CASIMIR FORCES INDUCED BY BOSE-EINSTEIN CONDENSATION

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"Tis much better to do a little with certainty & leave the rest for others that come after, than to explain all things by conjecture without making sure of any thing."

Isaak Newton
H. B. G. Casimir
"On the attraction between two perfectly conducting plates"

\[ -\frac{\hbar c \pi^2}{r^4 240} \]

H. B. G. Casimir, D. Polder
"The influence of retardation on the London-van der Waals forces"
Phys. Rev. 73, 360-372 (1948).

\[ U(r) = -\frac{\alpha_1 \alpha_2 23\hbar c}{r^7} \frac{1}{4\pi} \]
TYPES OF THE CASIMIR EFFECT

- electromagnetic
- in quantum field theory
- in particle physics
- in cosmology
- in critical phenomena
- dynamical Casimir effect
STATING THE PROBLEM:

Derive the Casimir effect in an imperfect (interacting) Bose gas filling the volume contained between two infinite parallel plane walls.

Hamiltonian of the imperfect Bose gas:

\[ H = H_0 + \frac{a}{V} \frac{N^2}{2} \]

- \( H_0 \) = kinetic energy (perfect gas Hamiltonian)
- \( a/V > 0 \) = repulsive mean-field interaction per pair of bosons
- \( V \) = volume occupied by the system.
- \( H \) is superstable!
Bose gas occupies volume \( V = L^2 D \) of a rectangular box with linear dimensions \( L \times L \times D \). \( D \) denotes the distance between two \( L \times L \) square walls. The excess grand canonical free energy per unit wall area is defined by

\[
\omega_s(T, D, \mu) = \lim_{L \to \infty} \left[ \frac{\Omega(T, L, D, \mu)}{L^2} \right] - D \omega_b(T, \mu)
\]

where \( \omega_b(T, \mu) \) denotes the grand canonical potential per unit volume evaluated in the thermodynamic limit. The Casimir force equals

\[
F(T, D, \mu) = -\frac{\partial \omega_s(T, D, \mu)}{\partial D}
\]
BOUNDARY CONDITIONS

One-particle kinetic energy $\epsilon(k) = (k_x^2 + k_y^2 + k_z^2)\hbar^2 / 2m$

z-axis perpendicular to $L \times L$ walls

- **periodic**
  
  $$k_z = \frac{2\pi}{D} n_z, \quad n_z = 0, \pm 1, \pm 2, \ldots$$

- **Dirichlet**
  
  $$k_z = \frac{\pi}{D} n_z, \quad n_z = 1, 2, \ldots$$

- **Neumann**
  
  $$k_z = \frac{\pi}{D} n_z, \quad n_z = 0, 1, 2, \ldots$$

$k_x, k_y$ -periodic b.c.
Grand canonical potential

\[ \Omega(T, L, D, \mu) = -k_B T \ln \Xi(T, L, D, \mu) \]

\( \Xi(T, L, D, \mu) \) is related to the analytic continuation of the perfect gas partition function \( \Xi_0 \) by

\[ \Xi(T, L, D, \mu) = \exp \left[ \frac{\beta L^2 D}{2a} \mu^2 \right] \sqrt{\frac{L^2 D \beta}{2\pi a}} \times (-i) \int_{\alpha-i\infty}^{\alpha+i\infty} dt \ \exp \left[ \frac{\beta L^2 D}{a} \left( \frac{t^2}{2} - t\mu \right) \right] \Xi_0(T, L, D, t) \]

(\( \alpha < 0 \))
The bulk grand canonical free-energy density

\[ \omega_b(T, \mu) = - \lim_{L \to \infty} \frac{1}{L^3} k_B T \ln \Xi(T, L, L, \mu) = -p(T, \mu) \]

can be calculated with the use of the steepest descent method.

If \( \mu < \mu_c = a n_{0,c} \)

\[ p(T, \mu) = \frac{1}{2} a n^2(T, \mu) + p_0(T, \mu - a n(T, \mu)) \]

where \( n(T, \mu) \) is the unique solution of the equation

\[ n = n_0(T, \mu - a n) \]
If $\mu > \mu_c = an_{0,c}$

$$p(T, \mu) = \frac{\mu^2}{2a} + p_0(T, 0)$$

In the two-phase region

$$n = \frac{\mu}{a}$$

and the density of condensate is equal to

$$\left( \frac{\mu}{a} - n_{0,c} \right)$$
IMPERFECT BOSE GAS: CONDENSATION

\[ n = \lambda^3 \rho \]

\[ m \geq \zeta(3/2) \]

\[ n = m \]

\[ m = \beta \mu \]

\[ \beta a / \lambda^3 = 1 \]
The steepest descent method yields the asymptotic form of the excess free energy density. The Casimir force in the one-phase region near the condensation point equals

$$F(T, D, \mu) = \frac{1}{k_B T} \left[ 2 \Psi(x) - x \Psi'(x) \right]$$

with

$$\Psi(x) = \sum_{n=1}^{\infty} \frac{1 + 2nx}{n^3} \exp(-2nx)$$

$$x = \frac{D}{\kappa_{per}}$$

$$\kappa_{per} = \lambda \frac{an_c}{(an_c - \mu)} \frac{2\pi^{1/2}}{\zeta(3/2)}$$
In the two-phase region (in the presence of condensate) one observes a power-law decay

\[
\frac{F(T, D, \mu)}{k_B T} = - \frac{2\zeta(3)}{\pi} \frac{1}{D^3}, \quad \mu > an_c
\]

exactly the same, and with the same amplitude as in the perfect Bose gas.
Divergence of the range of exponential forces at the approach to condensation:

imperfect (mean-field) Bose gas

\[ \kappa \sim (an_c - \mu)^{-1} \]

perfect Bose gas

\[ \kappa_0 \sim (-\mu)^{-1/2} \]
ONE-PARTICLE DENSITY MATRIX

FOR $\alpha = -\mu / K_B T > 0$

THE CASE OF A PERFECT GAS:

$$< x_2 | \hat{\rho}_1 | x_1 > = F(|x_2 - x_1|)$$

$$\lambda^3 F(x) = \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \exp \left[ -\alpha j - \frac{\pi x^2}{j \lambda^2} \right]$$

$$= \frac{\lambda}{x} \exp \left( -2 \frac{\sqrt{\pi} \alpha x}{\lambda} \right) + \sum_{s=1}^{\infty} \frac{\lambda}{x} \exp \left[ -A^+(s) \frac{x}{\lambda} \right] 2 \cos \left[ -A^-(s) \frac{x}{\lambda} \right]$$

with

$$A^{\pm}(s) = \sqrt{2\pi} (\alpha^2 + 4\pi^2 s^2)^{1/4} \left[ 1 \pm \frac{\alpha}{(\alpha^2 + 4\pi^2 s^2)^{1/2}} \right].$$
Correlation function of a perfect Bose gas

$$\lambda^6 [n_2(r; \mu, T) - n^2] = \left[ \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \exp \left[ -\alpha j - \frac{\pi r^2}{j \lambda^2} \right] \right]^2$$

$$\alpha = -\frac{\mu}{k_B T}, \quad \lambda = \frac{h}{\sqrt{2\pi mk_B T}}$$

Large distance ($r \gg \lambda$) asymptotics

$$\lambda^6 [n_2(r; \mu, T) - n^2] \approx \left( \frac{\lambda}{r} \right)^2 \exp \left( -\frac{r}{\xi_0(\mu)} \right).$$

$$\xi_0(\mu) = \frac{h}{4\pi\sqrt{2m}} \frac{1}{\sqrt{-\mu}} = \text{range of Casimir force}!$$
The hierarchy equations for the thermodynamic Green functions in the one-phase region $\mu < an_c$ imply the equality between the imperfect gas correlation function and the perfect gas correlation function calculated for the shifted chemical potential $[\mu - an(T, \mu)]$.

The range of exponentially decaying correlations equals

$$\xi_{imp} = \frac{\lambda}{4} \left( -\frac{k_BT}{\pi[\mu - an(T, \mu)]} \right)^{1/2}$$

and diverges $\sim \frac{\lambda}{2\zeta(3/2)} \left( 1 - \frac{\mu}{an_{0,c}} \right)^{-1}$, when $\mu$ approaches its critical value $\mu_c = an_{0,c}$ from below.
Knowledge of the asymptotic behavior of Bose functions

$$g_r(\alpha) = \sum_{q=1}^{\infty} \frac{\exp(-\alpha q)}{q^r}$$

when $\alpha \to 0$

$$g_{1/2}(\alpha) \approx \sqrt{\frac{\pi}{\alpha}}, \quad g_{-1/2}(\alpha) \approx \frac{1}{\alpha} \sqrt{\frac{\pi}{\alpha}}$$

permits to evaluate derivatives of the density with respect to the chemical potential at the condensation point

$$\mu = \mu_{imp,c} = n_c a.$$
\[ \kappa = \text{range of Casimir forces.} \]
\[ \xi = \text{range of bulk correlations} \]

- **perfect gas**
  \[ \kappa_{0, \text{periodic}} = 2\kappa_{0, \text{Dirichlet}} = 2\kappa_{0, \text{Neumann}} = \xi_0 \]
  critical exponent \( \nu = 1/2 \)

- **imperfect (mean field) gas**
  \[ \kappa_{\text{periodic}} = 2\kappa_{\text{Dirichlet}} = 2\kappa_{\text{Neumann}} = 2\sqrt{\pi}\xi \]
  critical exponent \( \nu = 1 \)
SOME REFERENCES


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