau_1 \times \tau_2 : K \longrightarrow K_1 \times_M K_2$$

  modeled on the pull-back of the core

$$K_1 \times_M K_2 \times_M C \longrightarrow K_1 \times_M K_2$$

First example is usually $\mathrm{T}E$...

$$\tau : E \longrightarrow M$$
$$(x^a, y^i) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M$$
$$(x^a, \xi_i) \longmapsto (x^a)$$

$$\tau_M : \mathrm{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi_M : \mathrm{T}^*M \longrightarrow M$$
$$(x^a, p_b) \longmapsto (x^a)$$

$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M$$
$$(x^a, y^i) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M$$
$$(x^a, \xi_i) \longmapsto (x^a)$$

$$\tau_M : \mathsf{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi_M : \mathsf{T}^*M \longrightarrow M$$
$$(x^a, p_b) \longmapsto (x^a)$$

$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M$$
$$(x^a, y^i) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M$$
$$(x^a, \xi_i) \longmapsto (x^a)$$

$$\tau_M : \mathsf{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi_M : \mathsf{T}^*M \longrightarrow M$$
$$(x^a, p_b) \longmapsto (x^a)$$

$$\tau_{E^*} : \mathsf{T}E^* \longrightarrow E^*$$
$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \longmapsto (x^a, \xi_i)$$

$$\nabla_1 = \dot{x}^a \partial_{x^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\mathsf{T}\pi_M : \mathsf{T}E^* \longrightarrow \mathsf{T}M$$
$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \longmapsto (x^a, \dot{x}^b)$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M$$
$$(x^a, y^i) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M$$
$$(x^a, \xi_i) \longmapsto (x^a)$$

$$\tau_M : \mathsf{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi_M : \mathsf{T}^*M \longrightarrow M$$
$$(x^a, p_b) \longmapsto (x^a)$$

$$\tau_{E^*} : \mathsf{T}E^* \longrightarrow E^*$$
$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \longmapsto (x^a, \xi_i)$$

$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\mathsf{T}\pi_M : \mathsf{T}E^* \longrightarrow \mathsf{T}M$$
$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \longmapsto (x^a, \dot{x}^b)$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

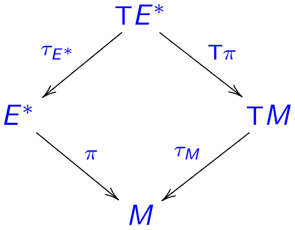
First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M$$
$$(x^a, y^i) \longmapsto (x^a)$$

$$\tau_M : \mathsf{T}M \longrightarrow M$$
$$(x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M$$
$$(x^a, \xi_i) \longmapsto (x^a)$$

$$\pi_M : \mathsf{T}^*M \longrightarrow M$$
$$(x^a, p_b) \longmapsto (x^a)$$



$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

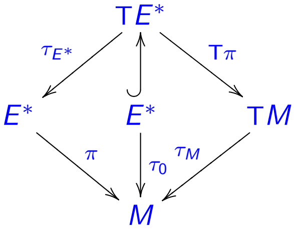
First example: $\mathsf{T}E^*$.

$$\tau : E \longrightarrow M \qquad (x^a, y^i) \longmapsto (x^a)$$

$$\pi : E^* \longrightarrow M \qquad (x^a, \xi_i) \longmapsto (x^a)$$

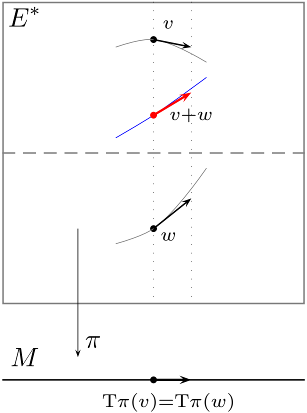
$$\tau_M : \mathsf{T}M \longrightarrow M \qquad (x^a, \dot{x}^b) \longmapsto (x^a)$$

$$\pi_M : \mathsf{T}^*M \longrightarrow M \qquad (x^a, p_b) \longmapsto (x^a)$$



$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}$$

$$\nabla_2 = \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$



- We can add vectors v, w such that

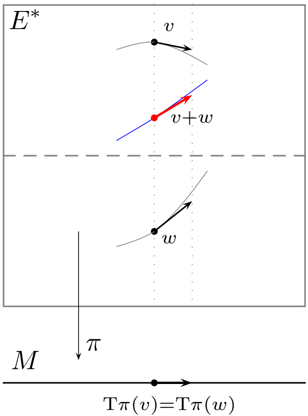
$$\mathsf{T}\pi(v) = \mathsf{T}\pi(w)\,.$$

- For v, w take curves γ_v, γ_w:

$$\pi \circ \gamma_v = \pi \circ \gamma_w\,.$$

- $v \dot{+} w$ is tangent to

$$t \mapsto \gamma_v(t) + \gamma_w(t)\,.$$



- We can add vectors v, w such that

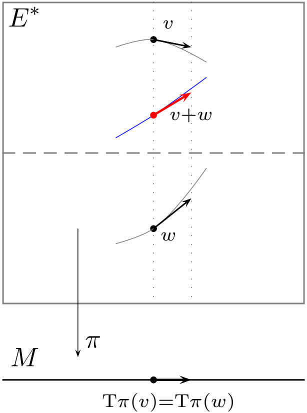
$$\mathsf{T}\pi(v) = \mathsf{T}\pi(w)\,.$$

- For v, w take curves γ_v, γ_w:

$$\pi \circ \gamma_v = \pi \circ \gamma_w\,.$$

- $v \dot{+} w$ is tangent to

$$t \mapsto \gamma_v(t) + \gamma_w(t)\,.$$



- We can add vectors v, w such that

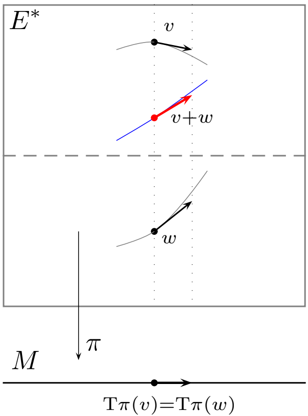
$$\mathsf{T}\pi(v) = \mathsf{T}\pi(w)\,.$$

- For v, w take curves γ_v, γ_w:

$$\pi \circ \gamma_v = \pi \circ \gamma_w\,.$$

- $v \dot{+} w$ is tangent to

$$t \mapsto \gamma_v(t) + \gamma_w(t)\,.$$



- We can add vectors v, w such that

$$\mathrm{T}\pi(v) = \mathrm{T}\pi(w)\,.$$

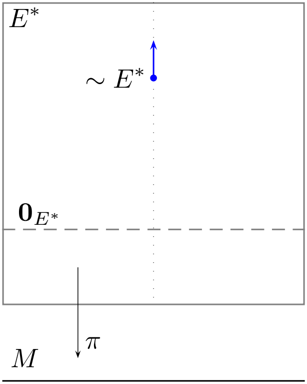
- For v, w take curves γ_v, γ_w:

$$\pi \circ \gamma_v = \pi \circ \gamma_w\,.$$

- $v \dot{+} w$ is tangent to

$$t \mapsto \gamma_v(t) + \gamma_w(t)\,.$$

Second example: T^*E^*.



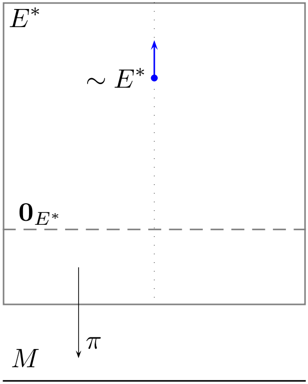
$$\pi_{E^*} : \mathsf{T}^*E^* \longrightarrow E^*$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, \xi_i)$$

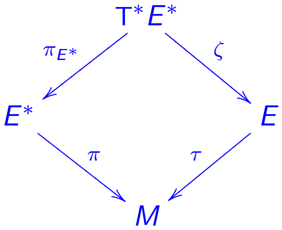
$$\zeta : \mathsf{T}^*E^* \longrightarrow E$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, y^j)$$

$$\nabla_1 = p_a \partial_{p_a} + y^i \partial_{y^i}$$

$$\nabla_2 = p_a \partial_{p_a} + \xi_i \partial_{\xi_i}$$

Second example: T^*E^*.

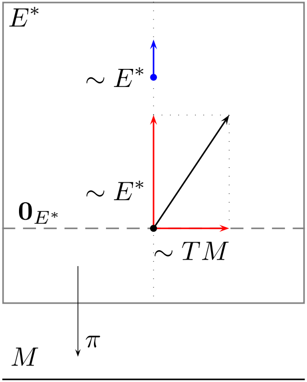


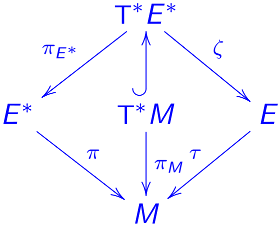


$$\nabla_1 = p_a\partial_{p_a} + y^i\partial_{y^i}$$

$$\nabla_2 = p_a\partial_{p_a} + \xi_i\partial_{\xi_i}$$

Second example: T^*E^*.



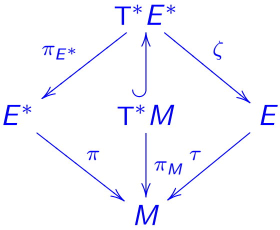


$$\mathsf{T}_0E^* \simeq E^*_x \oplus \mathsf{T}_xM$$
$$\mathsf{T}^*_0E^* \simeq E_x \oplus \mathsf{T}^*_xM$$

$$\nabla_1 = p_a\partial_{p_a} + y^i\partial_{y^i}$$

$$\nabla_2 = p_a\partial_{p_a} + \xi_i\partial_{\xi_i}$$

Second example: T^*E^*.



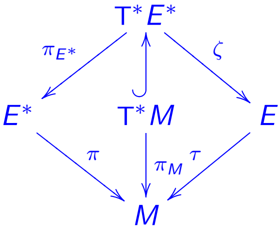
(x^a, ξ_i, p_b, y^j) $\qquad$ (x^a, y^i, p_b, ξ_j)

T^*E^* is isomorphic to T^*E. The graph of the canonical isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

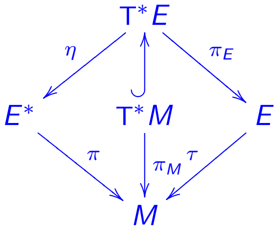
$$\mathrm{T}^*(E^* \times E) \simeq \mathrm{T}^*E^* \times \mathrm{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$

$$\mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j)$$

Second example: T^*E^*.



$$(x^a, \xi_i, p_b, y^j)$$



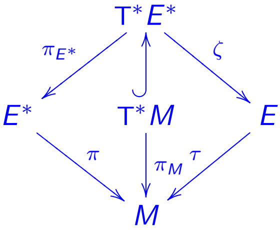
$$(x^a, y^i, p_b, \xi_j)$$

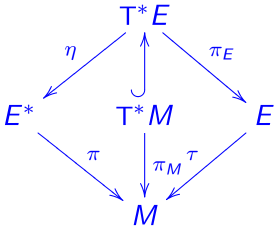
T^*E^* is isomorphic to T^*E. The graph of the canonical isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$\mathrm{T}^*(E^* \times E) \simeq \mathrm{T}^*E^* \times \mathrm{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$

$$\mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j)$$

Second example: T^*E^*.





(x^a, ξ_i, p_b, y^j) $\qquad$ (x^a, y^i, p_b, ξ_j)

T^*E^* is isomorphic to T^*E. The graph of the canonical isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$\mathsf{T}^*(E^* \times E) \simeq \mathsf{T}^*E^* \times \mathsf{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$

$$\mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j)$$

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure τ.

- Euler vector field is tangent to F;
- F can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle K is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures τ_1 and τ_2.

- Both Euler vector fields are tangent to D;
- D defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$.

Dirac algebroids

Linearity of different geometrical structures is connected with double vector bundles.

- A connection Γ on a vector bundle $F \to M$ is *linear* if the map

$$\widetilde{\Gamma} : \mathsf{T}F \longrightarrow VF \oplus_F (F \times_M \mathsf{T}M) = F \times_M F \times_M \mathsf{T}M$$

  is a double vector bundle morphism.

- A Poisson tensor Π on a vector bundle F is linear if the corresponding map

$$\widetilde{\Pi} : \mathsf{T}^*F \longrightarrow \mathsf{T}F$$

  is a double vector bundle morphism.

- ...

Dirac algebroids

Linearity of different geometrical structures is connected with double vector bundles.

- A connection Γ on a vector bundle $F \to M$ is *linear* if the map

$$\widetilde{\Gamma} : \mathsf{T}F \longrightarrow VF \oplus_F (F \times_M \mathsf{T}M) = F \times_M F \times_M \mathsf{T}M$$

  is a double vector bundle morphism.

- A Poisson tensor Π on a vector bundle F is linear if the corresponding map

$$\widetilde{\Pi} : \mathsf{T}^*F \longrightarrow \mathsf{T}F$$

  is a double vector bundle morphism.

- ...

Dirac algebroids

Linearity of different geometrical structures is connected with double vector bundles.

- A connection Γ on a vector bundle $F \to M$ is *linear* if the map

$$\widetilde{\Gamma} : \mathsf{T}F \longrightarrow VF \oplus_F (F \times_M \mathsf{T}M) = F \times_M F \times_M \mathsf{T}M$$

  is a double vector bundle morphism.

- A Poisson tensor Π on a vector bundle F is linear if the corresponding map

$$\widetilde{\Pi} : \mathsf{T}^*F \longrightarrow \mathsf{T}F$$

  is a double vector bundle morphism.

- ...

Dirac algebroids

Linearity of different geometrical structures is connected with double vector bundles.

- A connection Γ on a vector bundle $F \to M$ is *linear* if the map

$$\widetilde{\Gamma} : \mathsf{T}F \longrightarrow VF \oplus_F (F \times_M \mathsf{T}M) = F \times_M F \times_M \mathsf{T}M$$

  is a double vector bundle morphism.

- A Poisson tensor Π on a vector bundle F is linear if the corresponding map

$$\widetilde{\Pi} : \mathsf{T}^*F \longrightarrow \mathsf{T}F$$

  is a double vector bundle morphism.

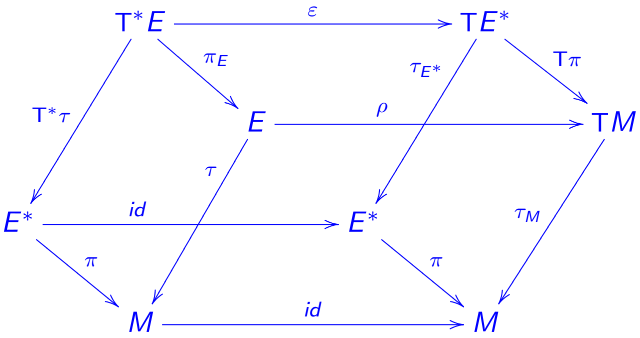
- ...

- A general algebroid is a double vector bundle morphism covering the identity on E^*:

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x) y^k, c^k_{ij}(x) y^i \xi_k + \sigma^a_j(x) p_a)$$

$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}.$$

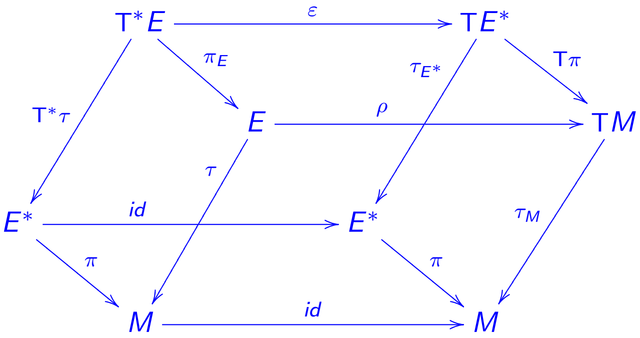
- A general algebroid is a double vector bundle morphism covering the identity on E^*:



$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x) y^k, c^k_{ij}(x) y^i \xi_k + \sigma^a_j(x) p_a)$$

$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}.$$

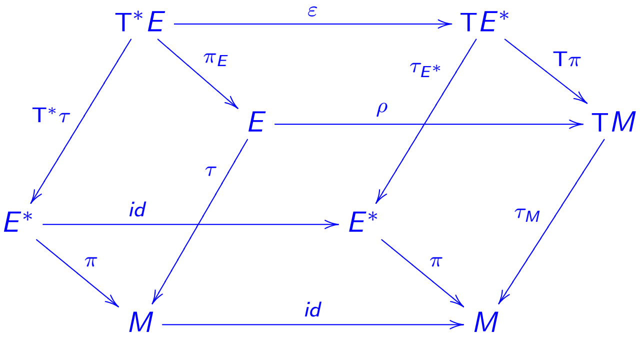
- A general algebroid is a double vector bundle morphism covering the identity on E^*:



$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x) y^k, c^k_{ij}(x) y^i \xi_k + \sigma^a_j(x) p_a)$$

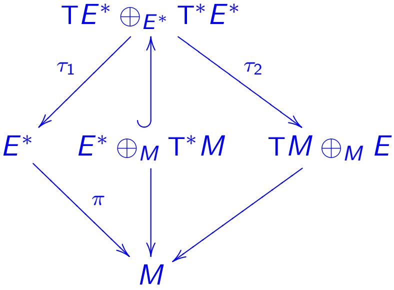
$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}.$$

- A general algebroid is a double vector bundle morphism covering the identity on E^*:



$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x)y^k, c^k_{ij}(x)y^i\xi_k + \sigma^a_j(x)p_a)$$

$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^a_j(x)\partial_{x^a} \otimes \partial_{\xi_j}.$$

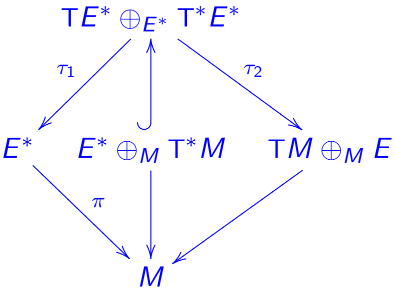


$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b}\,,$$
$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.



$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$

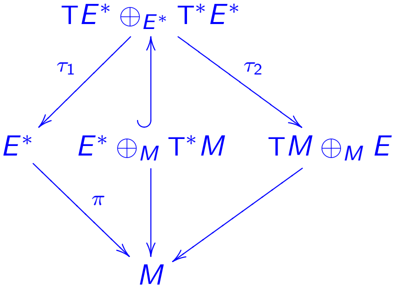
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b}\,,$$

$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.



$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$

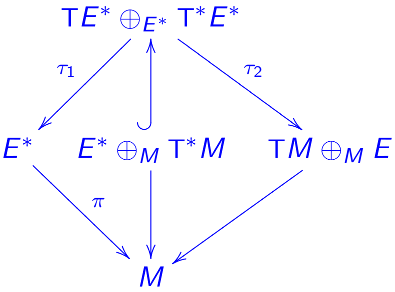
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b},$$

$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.



$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$

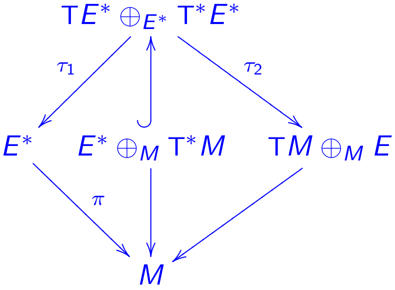
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b},$$

$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathsf{T}M \oplus_M E$.

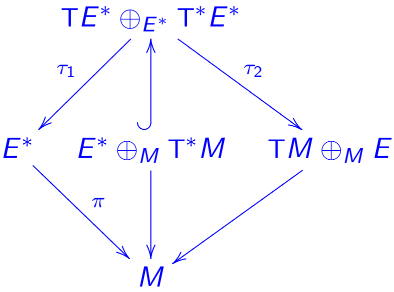


$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b} ,$$
$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to TM \oplus_M E$.

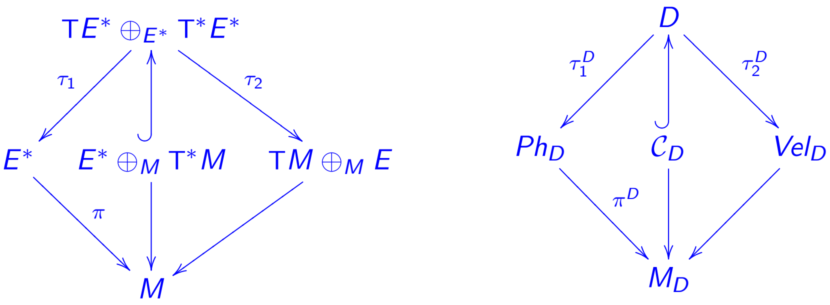


$$\tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i),$$
$$\tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k),$$

$$\nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b}\,,$$
$$\nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j}$$

Definition

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E is an almost Dirac (resp., Dirac) subbundle D of $\mathcal{T}E^*$ being a double vector subbundle, i.e., D is not only a subbundle of $\tau_1 : \mathcal{T}E^* \to E^*$ but also a vector subbundle of the vector bundle $\tau_2 : \mathcal{T}E^* \to \mathrm{T}M \oplus_M E$.



- $Ph_D = \tau_1(D)$ - phase bundle
- $Vel_D = \tau_2(D)$ - velocity bundle (anchor relation)
- $\mathcal{C}_D \subset E^* \oplus_M \mathsf{T}^*M$ - core bundle for D

- The graph of any linear bivector field is a Dirac algebroid,

$$\Pi = \frac{1}{2}c^k_{ij}(x)\xi_k\partial_{\xi_i} \wedge \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \wedge \partial_{x^b}, \qquad c^k_{ij}(x) = -c^k_{ji}(x)\,,$$

$$D_\Pi = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : \dot{x}^b = \rho^b_k(x)y^k,\ \dot{\xi}_j = c^k_{ij}(x)y^i\xi_k - \rho^a_j(x)p_a\}\,.$$

  The phase bundle is E^*, the velocity bundle is the graph of $\rho : E \to \mathsf{T}M$, the core bundle is the graph of $-\rho^*$.

- The graph of any linear two-form is a Dirac algebroid,

$$\omega = \frac{1}{2}c^k_{ab}(x)\xi_k \mathrm{d}x^a \wedge \mathrm{d}x^b + \rho^i_b(x)\mathrm{d}\xi_i \wedge \mathrm{d}x^b, \qquad c^k_{ab}(x) = -c^k_{ba}(x)\,,$$

$$D_\omega = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : y^i = \rho^i_a(x)\dot{x}^a,\ p_a = c^k_{ab}(x)\xi_k\dot{x}^b - \rho^i_a(x)\dot{\xi}_i\}.$$

  The phase bundle is E^*, the velocity bundle is the graph of $\rho : \mathsf{T}M \to E$, the core bundle is the graph of $-\rho^*$.

- The graph of any linear bivector field is a Dirac algebroid,

$$\Pi = \frac{1}{2}c^k_{ij}(x)\xi_k\partial_{\xi_i}\wedge\partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i}\wedge\partial_{x^b}, \qquad c^k_{ij}(x) = -c^k_{ji}(x)\,,$$

$$D_\Pi = \{(x^a,\xi_i,\dot{x}^b,\dot{\xi}_j,p_c,y^k) : \dot{x}^b = \rho^b_k(x)y^k,\ \dot{\xi}_j = c^k_{ij}(x)y^i\xi_k - \rho^a_j(x)p_a\}\,.$$

The phase bundle is E^*, the velocity bundle is the graph of $\rho : E \to \mathsf{T}M$, the core bundle is the graph of $-\rho^*$.

- The graph of any linear two-form is a Dirac algebroid,

$$\omega = \frac{1}{2}c^k_{ab}(x)\xi_k \mathrm{d}x^a\wedge\mathrm{d}x^b + \rho^i_b(x)\mathrm{d}\xi_i\wedge\mathrm{d}x^b, \qquad c^k_{ab}(x) = -c^k_{ba}(x)\,,$$

$$D_\omega = \{(x^a,\xi_i,\dot{x}^b,\dot{\xi}_j,p_c,y^k) : y^i = \rho^i_a(x)\dot{x}^a,\ p_a = c^k_{ab}(x)\xi_k\dot{x}^b - \rho^i_a(x)\dot{\xi}_i\}.$$

The phase bundle is E^*, the velocity bundle is the graph of $\rho : \mathsf{T}M \to E$, the core bundle is the graph of $-\rho^*$.

Due to isotropy condition for D we have the following.

Theorem

The core bundle of a Dirac algebroid D is the annihilator subbundle $Vel_D^0 \subset \mathsf{T}^*M \oplus_M E^*$ of the anchor relation Vel_D:

$$\mathcal{C}_D = {Vel_D}^0 .$$

For Dirac-Lie algebroids we have the following.

Theorem

If D is a Dirac-Lie algebroid, then it induces a Lie algebroid structure on the bundle Vel_D.

Due to isotropy condition for D we have the following.

Theorem

The core bundle of a Dirac algebroid D is the annihilator subbundle $Vel_D^0 \subset \mathsf{T}^*M \oplus_M E^*$ of the anchor relation Vel_D:

$$\mathcal{C}_D = {Vel_D}^0 \,.$$

For Dirac-Lie algebroids we have the following.

Theorem

If D is a Dirac-Lie algebroid, then it induces a Lie algebroid structure on the bundle Vel_D.

$$
\begin{array}{ccccc}
 & & \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* & & \\
 & \swarrow^{\tau_1} & \uparrow & \searrow^{\tau_2} & \\
E^* & & E^* \oplus_M \mathsf{T}^*M & & \mathsf{T}M \oplus_M E \\
 & \searrow^{\pi} & \downarrow & \swarrow & \\
 & & M & &
\end{array}
$$

$$
\begin{array}{c}
\mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* \\
\downarrow \\
E^* \times_M (E \oplus_M \mathsf{T}M)
\end{array}
$$

$$
\begin{array}{c}
E^* \times_M (E \oplus_M \mathsf{T}M) \times_M (\mathsf{T}^*M \oplus_M E^*) \\
\downarrow \\
E^* \times_M (E \oplus_M \mathsf{T}M)
\end{array}
$$

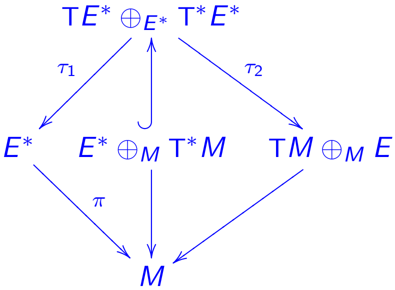
$$
\begin{array}{c}
(x, \xi, \dot{x}, \dot{\xi}, p, y) \\
\downarrow \\
(x, \xi, \dot{x}, y)
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* & & \\
 & \swarrow^{\tau_1} & \uparrow & \searrow^{\tau_2} & \\
E^* & & E^* \oplus_M \mathsf{T}^*M & & \mathsf{T}M \oplus_M E \\
 & \searrow^{\pi} & \downarrow & \swarrow & \\
 & & M & &
\end{array}
$$

$$
\begin{array}{c}
\mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* \\
\downarrow \\
E^* \times_M (E \oplus_M \mathsf{T}M)
\end{array}
$$

$$
\begin{array}{c}
E^* \times_M (E \oplus_M \mathsf{T}M) \times_M (\mathsf{T}^*M \oplus_M E^*) \\
\downarrow \\
E^* \times_M (E \oplus_M \mathsf{T}M)
\end{array}
$$

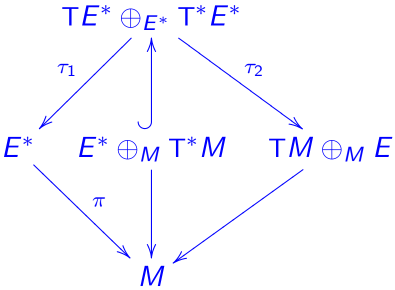
$$
\begin{array}{c}
(x, \xi, \dot{x}, \dot{\xi}, p, y) \\
\downarrow \\
(x, \xi, \dot{x}, y)
\end{array}
$$



$$\begin{array}{c} TE^* \oplus_{E^*} T^*E^* \\ \big\downarrow \\ E^* \times_M (E \oplus_M TM) \end{array}$$

$$\begin{array}{c} E^* \times_M (E \oplus_M TM) \times_M (T^*M \oplus_M E^*) \\ \big\downarrow \\ E^* \times_M (E \oplus_M TM) \end{array}$$

$$\begin{array}{c} (x, \xi, \dot{x}, \dot{\xi}, p, y) \\ \big\downarrow \\ (x, \xi, \dot{x}, y) \end{array}$$



$$\begin{array}{c} \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* \\ \downarrow \\ E^* \times_M (E \oplus_M \mathsf{T}M) \end{array}$$

$$\begin{array}{c} E^* \times_M (E \oplus_M \mathsf{T}M) \times_M (\mathsf{T}^*M \oplus_M E^*) \\ \downarrow \\ E^* \times_M (E \oplus_M \mathsf{T}M) \end{array}$$

$$\begin{array}{c} (x, \xi, \dot{x}, \dot{\xi}, p, y) \\ \downarrow \\ (x, \xi, \dot{x}, y) \end{array}$$

$$E^* \times_M (E \oplus_M \mathsf{T}M) \times_M (\mathsf{T}^*M \oplus_M E^*)$$

$$\downarrow$$

$$E^* \times_M (E \oplus_M \mathsf{T}M)$$

$$(x, \xi, \dot{x}, \dot{\xi}, p, y) \quad \longleftrightarrow \quad (x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$$

- Coordinates in $E \oplus_M \mathsf{T}M = Vel_D \oplus V$,
$$(x, \eta, \widehat{\eta}): \quad Vel_D = \{\widehat{\eta} = 0\}.$$
- Dual oordinates in $\mathsf{T}^*M \oplus_M E = V^0 \oplus {Vel_D}^0$,
$$(x, \zeta, \widehat{\zeta}): \quad {Vel_D}^0 = \{\zeta = 0\}.$$
- For D we have now
$$(x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta}): \quad \widehat{\eta} = 0, \quad \widehat{\zeta} \quad \text{arbitrary}, \quad \dots$$

$$E^* \times_M (E \oplus_M \mathsf{T}M) \times_M (\mathsf{T}^*M \oplus_M E^*)$$

$$\downarrow$$

$$E^* \times_M (E \oplus_M \mathsf{T}M)$$

$$(x, \xi, \dot{x}, \dot{\xi}, p, y) \quad \longleftrightarrow \quad (x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$$

- Coordinates in $E \oplus_M \mathsf{T}M = Vel_D \oplus V$,

$$(x, \eta, \widehat{\eta}) : \quad Vel_D = \{\widehat{\eta} = 0\}\,.$$

- Dual oordinates in $\mathsf{T}^*M \oplus_M E = V^0 \oplus {Vel_D}^0$,

$$(x, \zeta, \widehat{\zeta}) : \quad {Vel_D}^0 = \{\zeta = 0\}\,.$$

- For D we have now

$$(x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta}) : \quad \widehat{\eta} = 0, \quad \widehat{\zeta} \ \text{ arbitrary}, \quad \ldots$$

$$E^* \times_M (E \oplus_M \mathrm{T}M) \times_M (\mathrm{T}^*M \oplus_M E^*)$$

$$\downarrow$$

$$E^* \times_M (E \oplus_M \mathrm{T}M)$$

$$(x, \xi, \dot{x}, \dot{\xi}, p, y) \quad \longleftrightarrow \quad (x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$$

- Coordinates in $E \oplus_M \mathrm{T}M = Vel_D \oplus V$,

$$(x, \eta, \widehat{\eta}): \quad Vel_D = \{\widehat{\eta} = 0\}.$$

- Dual oordinates in $\mathrm{T}^*M \oplus_M E = V^0 \oplus {Vel_D}^0$,

$$(x, \zeta, \widehat{\zeta}): \quad {Vel_D}^0 = \{\zeta = 0\}.$$

- For D we have now

$$(x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta}): \quad \widehat{\eta} = 0, \quad \widehat{\zeta} \quad \text{arbitrary}, \quad \ldots$$

$$E^* \times_M (E \oplus_M \mathrm{T}M) \times_M (\mathrm{T}^*M \oplus_M E^*)$$

$$\downarrow$$

$$E^* \times_M (E \oplus_M \mathrm{T}M)$$

$$(x, \xi, \dot{x}, \dot{\xi}, p, y) \quad \longleftrightarrow \quad (x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$$

- Coordinates in $E \oplus_M \mathrm{T}M = Vel_D \oplus V$,

$$(x, \eta, \widehat{\eta}) : \quad Vel_D = \{\widehat{\eta} = 0\}.$$

- Dual oordinates in $\mathrm{T}^*M \oplus_M E = V^0 \oplus {Vel_D}^0$,

$$(x, \zeta, \widehat{\zeta}) : \quad {Vel_D}^0 = \{\zeta = 0\}.$$

- For D we have now

$$(x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta}) : \quad \widehat{\eta} = 0, \quad \widehat{\zeta} \quad \text{arbitrary}, \quad \ldots$$

$$E^* \times_M (E \oplus_M \mathrm{T}M) \times_M (\mathrm{T}^*M \oplus_M E^*)$$

$$\downarrow$$

$$E^* \times_M (E \oplus_M \mathrm{T}M)$$

$$(x, \xi, \dot{x}, \dot{\xi}, p, y) \quad \longleftrightarrow \quad (x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta})$$

- Coordinates in $E \oplus_M \mathrm{T}M = Vel_D \oplus V$,

$$(x, \eta, \widehat{\eta}) : \quad Vel_D = \{\widehat{\eta} = 0\}\,.$$

- Dual oordinates in $\mathrm{T}^*M \oplus_M E = V^0 \oplus {Vel_D}^0$,

$$(x, \zeta, \widehat{\zeta}) : \quad {Vel_D}^0 = \{\zeta = 0\}\,.$$

- For D we have now

$$(x, \xi, \eta, \widehat{\eta}, \zeta, \widehat{\zeta}) : \quad \widehat{\eta} = 0, \quad \widehat{\zeta} \quad \text{arbitrary}, \quad \ldots$$

- For D we have now

$$(x,\xi,\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \quad \widehat{\eta}=0, \quad \widehat{\zeta} \text{ arbitrary}, \quad \dots$$

- More conditions

$$\zeta_k = c^i_{jk}(x)\eta^j\xi_i, \quad \text{isotropy gives} \quad c^i_{jk}(x) = -c^i_{kj}(x)\,.$$

- If $Ph_D \subsetneq E^*$

$$(x,\widehat{x},\xi,\widehat{\xi}): \quad M_D = \{\widehat{x}=0\},\ Ph_D = \{\widehat{x}=0,\widehat{\xi}=0\}\,.$$

Theorem

In the introduced coordinates we have

$$D = \{(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta}):\ \widehat{x}=0,\ \widehat{\xi}=0,\ \widehat{\eta}=0,\ \zeta_k = c^i_{jk}(x)\eta^j\xi_i\}\,.$$

- For D we have now

$$(x,\xi,\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \quad \widehat{\eta}=0, \quad \widehat{\zeta} \quad \text{arbitrary}, \quad \ldots$$

- More conditions

$$\zeta_k = c^i_{jk}(x)\eta^j\xi_i, \quad \text{isotropy gives} \quad c^i_{jk}(x) = -c^i_{kj}(x)\,.$$

- If $Ph_D \subsetneq E^*$

$$(x,\widehat{x},\xi,\widehat{\xi}): \quad M_D = \{\widehat{x}=0\},\ Ph_D = \{\widehat{x}=0,\widehat{\xi}=0\}\,.$$

Theorem

In the introduced coordinates we have

$$D = \{(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \ \widehat{x}=0,\ \widehat{\xi}=0,\ \widehat{\eta}=0,\ \zeta_k = c^i_{jk}(x)\eta^j\xi_i\}\,.$$

- For D we have now

$$(x,\xi,\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \quad \widehat{\eta}=0, \quad \widehat{\zeta} \text{ arbitrary}, \quad \dots$$

- More conditions

$$\zeta_k = c^i_{jk}(x)\eta^j\xi_i, \quad \text{isotropy gives} \quad c^i_{jk}(x) = -c^i_{kj}(x)\,.$$

- If $Ph_D \subsetneq E^*$

$$(x,\widehat{x},\xi,\widehat{\xi}): \quad M_D = \{\widehat{x}=0\},\ Ph_D = \{\widehat{x}=0,\widehat{\xi}=0\}\,.$$

Theorem

In the introduced coordinates we have

$$D = \{(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \ \widehat{x}=0,\ \widehat{\xi}=0,\ \widehat{\eta}=0,\ \zeta_k = c^i_{jk}(x)\eta^j\xi_i\}\,.$$

- For D we have now

$$(x,\xi,\eta,\widehat{\eta},\zeta,\widehat{\zeta}): \quad \widehat{\eta}=0, \quad \widehat{\zeta} \ \text{ arbitrary}, \quad \dots$$

- More conditions

$$\zeta_k = c^i_{jk}(x)\eta^j\xi_i, \quad \text{isotropy gives} \quad c^i_{jk}(x) = -c^i_{kj}(x)\,.$$

- If $Ph_D \subsetneq E^*$

$$(x,\widehat{x},\xi,\widehat{\xi}): \quad M_D=\{\widehat{x}=0\},\ Ph_D=\{\widehat{x}=0,\widehat{\xi}=0\}\,.$$

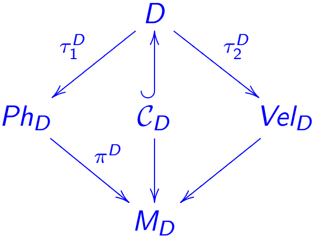
Theorem

In the introduced coordinates we have

$$D=\{(x,\widehat{x},\xi,\widehat{\xi},\eta,\widehat{\eta},\zeta,\widehat{\zeta}):\ \widehat{x}=0,\ \widehat{\xi}=0,\ \widehat{\eta}=0,\ \zeta_k=c^i_{jk}(x)\eta^j\xi_i\}\,.$$

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.



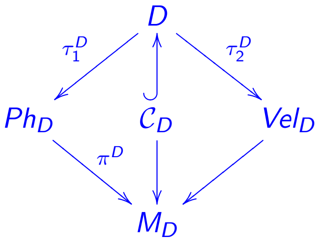
- $V \subset Vel_D \subset E \oplus_M \mathsf{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathsf{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.



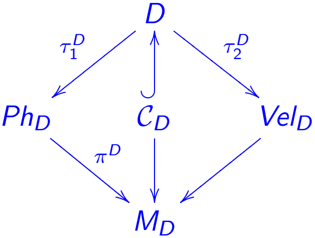
- $V \subset Vel_D \subset E \oplus_M \mathsf{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathsf{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.



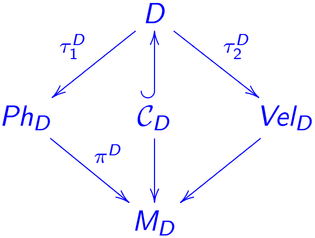
- $V \subset Vel_D \subset E \oplus_M \mathsf{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathsf{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.



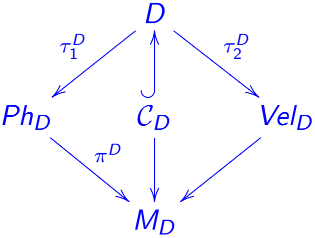
- $V \subset Vel_D \subset E \oplus_M \mathsf{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathsf{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.



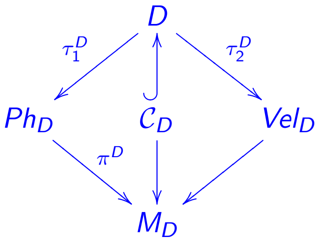
- $V \subset Vel_D \subset E \oplus_M \mathrm{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathrm{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

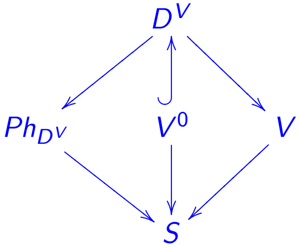
Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Example with application

We start with a Dirac algebroid D and a vector subbundle $V \subset Vel_D$.





- $V \subset Vel_D \subset E \oplus_M \mathsf{T}M$
- $\widetilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset \mathsf{T}^*M \oplus_M E^*$, $V^0 \supset \mathcal{C}_D$
- $D^V = \widetilde{V} + V^0$

Definition

The Dirac algebroid D^V is called induced from D by the subbundle V.

Dirac algebroids in mechanics

How to obtain phase equations from a Lagrangian (or Hamiltonian):

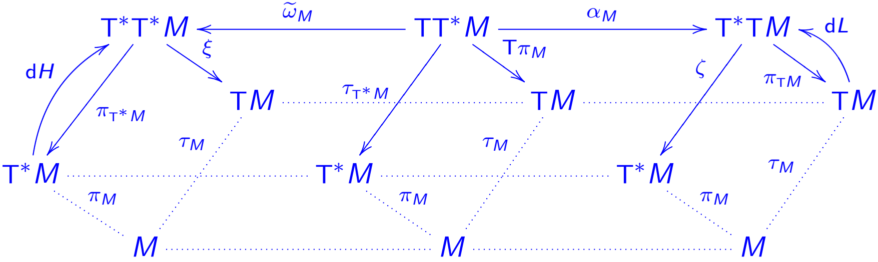
- Bundle of configurations: $\mathsf{T}M$, phase bundle: T^*M.

$$D_L = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)), \qquad D_H = \widetilde{\omega}_M^{-1}(\mathrm{d}H(\mathsf{T}^*M)).$$

Dirac algebroids in mechanics

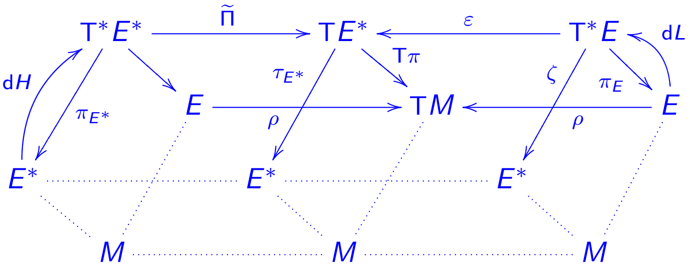
How to obtain phase equations from a Lagrangian (or Hamiltonian):

- Bundle of configurations: $\mathsf{T}M$, phase bundle: T^*M.



$$D_L = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)), \qquad D_H = \widetilde{\omega}_M^{-1}(\mathrm{d}H(\mathsf{T}^*M))\,.$$

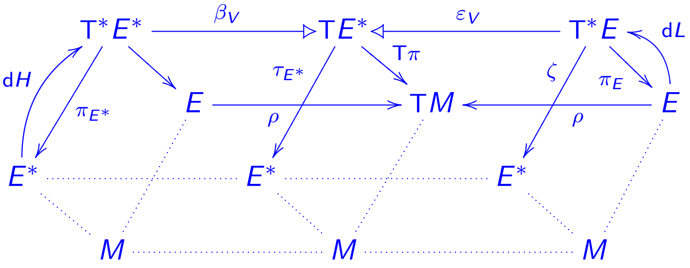
- Bundle of configurations: E (skew-algebroid), phase bundle: E^*.



$$D_L = \varepsilon(\mathrm{d}L(E)), \qquad D_H = \widetilde{\Pi}(\mathrm{d}H(E^*)) .$$

- Bundle of configurations: E (skew-algebroid), constraints $W \subset E$.
  - W defines $V = \{y + v \in E \oplus_M \mathsf{T}M : \; y \in W, \; v = \rho(y)\} \subset Vel_{D_\Pi}$.
  - We induce D_Π^V.
  - D_Π^V gives the relations

$$\varepsilon_V : \mathsf{T}^*E \longrightarrow\!\!\rhd \mathsf{T}E^*, \qquad \beta_V : \mathsf{T}^*E^* \longrightarrow\!\!\rhd \mathsf{T}E^*.$$



$$D_L = \varepsilon_V(\mathrm{d}L(E)), \qquad D_H = \beta_V(\mathrm{d}H(E^*))$$

J. Grabowski, K. Grabowska: "Dirac Algebroids in Lagrangian and Hamiltonian Mechanics" *arXiv: 1101.2497*.

I. Y. Dorfman: Dirac structures of integrable evolution equations. *Phys. Lett. A* **125** (1987), no. 5, 240–246.

T. J. Courant: Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.

J. Pradines: Fibrés vectoriels doubles et calcul des jets non holonomes (French), Notes polycopiées, Amiens, 1974.

K. Konieczna and P. Urbański: Double vector bundles and duality, *Arch. Math. (Brno)* **35**, (1999), 59–95.

J. Grabowski and M. Rotkiewicz: Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285-1305.

H. Yoshimura, J. E. Marsden: "Dirac structures in Lagrangian Mechanics", *J. Geom. Phys.*, **57**, (2006) Part **I** 133–156, Part **II** 209-250.

J. Grabowski, K. Grabowska: "Dirac Algebroids in Lagrangian and Hamiltonian Mechanics" *arXiv: 1101.2497*.

I. Y. Dorfman: Dirac structures of integrable evolution equations. *Phys. Lett. A* **125** (1987), no. 5, 240–246.

T. J. Courant: Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.

J. Pradines: Fibrés vectoriels doubles et calcul des jets non holonomes (French), Notes polycopiées, Amiens, 1974.

K. Konieczna and P. Urbański: Double vector bundles and duality, *Arch. Math. (Brno)* **35**, (1999), 59–95.

J. Grabowski and M. Rotkiewicz: Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285-1305.

H. Yoshimura, J. E. Marsden: "Dirac structures in Lagrangian Mechanics", *J. Geom. Phys.*, **57**, (2006) Part I 133–156, Part II 209-250.

J. Grabowski, K. Grabowska: "Dirac Algebroids in Lagrangian and Hamiltonian Mechanics" *arXiv: 1101.2497*.

I. Y. Dorfman: Dirac structures of integrable evolution equations. *Phys. Lett. A* **125** (1987), no. 5, 240–246.

T. J. Courant: Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.

J. Pradines: Fibrés vectoriels doubles et calcul des jets non holonomes (French), Notes polycopiées, Amiens, 1974.

K. Konieczna and P. Urbański: Double vector bundles and duality, *Arch. Math. (Brno)* **35**, (1999), 59–95.

J. Grabowski and M. Rotkiewicz: Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285-1305.

H. Yoshimura, J. E. Marsden: "Dirac structures in Lagrangian Mechanics", *J. Geom. Phys.*, **57**, (2006) Part I 133–156, Part II 209-250.

- J. Grabowski, K. Grabowska: "Dirac Algebroids in Lagrangian and Hamiltonian Mechanics" *arXiv: 1101.2497*.
- I. Y. Dorfman: Dirac structures of integrable evolution equations. *Phys. Lett. A* **125** (1987), no. 5, 240–246.
- T. J. Courant: Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.
- J. Pradines: Fibrés vectoriels doubles et calcul des jets non holonomes (French), Notes polycopiées, Amiens, 1974.
- K. Konieczna and P. Urbański: Double vector bundles and duality, *Arch. Math. (Brno)* **35**, (1999), 59–95.
- J. Grabowski and M. Rotkiewicz: Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285-1305.
- H. Yoshimura, J. E. Marsden: "Dirac structures in Lagrangian Mechanics", *J. Geom. Phys.*, **57**, (2006) Part I 133–156, Part II 209-250.