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Control of linear systems

e di P. URBANSKI**

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Summary. *The general theory of linear symplectic relations presented in* [1] *is applied to the analysis of symplectic relations representing physical devices controlling linear static systems. The analysis of positive relations* [3] *is used to single out those symplectic relations which can represent real physical devices. Applications of symplectic geometry to control theory were initiated in* [2].

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1. Special symplectic spaces. Lagrangian subspaces

Let Q be a real vector space and let Q^* denote the dual space. The canonical pairing of Q with Q^* is a mapping

$$\langle\ ,\ \rangle : Q \oplus Q^* \to R$$

defined by

$$\langle q, f\rangle = f(q)\ .$$

We denote the direct sum $Q \oplus Q^*$ by P. The canonical projections of

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P onto Q and Q^* are denoted by

$$\pi_Q : P \to Q$$

and

$$\pi_{Q^*} : P \to Q^* .$$

The 2-form $\omega \in P^* \oplus P^*$ defined by

$$\langle (q \oplus f) \wedge (q' \oplus f'), \omega \rangle = \langle q', f \rangle - \langle q, f' \rangle$$

gives P the structure of a symplectic vector space. For each subspace K of P we denote by $K^{\S}$ the subspace

$$\{q \oplus f \in P; \langle (q \oplus f) \wedge (q' \oplus f'), \omega \rangle = 0 \text{ for each } q' \oplus f' \in K\} .$$

We have the following easy to verify relations

$$K^{\S\S} = K,$$

$$\dim(K) + \dim(K^{\S}) = \dim(P),$$

$$(K + L)^{\S} = K^{\S} \cap L^{\S} ,$$

$$(K \cap L)^{\S} = K^{\S} + L^{\S} ,$$

$$O^{\S} = P,$$

$$P^{\S} = O,$$

where K and L are subspaces of P and O is the subspace of P containing only the zero vector.

Definition 1.1. - A subspace K of P is said to be

a) *isotropic* if $K^{\S} \supset K$,
b) *coisotropic* if $K^{\S} \subset K$,
c) *Lagrangian* if $K^{\S} = K$.

Proposition 1.1. - To each subspace K of P there corresponds a mapping

$$x : C \to D^*, \quad C = \pi_Q(K), D = \pi_Q(K^{\S})$$

such that

$$K = \{q \oplus f \in P; q \in C \text{ and}$$
$$\langle q', f\rangle = \langle q', x(q)\rangle \text{ for each } q' \in D\}.$$

Proof. If $q \in C$ then there exists an element f of Q^* such that $q \oplus f \in K$. The equation

$$\langle q', x(q)\rangle = \langle q', f\rangle \quad \text{for each} \quad q' \in D$$

defines a mapping $x : C \to D^*$ because if f_1 and f_2 are two elements of Q^* such that $q \oplus f_1 \in K$ and $q \oplus f_2 \in K$, and q' belongs to D then there exists an element f' of Q^* such that $q' \oplus f' \in K^{\S}$ and

$$\begin{aligned}\langle q', f_2\rangle - \langle q', f_1\rangle &= \langle q', f_2\rangle - \langle q, f'\rangle + \langle q, f'\rangle - \langle q', f_1\rangle \\ &= \langle (q \oplus f_2) \wedge (q' \oplus f'), \omega\rangle - \langle (q \oplus f_1) \wedge \\ &\wedge (q' \oplus f'), \omega\rangle = 0.\end{aligned}$$

From this construction of x it follows already that

$$K \subset \{q \oplus f \in P; \ q \in C \text{ and}$$
$$\langle q', f\rangle = \langle q', x(q)\rangle \text{ for each } q' \in D\}.$$

Now let $q \in C$ and let f satisfy $\langle q', f\rangle = \langle q', x(q)\rangle$ for each $q' \in D$. Then there exists $f'' \in Q^*$ such that $q \oplus f'' \in K$. Hence, for each $q' \oplus f' \in K^{\S}$

$$\begin{aligned}\langle (q \oplus f) \wedge (q' \oplus f'), \omega\rangle &= \langle q', f\rangle - \langle q, f'\rangle \\ &= \langle q', x(q)\rangle - \langle q, f'\rangle \\ &= \langle q', f''\rangle - \langle q, f'\rangle \\ &= \langle (q \oplus f'') \wedge (q' \oplus f'), \omega\rangle = 0.\end{aligned}$$

It follows that $q \oplus f$ belongs to $K^{\S\S} = K$.

Q.E.D.

Definition 1.2. - The mapping $x : C \to D^*$ is called the *generating form* of the subspace

$$K = \{q \oplus f \in P \; ; \; q \in C \text{ and } \langle q', f \rangle = \langle q', x(q) \rangle \text{ for each } q' \in D\}$$

and the subspace K is said to be *generated* by x.

Proposition 1.2. - If K is generated by a form $x : C \to D^*$ then $K^{\S}$ is generated by the adjoint form $x^* : D \to C^*$.

Proof. Let K' be the subspace of P generated by x^*. Let $q \oplus f \in K$ and $q' \oplus f' \in K^{\S}$. Then

$$\langle q', f \rangle - \langle q, f' \rangle = \langle (q \oplus f) \wedge (q' \oplus f'), \omega \rangle = 0.$$

Hence

$$\langle q, f' \rangle = \langle q', f \rangle = \langle q', x(q) \rangle = \langle q, x^*(q') \rangle .$$

Since q can be any element of C it follows that $q' \oplus f' \in K'$. Consequently $K^{\S} \subset K'$. Now let $q \oplus f \in K$ and $q' \oplus f' \in K'$. Then

$$\begin{aligned} \langle (q \oplus f) \wedge (q' \oplus f'), \omega \rangle &= \langle q', f \rangle - \langle q, f' \rangle \\ &= \langle q', x(q) \rangle - \langle q, x^*(q') \rangle \\ &= 0. \end{aligned}$$

Hence $q' \oplus f' \in K^{\S}$. Consequently $K' \subset K^{\S}$. Q.E.D.

Proposition 1.3. - Let K and K' be subspaces of P generated by forms $x : C \to D^*$ and $x' : C' \to D'^*$ respectively. Then $K' \subset K$ if and only if $C' \subset C$, $D' \supset D$ and

$$\langle q, x(q') \rangle = \langle q, x'(q') \rangle$$

for each $q \in D$ and $q' \in C'$.

Proof. a) Let relations $C' \subset C$, $D' \supset D$ and $\langle q, x(q') \rangle = \langle q, x'(q') \rangle$ for each $q \in D$ and $q' \in C'$ hold. If $q' \oplus f' \in K'$ then for each $q \in D$

$$\langle q, f' \rangle = \langle q, x'(q') \rangle = \langle q, x(q') \rangle.$$

It follows that $q' \oplus f' \in K$. b) Let $K' \subset K$. It follows that $C' \subset C$ and $D' \supset D$. If $q \in D$ and $q' \in C'$ then there exists an element f' of Q^* such that $q' \oplus f' \in K'$ and

$$\langle q, x'(q')\rangle = \langle q, f'\rangle = \langle q, x(q')\rangle .$$

Q.E.D.

Corollary 1.1. - Let K be a subspace of P generated by a form $x : C \to D^*$. Then
a) K is isotropic if and only if $C \subset D$ and $\langle q', x(q)\rangle = \langle q, x(q')\rangle$ for all $q, q' \in C$,
b) K is coisotropic if and only if $C \supset D$ and $\langle q', x(q)\rangle = \langle q, x(q')\rangle$ for all $q, q' \in D$,
c) K is Lagrangian if and only if $C = D$ and x is selfadjoint.

Let K be a Lagrangian subspace of P generated by a form $x : C \to C^*$. Since x is selfadjoint it is equal to the differential dF of a quadratic function $F : C \to R : q \mapsto \frac{1}{2}\langle q, x(q)\rangle$. A function $F : C \to R$ is quadratic it the mapping

$$\delta F : C \times C \to R : (q, q') \mapsto F(q + q') - F(q) - F(q')$$

is bilinear and $F(q) = \frac{1}{2}\,\delta F(q, q)$. The differential $dF : C \to C^*$ of a quadratic function $F : C \to R$ is defined by

$$\langle q', dF(q)\rangle = \delta F(q, q') .$$

Definition 1.3. - The Lagrangian subspace K of P generated by the differential of a quadratic function $F : C \to R$ is said to be *generated* by F and F is called the *generating function* of K.

2. Physical interpretation

Lagrangian subspaces can be used to describe the behaviour of physical systems. Let Q be the *configuration space* of a linear static physical system. *Virtual displacements* of the system are also elements of Q. The dual space Q^* is the *force space*. The *constitutive law* of the system is a relation between configurations and external forces which must be applied to the system in order to maintain these configurations. The constitutive law is represented geometrically as a subspace S of the *phase space* $P = Q \oplus Q^*$. In the simplest case the constitutive law associates with each configuration a unique force. This means that S in the graph of a mapping $\sigma : Q \to Q^*$. The system is said to be *reciprocal* if σ is selfadjoint. The concept of reciprocity has a natural generalization to the general case of a static physical system characterized by a subspace S of the phase space P.

Definition 2,1. - A linear physical system is said to be *reciprocal* if its constitutive law is represented by a Lagrangian subspace S of the phase space P.

Definition 2.2. - The generating function U of a Lagrangian subspace S representing the constitutive law of a linear physical system is called the *internal energy*.

The internal energy of linear physical systems is usually positive. We will examin consequences of this fact. Numerous examples of physical systems and their constitutive laws can be found in [2].

3. Elementary operations

Let Q and Q' be vector spaces and let $\iota : Q' \to Q$ be an injection. Then $\iota^* : Q^* \to Q'^*$ is a surjection whose kernel is the anihilator of the image of ι.

Definition 3.1. - Let K be a subspace of $P = Q \oplus Q^*$ and let $\rho_\iota(K)$ be a subspace of $P' = Q' \oplus Q'^*$ defined by

$$\rho_\iota(K) = \{q' \oplus f' \in P' ;\ \iota(q') \oplus f \in K \text{ for some } f \in Q^* \text{ such that } \iota^*(f) = f'\} .$$

The transition from the space K to $\rho_\iota(K)$ is called the *reduction* of K with respect to the injection ι.

Proposition 3.1. - If K is a subspace of P generated by a form $x : C \to D^*$ then $K' = \rho_\iota(K)$ is generated by a form $x' : C' \to D'^*$, where $C' = \iota^{-1}(C), D' = \iota^{-1}(D)$ and x' is defined by

$$\langle q, x'(q') \rangle = \langle \iota(q), x(\iota(q')) \rangle$$

for each $q \in D'$ and each $q' \in C'$.

Proof. - Let $\bar{K}$ be the subspace of P' generated by x'. If $q' \oplus f' \in K'$ then there exists an $f \in Q^*$ such that $f' = \iota^*(f)$, $\iota(q') \oplus f \in K$ and for each $q \in D'$ we have

$$\begin{aligned}\langle q, f'\rangle &= \langle q, \iota^*(f)\rangle \\ &= \langle \iota(q), f\rangle \\ &= \langle \iota(q), x(\iota(q'))\rangle \\ &= \langle q, x'(q')\rangle .\end{aligned}$$

Hence, $q' \oplus f' \in \bar{K}$. It follows that $K' \subset \bar{K}$. Now let $q' \oplus f' \in \bar{K}$. Then for each $q \in D'$ we have

$$\langle q, f'\rangle = \langle q, x'(q')\rangle = \langle \iota(q), x(\iota(q'))\rangle .$$

It is possible to find an element f of Q^* such that $\iota^*(f) = f'$ and $\langle q, f\rangle = \langle q, x(\iota(q'))\rangle$ for each q in D. Hence, $\iota(q') \oplus f \in K$. It follows that $q' \oplus f' \in K'$ and $\bar{K} \subset K'$. Q.E.D.

The following statements are corollaries of Proposition 1.2 and Proposition 3.1.

Corollary 3.1. - For each subspace K of P we have

$$(\rho_\iota(K))^\S = \rho_\iota(K^\S) .$$

Corollary 3.2. - Let C be a subspace of Q and let K be a Lagrangian subspace of P generated by a quadratic function $F : C \to R$. Then $\rho_\iota(K)$ is a Lagrangian subspace of P' generated by the pullback $F' : C' \to R$ of F to $C' = \iota^{-1}(C)$.

Let $\pi : Q \to Q'$ be a surjection.

Definition 3.2. - Let K be a subspace of $P = Q \oplus Q^*$ and let $\rho_\pi(K)$ be a subspace of $P' = Q' \oplus Q'^*$ defined by

$$\begin{aligned}\rho_\pi(K) = \{q' \oplus f' \in P'; q \oplus \pi^*(f') \in K \text{ for some} \\ q \in Q \text{ such that } \pi(q) = q'\} .\end{aligned}$$

The transition from the space K to $\rho_\pi(K)$ is called the *reduction* of K with respect to the surjection π.

Proposition 3.2. - If K is a subspace of P generated by a form $x : C \to \to D^*$ then $K' = \rho_\pi(K)$ is generated by the form $x' : C' \to D'^*$, where

$$C' = \{q_1' \in Q' ; q_1' \in \pi(C), \text{ there exists } q_1 \in C \text{ such that } \pi(q_1) = q_1' \text{ and } \langle q'', x(q_1)\rangle = 0 \text{ for each } q'' \in D \text{ such that } \pi(q'') = 0\} ,$$

$$D' = \{q_2' \in Q' ; q_2' \in \pi(D), \text{ there exist } q_2 \in D \text{ such that } \pi(q_2) = q_2' \text{ and } \langle q_2, x(q'')\rangle = 0 \text{ for each } q'' \in C \text{ such that } \pi(q'') = 0\} ,$$

and x' is defined by

$$\langle q_2', x'(q_1')\rangle = \langle q_2, x(q_1)\rangle ,$$

where q_1, q_2, q_1' and q_2' are the elements used in the definitions of C' and D'.

Proof. - Let $x'' : C'' \to D''$ be the generating form of K'. Since the image of π^* is the anihilator of the kernel of π is follows from Definition 1.2 that

$$K' = \{q' \oplus f' \in P' ; q' \in \pi(C) \text{ and there exists } q \in Q \text{ such that } \pi(q) = q', \langle q'', x(q)\rangle = 0 \text{ for each } q'' \in D \text{ such that } \pi(q'') = 0 \text{ and } \langle q'', x(q)\rangle = \langle \pi(q''), f'\rangle \text{ for each } q'' \in D\} .$$

Hence, $C'' = C'$. Moreover, since $0 \oplus f' \in K'$ if and only if $f' \in (D')^\circ$, it follows that $\pi(D) \supset D'' \supset D'$. It follows already that for $q' \in C'$ and $q'' \in D'$ we have $\langle q'', x'(q')\rangle = \langle q'', x''(q')\rangle$. It remains to be shown that $D' = D''$. From Proposition 1.2 and Proposition 1.3 it follows that $D' \supset D''$ is equivalent to $\rho_\pi(K^\S) \supset (\rho_\pi(K))^\S$. Let $q' \oplus f' \in (\rho_\pi(K))^\S$. Then $\langle q'', f'\rangle - \langle q', f''\rangle = 0$ for each $q'' \oplus f'' \in \rho_\pi(K)$. It follows that there exists an element $q \oplus f \in K^\S$ such that

$$\langle q'', f\rangle - \langle q, \pi^*(f'')\rangle = \langle \pi(q''), f'\rangle - \langle q', f''\rangle$$

for each $q'' \in Q$ and $f'' \in Q'^*$. Consequently $f = \pi^* f'$, $\pi(q) = q'$ and $D' = D''$.

Q.E.D.

Corollary 3.3. - For each subspace K of P we have

$$(\rho_\pi(K))^\S = \rho_\pi(K^\S).$$

Corollary 3.4. - Let C be a subspace of Q and let K be a Lagrangian subspace of P generated by a quadratic function $F: C \to R$. Then $\rho_\pi(K)$ is a Lagrangian subspace of P' generated by the function $F': C' \to R$, where

$$C' = \{q' \in Q';\ \text{there exists } q \in Q \text{ such that } \pi(q) = q' \text{ and}$$
$$\langle q'', dF(q)\rangle = 0 \ \text{ for each } q'' \in C \text{ such that } \pi(q'') = 0\}$$

and $F'(q') = F(q)$, where q and q' are the elements used in the definition of C'.

Let Q_1 and Q_2 be vector spaces. We denote by Q the space $Q_1 \oplus Q_2$. The space Q^* is canonically isomorphic to the space $Q_1^* \oplus Q_2^*$. The isomorphism

$$\gamma: Q_1^* \oplus Q_2^* \to Q^*$$

is defined by

$$\langle q_1 \oplus q_2, \gamma(f_1 \oplus f_2)\rangle = \langle q_1, f_1\rangle + \langle q_2, f_2\rangle.$$

Spaces $Q_1 \oplus Q_2 \oplus Q_1^* \oplus Q_2^*$ and $P_1 \oplus P_2 = Q_1 \oplus Q_1^* \oplus Q_2 \oplus Q_2^*$ are also isomorphic. We will identify the space $P = Q \oplus Q^*$ with the space $P_1 \oplus P_2$.

The following proposition is an immediate consequence of the definition of the generating form of a subspace.

Proposition 3.3. - Let K_1 and K_2 be subspaces of P_1 and P_2 respectively generated by forms $x_1: C_1 \to D_1^*$ and $x_2: C_2 \to D_2^*$, where C_1 and D_1 are subspaces of Q_1, and C_2 and D_2 are subspaces of Q_2. Then $K = K_1 \oplus K_2$ is a subspace of P generated by the form

$$x: C_1 \oplus C_2 \to D_1^* \oplus D_2^*: q_1 \oplus q_2 \mapsto x_1(q_1) \oplus x_2(q_2).$$

Corollary 3.5. - If K_1 and K_2 are subspaces of P_1 and P_2 respectively then

$$(K_1 \oplus K_2)^\S = K_1^\S \oplus K_2^\S.$$

Corollary 3.6. - Let K_1 and K_2 be Lagrangian subspaces of P_1 and P_2 respectively generated by functions $F_1 : C_1 \to R$ and $F_2 : C_2 \to R$, where C_1 and C_2 are subspaces of Q_1 and Q_2 respectively. Then $K = K_1 \oplus K_2$ is a Lagrangian submanifold of P generated by the function

$$F : C_1 \oplus C_2 \to R : q_1 \oplus q_2 \mapsto F_1(q_1) + F_2(q_2).$$

4. Composition of physical systems

Let S and S' be constitutive sets of static systems with configuration manifolds Q and Q'. The *combined system*, composed of the two systems, is a static system with configuration manifold $Q \oplus Q'$ and constitutive set $S \oplus S' \subset Q \oplus Q^* \oplus Q' \oplus Q'^* = (Q \oplus Q') \oplus (Q \oplus Q')^*$.

Let S_1 and S_2 be constitutive sets of two static systems with configuration spaces $Q \oplus Q_1$ and $Q_2 \oplus Q$ respectively. The constitutive set $S_2 \circ S_1$ of the *coupled system* is defined by

$$\begin{aligned} S_2 \circ S_1 = \{&(q_2 \oplus f_2) \oplus (q_1 \oplus f_1) \in (Q_2 \oplus Q_2^*) \oplus (Q_1 \oplus Q_1^*);\\ &\text{there exists } q \oplus f \in Q \oplus Q^* \text{ such that}\\ &(q \oplus f) \oplus (q_1 \oplus f_1) \in S_1\\ &\text{and } (q_2 \oplus f_2) \oplus (q \oplus (-f)) \in S_2\}. \end{aligned}$$

If $S_1 \subset Q \oplus Q^*$ and $S_2 \subset (Q' \oplus Q) \oplus (Q' \oplus Q)^*$, the the constitutive set $S_2 \circ S_1$ of the coupled system is defined by

$$\begin{aligned} S_2 \circ S_1 = \{&q' \oplus f' \in Q' \oplus Q'^*; \text{ there exists } q \oplus f \in S_1\\ &\text{such that } (q' \oplus f') \oplus (q \oplus (-f)) \in S_2\}. \end{aligned}$$

It is useful to observe that the coupled system is obtained by applying two reductions to the constitutive set $S_2 \oplus S_1$ of the combined system. The first reduction is with respect to the injection

$$Q_2 \oplus Q \oplus Q_1 \to Q_2 \oplus Q \oplus Q \oplus Q_1 : q_2 \oplus q \oplus q_1 \mapsto q_2 \oplus q \oplus q \oplus q_1 .$$

This is followed by the reduction with respect to the canonical projection of $Q_2 \oplus Q \oplus Q_1$ onto $Q_2 \oplus Q_1$.

This observation together with Proposition 3.1, 3.2 and 3.3 leads to the following proposition.

Proposition 4.1. Let S_1 and S_2 be constitutive sets of two static systems with configuration spaces $Q \oplus Q_1$ and $Q_2 \oplus Q$ respectively. Let $x_1 : C_1 \to D_1^*$ and $x_2 : C_2 \to D_2^*$ be generating forms of S_1 and S_2. The constitutive set of the coupled system $S_2 \circ S_1$ is generated by the form $x : C \to D^*$, where

$$\begin{aligned} C = \{ & q_2 \oplus q_1 \in Q_2 \oplus Q_1 \text{; there exists } q \in Q \text{ such that} \\ & q \oplus q_1 \in C_1 \text{ and } q_2 \oplus q \in C_2, \text{ and} \\ & \langle 0 \oplus q'', x_2(q_2 \oplus q)\rangle + \langle q'' \oplus 0, x_1(q \oplus q_1)\rangle = 0 \\ & \text{for each } q'' \in Q \text{ such that } 0 \oplus q'' \in D_2 \text{ and } q'' \oplus 0 \in D_1 \} \end{aligned}$$

$$\begin{aligned} D = \{ & q_2' \oplus q_1' \in Q_2 \oplus Q_1\text{; there exists } q' \in Q \text{ such that} \\ & q' \oplus q_1' \in D_1 \text{ and } q_2' \oplus q' \in D_2, \text{ and} \\ & \langle q_2' \oplus q', x_2(0 \oplus q'')\rangle + \langle q' \oplus q_1', x_1(q'' \oplus 0)\rangle = 0 \\ & \text{for each } q'' \in Q \text{ such that } 0 \oplus q'' \in C_2 \text{ and } q'' \oplus 0 \in C_1 \} \end{aligned}$$

and x is defined by

$$\begin{aligned} & \langle q_2' \oplus q_1', x(q_2 \oplus q_1)\rangle = \\ & = \langle q_2' \oplus q', x_2(q_2 \oplus q)\rangle + \langle q' \oplus q_1', x_1(q \oplus q_1)\rangle, \end{aligned}$$

where q_1, q_2, q, q_1', q_2' and q' are elements related as in the definitions of C and D.

Corollary 4.1. If S_1 and S_2 are subspaces of $(Q \oplus Q_1) \oplus (Q \oplus Q_1)^*$ and $(Q \oplus Q_1) \oplus (Q \oplus Q_1)^*$ respectively then

$$(S_2 \circ S_1)^\S = S_2^\S \circ S_1^\S .$$

Corollary 4.2. If S_1 and S_2 are constitutive sets of reciprocal systems generated by functions $F_1 : C_1 \to R$ and $F_2 : C_2 \to R$ respectively then the coupled system is reciprocal and the constitutive set $S_2 \circ S_1$ is generated by the function $F : C \to R$, where

$$C = \{q_2 \oplus q_1 \in Q_2 \oplus Q_1\,;\ \text{there exists } q \in Q \text{ such that}$$
$$q \oplus q_1 \in C_1,\ q_2 \oplus q \in C_2 \text{ and}$$
$$\langle 0 \oplus q', dF_2\,(q_2 \oplus q)\rangle + \langle q' \oplus 0, dF_1\,(q \oplus q_1)\rangle = 0$$
$$\text{for each } q' \text{ such that } q' \oplus 0 \in C_1 \text{ and } 0 \oplus q' \in C_2\}$$

and F is defined by

$$F(q_2 \oplus q_1) = F_2\,(q_2 \oplus q) + F_1\,(q \oplus q_1),$$

where q_1, q_2 and q are related as in the definition of C.

If $S_1 \subset Q \oplus Q^*$ and $S_2 \subset (Q' \oplus Q) \oplus (Q' \oplus Q)^*$, the Proposition 4.1 and the two corollaries hold in suitably modified versions.

5. Symplectic relations

Let Q and Q' be vector spaces. We denote by P and P' the symplectic spaces $Q \oplus Q^*$ and $Q' \oplus Q'^*$ respectively. For each subspace S of $P' \oplus P$ we denote by $\overline{S}$ the subspace

$$\overline{S} = \{(q' \oplus f') \oplus (q \oplus f) \in P' \oplus P;$$
$$(q' \oplus f') \oplus (q \oplus (-f)) \in S\}\,.$$

Definition 5.1. The *generating form* of a linear relation $\rho : P \to P'$ is the generating form of the subspace $\overline{\text{graph } \rho} \subset P' \oplus P$.

Definition 5.2. A linear relation $\rho : P \to P'$ is said to be *symplectic* if $\overline{\text{graph } \rho}$ is a Lagrangian subspace of $P' \oplus P$.

Definition 5.3. The *generating function* of a symplectic relation $\rho : P \to P'$ is the generating function of the Lagrangian subspace $\overline{\text{graph } \rho}$.

Example 5.1. Let $\iota : Q' \to Q$ be an injection. The relation $\rho_\iota : P \to$ $\to P'$ whose graph is defined by

$$\text{graph } \rho_\iota = \{(q' \oplus f') \oplus (q \oplus f) \in P' \oplus P;$$
$$q = \iota(q'),\ f' = \iota^*\,(f)\}$$

is a symplectic relation. The symbol $\rho_t(K)$ used in Section 3 denotes the image of K by the relation ρ_t.

Example 5.2. Let $\pi : Q \to Q'$ be a surjection. A symplectic relation ρ_π is defined by

$$\text{graph } \rho_\pi = \{(q' \oplus f') \oplus (q \oplus f) \in P' \oplus P; \\ q' = \pi(q), f = \pi^*(f')\} .$$

If $\overline{\text{graph } \rho} = S$ then the relation ρ will be denoted by ρ_S.

Proposition 5.1. Let S a be subspace of $P' \oplus P$ and K a subspace of P. Then

$$\rho_S(K) = S \circ K .$$

Proof. From the definition of ρ_S we have

$$\rho_S(K) = \{q' \oplus f' \in P'; \text{ there exists } q \oplus f \in P \text{ such that} \\ q \oplus f \in K \text{ and } (q' \oplus f') \oplus (q \oplus f) \in S\} .$$

By comparing this with the definition of a coupled system we obtain the equality $\rho_S(K) = S \circ K$. Q.E.D.

The following corollary is a direct consequence of Proposition 5.1 and Corollary 4.1.

Corollary 5.1. If $\rho : P \to P'$ is a symplectic relation and K is a subspace of P then

$$\rho(K^\S) = (\rho(K))^\S ,$$

$\rho(P)$ is coisotropic,

$\rho(0)$ is sotropic .

The proof of the following proposition is analogous to the proof of Proposition 5.1.

Proposition 5.2. If S and S' are subspaces of $P' \oplus P$ and $P'' \oplus P'$ respectively then

$$\rho_{S'} \circ \rho_S = \rho_{S' \circ S} \ .$$

Corollary 5.2. If $\rho_1 : P \to P'$ and $\rho_2 : P' \to P''$ are symplectic relations then $\rho_2 \circ \rho_1$ is symplectic.

For each subspace K of a direct sum $Q_1 \oplus Q_2$ we denote by ${}^tK$ the subspace of $Q_2 \oplus Q_1$ defined by

$${}^tK = \{q_2 \oplus q_1 \in Q_2 \oplus Q_1 \ ; \ q_1 \oplus q_2 \in K\} \ .$$

If $\rho : Q_1 \to Q_2$ is a linear relation then ${}^t\rho : Q_2 \to Q_1$ is the relation defined by graph ${}^t\rho = {}^t$(graph ρ).

Proposition 5.3. Let S be a subspace of $P' \oplus P$ generated by a form $x : C \to D^*$. Then ${}^tS$ is generated by the form $\widetilde{x} : {}^tC \to ({}^tD)^*$ defined by

$$\langle q_1 \oplus q_1', \widetilde{x}(q \oplus q') \rangle = \langle q_1' \oplus q_1, x(q' \oplus q) \rangle \ .$$

Proof. Obvious.

Corollary 5.3. If $S \subset P' \oplus P$ is a Lagrangian subspace generated by a function $F : C \to R$ then ${}^tS$ is a Lagrangian subspace generated by the function $\widetilde{F} : {}^tC \to R$ defined by

$$\widetilde{F}(q \oplus q') = F(q' \oplus q).$$

Corollary 5.4. If $\rho : P \to P'$ is a linear relation generated by a form $x \ \ C \to D^*$ then ${}^t\rho$ is generated by $-\widetilde{x}$.

Corollary 5.5. If $\rho : P \to P'$ is a symplectic relation generated by a function $F : C \to R$ then ${}^t\rho$ is a symplectic relation generated by $-\widetilde{F}$.

6. Control modes.

Let Q be a vector space.

Definition 6.1. A *control system* (I, R) for physical systems with configuration space Q is a pair of reciprocal physical systems with constitutive sets I and R and configuration space $A \oplus Q$ and Q respectively. We associate with I a symplectic relation

$$\rho_I : Q \oplus Q^* \to A \oplus A^* .$$

The following conditions are satisfied:

$$\rho_I (Q \oplus Q^*) = A \oplus A^*$$
$$\pi_A \ (\rho_I (R)) = A.$$

The system with constitutive set I is called the *control interface* and the system with constitutive set R is called the *response reference*.

Let I and $I \circ R$ be generated by functions $F_I : C \to R$ and $F_{I \circ R}$: $A \to R$. We asociate with the pair (I, R) a relation

$$\rho_{(I,R)} : Q \oplus Q^* \to A \oplus A^*$$

generated by the function

$$F_{(I,R)} : C \to R : (a \oplus q) \mapsto F_I (a \oplus q) - F_{I \circ R} \ (a).$$

Definition 6.2. Two control system (I, R) and (I', R') are said to be *equivalent* if $\rho_{(I', R')} = \rho_{(I,R)}$. An equivalence class of control system is called a *control mode.*

Proposition 6.1. Two control systems (I, R) and (I', R') are equivalent if and only if

$$\pi_A \circ \rho_{(I',R')} = \pi_A \circ \rho_{(I,R)}$$

and

$${}^t\rho_{(I',R')} (\rho_{(I',R')} (R')) = {}^t\rho_{(I,R)} (\rho_{(I,R)} (R)) .$$

Proof. Let two control systems (I, R) and $(I', R,)$ be equivalent. From $\rho_{(I',R')} = \rho_{(I,R)}$ it follows that $\pi_A \circ \rho_{(I',R')} = \pi_A \circ \rho_{(I,R)}$.

Proposition 4.1 and Proposition 5.1 imply

$$\rho_{(I',R')} (R') = A \oplus 0 \quad \text{and} \quad \rho_{(I,R)} (R) = A \oplus 0.$$

Consequently, ${}^t\rho_{(I',R')} (\rho_{(I',R')} (R')) = {}^t\rho_{(I,R)} (\rho_{(I,R)} (R))$.

Now, let (I, R) and (I', R') be control systems such that

$$\pi_A \circ \rho_{(I',R')} = \pi_A \circ \rho_{(I,R)}$$

and

$$^t\rho_{(I',R')}\,(\rho_{(I',R')}\,(R')) = {}^t\rho_{(I,R)}\,(\rho_{(I,R)}\,(R)) \,.$$

From

$$\pi_A \circ \rho_{(I',R')} = \pi_A \circ \rho_{(I,R)}$$

we have

$$\begin{aligned} ^t\rho_{(I',R')}\,(A \oplus A^*) &= {}^t\rho_{(I',R')}\,({}^t\pi_A\,(A)) \\ &= {}^t(\pi_A \circ \rho_{(I',R')})(A) \\ &= {}^t(\pi_A \circ \rho_{(I,R)})(A) \\ &= {}^t\rho_{(I,R)}\,(A \oplus A^*). \end{aligned}$$

It follows from the decomposition theorem [1] that $\rho_{(I',R')} = \sigma \circ \rho_{(I,R)}$ for some symplectomorphism $\sigma : A \oplus A^* \to A \oplus A^*$ such that $\pi_A \circ \sigma = \pi_A$, The equality

$$^t\rho_{(I',R')}\,(\rho_{(I',R')}\,(R')) = {}^t\rho_{(I,R)}\,(\rho_{(I,R)}\,(R))$$

implies

$$\begin{aligned} \sigma\,(A \oplus 0) &= \sigma\,(\rho_{(I,R)}\,({}^t\rho_{(I,R)}\,(A \oplus 0))) \\ &= \sigma\,(\rho_{(I,R)}\,({}^t\rho_{(I,R)}\,(\rho_{(I,R)}\,(R)))) \\ &= \rho_{(I',R')}\,({}^t\rho_{(I',R')}\,(\rho_{(I',R')}\,(R'))) \\ &= A \oplus 0. \end{aligned}$$

It follows that the generating function of σ is the zero function defined on the diagonal in $A \oplus A$. Hence, σ is the identity mapping.

Q.E.D.

It is evident that if (I, R) is a control system then (I, R'), where $R' = {}^t\rho_{(I,R)}\,(\rho_{(I,R)}\,(R))$, is an equivalent control system and $R' = {}^t\rho_{(I',R')}\,(\rho_{(I',R')}\,(R'))$. The linear relation $\pi = \pi_A \circ \rho_{(I,R)}$ and the Lagrangian subspace $R' = {}^t\rho\,(\rho(R))$ are said to represent the equivalence class of (I, R). Not every pair (π, R), where $\pi : P = Q \oplus Q^* \to A$ is a linear relation and R is a Lagrangian subspace of $Q \oplus Q^*$, represents a control mode.

Proposition 6.2 Let $\pi : P \to A$ be a linear relation and let R be a Lagrangian subspace of P. The pair (π, R) represents a control mode if and only if the following conditions are satisfied

(1) $\quad {}^t\pi(A)$ is a coisotropic subspace of P,

(2) $\quad {}^t\pi(0)$ is a Lagrangian subspace of P,

(3) $\quad \pi(R) = A \quad$ and $\quad \pi(0) = 0$,

(4) $\quad R \subset {}^t\pi(A)$.

Proof. Let (π, R) represent a control mode. Then there exists a control interface I such that $\pi = \pi_A \circ \rho_{(I,R)} = \pi_A \circ \rho_I$, (I, R) is a control system and

$$ {}^t\rho_{(I,R)}(A \oplus 0) = R . $$

Consequently, (1) and (2) follow from Corollary 5.2, (3) follows from Definition 6.1 and (4) is a consequence of

$$ R = {}^t\rho_{(I,R)} \circ \rho_{(I,R)}(R) $$

and

$$ {}^t\pi(A) = {}^t\rho_{(I,R)}(A \oplus A^*) . $$

Now, let conditions (1)-(4) be satisflied and let L denote ${}^t\pi(0)$. Since π is a linear relation, (3) and (4) imply that ${}^t\pi(A) = R + L$. We define a subspace

$$ \begin{aligned} I = \{ &(a \oplus b) \oplus (q \oplus f) \in (A \oplus A^*) \oplus (Q \oplus Q^*); \\ &q \oplus f \in {}^t\pi(A),\, a = \pi(q \oplus f) \\ &\text{and there exist } q' \oplus f' \in L \text{ and } q'' \oplus f'' \in R \text{ such that} \\ &q' \oplus f' + q'' \oplus f'' = q \oplus f \text{ and} \\ &\langle (q' \oplus f') \wedge (q_1 \oplus f_1), \omega \rangle + \langle a_1, b \rangle = 0 \text{ for each} \\ &q_1 \oplus f_1 \in R \text{ and } a_1 \in A \text{ such that } \pi(q_1 \oplus f_1) = a_1 \} . \end{aligned} $$

It is evident that I is an isotropic subspace of $(A \oplus A^*) \oplus (Q \oplus Q^*)$. We show that $\rho_I(P) = A \oplus A^*$. Since $\rho_I(R) = A \oplus 0$ it is enough to prove that $0 \oplus A^* \subset \rho_I(P)$. We have $\rho_I(L) \subset 0 \oplus A^*$ and for $q \oplus f \in P$, $q \oplus f \in R \cap L = ({}^t\pi(A))^\S$ if and only if $\rho_I(q \oplus f) = 0$. Comparison of dimensions shows that

$$\dim(\rho_I(L)) = \dim(\rho_I(R)) = \dim A = \dim A^*$$

and, consequently, $\rho_I(P) = A \oplus A^*$. It follows further that

$$\begin{aligned}\dim I &= \dim(\text{graph } \rho_I) = \dim A + \dim({}^t\rho_I(0)) \\ &= \dim A + \dim R = \dim Q + \dim A.\end{aligned}$$

Hence, I is a Lagrangian subspace. The pair (I, R) is a control system and the corresponding control mode is represented by (π, R).

Q.E.D.

7. Admissible control modes

In this section we examine consequences of the fact that the internal energy of linear physical systems is positive.

Definition 7.1. A control system (I, R) is said to be *admissible* if the generating functions of I and R are positive.

Definition 7.2. A control mode is said to be *admissible* if it can be represented by an admissible control system.

The following proposition is a corollary to the decomposition theorem for positive symplectic relations (Theorem 4.1 in [3]).

Proposition 7.1. Let I be a control interface generated by a positive function. Then there exist subspaces Q' and Q'' of Q such that $P = Q \oplus Q^* = P' \oplus P''$ where $P' = Q' \oplus (Q'')^\circ$ and $P'' = Q'' \oplus (Q')^\circ$, and the following conditions are satisfied:

(1) $I = I' \oplus I''$ where $I' \subset (A \oplus A^*) \oplus P'$ and $I'' \subset 0 \oplus P''$ are Lagrangian subspaces,

(2) ${}^t\rho_{I'}(A \oplus A^*) = C \oplus B$, where C and B are subspaces of Q' and $(Q'')^\circ$ respectively.

Corollary 7.1. Let (π, R) represent an admissible control mode. Then $\pi = \pi' \circ \pi''$, where $\pi' : P' \to A$ and $\pi'' : P'' \to 0$ are linear relations. Moreover $R = R' \oplus R''$, where $R'' = {}^t\pi''(0)$ and $R' \subset {}^t\pi'(A)$.

We note that for an admissible control mode represented by (π, R) the generating function of R is not necessarily positive.

Theorem 7.1. Let (π, R) represent a control mode. This control mode is admissible if and only if there exist subspaces Q' and Q'' of Q such that $Q = Q' \oplus Q''$ and the following conditions are satisfied

(1) $\pi = \pi' \circ \pi''$, where $\pi' : P' \to A$ and $\pi'' : P'' \to 0$ are linear relations,

(2) ${}^t\pi'(A) = C \oplus B$, where C is a subspace of Q' and B is a subspace of $(Q'')^\circ$,

(3) the generating function of ${}^t\pi(0)$ is negative,

(4) the generating function of R is positive on Q',

(5) $B = \{f \in (Q'')^\circ \; ; \langle q, f\rangle = 0$ for q such that $q \in Q'$ and $q \oplus 0 \in {}^t\pi(0)\}$.

Proof. Let (π, R) represent an admissible control mode. It follows from Proposition 7.1 and Corollary 7.1 that there exist Q' and Q'' such that $Q = Q' \oplus Q''$ and conditions (1) and (2) are satisfied. Condition (3) follows from Corollary 5.3 and Corollary 4.2. We then have

$$ {}^t\pi(A) = (C \oplus B) + {}^t\pi''(0). $$

It follows that

$$ B = \{f \in (Q'')^\circ; \langle q, f\rangle = 0 \text{ for } q \text{ such that } q \in Q' \text{ and } q \oplus 0 \in ({}^t\pi(A))^\S\} . $$

Let (I, R_1) be an admissible control system representing (π, R). Since the generating function of I is positive, $(0 \oplus b) \oplus (q \oplus 0) \in I$ implies that $(0 \oplus 0) \oplus (q \oplus 0) \in I$. This means that $q \oplus 0 \in {}^t\pi(0)$ if and only if $q \oplus 0 \in {}^t\rho_I(0) = ({}^t\pi(A))^\S$ and, consequently, (4) is satisfied. In order to prove (5) we note that $\rho_{(I,R)}(R_1) = \rho_{(I',R')}(R_1')$, where $R' = P' \cap R$ and $R_1' = P' \cap R_1$ $(P' = Q' \oplus (Q'')^\circ)$. I' and I'' are related as in Proposition 7.1. According to the decomposition theorem for sym plectic relations ([1]), $\rho_{(I',R')} = \rho_2 \circ \rho_1$, where

$$\rho_1 : P' \to (C/B^\circ) \oplus (B/C^\circ)$$

is the composition of reductions with respect to the canonical injection $\iota : C \to Q'$ and the canonical surjection $\varphi : C \to C/B^\circ$ and ρ_2 is an isomorphism. It follows that the generating function of $\rho_1 (R_1')$ is positive. Since ρ_2 is an isomorphism it follows that

$${}^t\rho_2 \circ \rho_2 \circ \rho_1 (R_1') = \rho_1 (R_1')$$

and, consequently, the generating function of the subspace ${}^t\rho_{(I',R')} \circ$ $\circ\, \rho_{(I',R')} (R_1')$ is positive. Hence,

$${}^t\rho_{(I,R)} \circ \rho_{(I,R)} (R_1) = ({}^t\rho_{(I',R')} \circ \rho_{(I',R')} (R_1')) \oplus R'' ,$$

where R'' is related to R as in Corollary 7.1 and, consequently, we have (5).

Now, suppose that a pair (π, R) represents a control mode and conditions (1)-(5) are satisfied. Then (1) and (3) imply that $\pi'' = \rho_{I''}$, where $I'' \subset 0 \oplus P''$ and its generating function is positive. We construct an appropriate Lagrangian subspace

$$I' \subset (A \oplus A^*) \oplus (Q' \oplus (Q'')^\circ).$$

Let a set C' be defined by

$$C' = \{a \oplus q \in A \oplus Q' ;\ \text{there exists } f \in (Q'')^\circ \ \text{ such that } \pi (q \oplus f) = a\}$$

and let $F_1 : C_1 \to R$ be the generating function of ${}^t\pi (0)$. From Proposition 6.2 it follows that for each $a \in A$ there exists $q_1 \oplus f_1$ such that $q_1 \in Q'$ and $\pi (q_1 \oplus f_1) = a$. Moreover, for two elements $q_1 \oplus f_1$, $q_1' \oplus f_1'$ such that $q_1, q_1' \in Q'$ and $\pi(q_1 \oplus f_1) = \pi(q_1' \oplus f_1')$ we have $(q_1 - q_1') \oplus (f_1 - f_1') \in ({}^t\pi (A))^\S$, i.e., $(q_1 - q_1') \in (B)^\circ$. Since F_1 is negative and $F_1 (q) = 0$ for $q \in Q' \cap (B)^\circ$ it follows that

$$F' : C' \to R : q \oplus a \mapsto - F_1 (q - q_1)$$

correctly defines a function on C' if $q_1 \in Q'$ is such that there exist

$f \in Q^*$ satisfying $q_1 \oplus f \in R$ and $\pi(q_1 \oplus f) = a$. We define I' as the subspace generated by F'. From (4) and Corollary 4.2 it follows that ${}^t\rho_{I'}(0 \oplus A^*) = {}^t\pi'(0)$ and ${}^t\rho_I(0 \oplus 0) = ({}^t\pi'(A))^{\S}$ and, consequently, $\pi_A \circ \rho_{I'} = \pi'$. It follows that $\pi = \pi_A \circ \rho_I$, where $I = I' \circ I''$. It is evident that (I, R) is equivalent to (I, R'), where $R' = (R \cap P') \oplus (Q'' \oplus 0)$. Condition (5) implies that (I, R') is an admissible control system and, consequently, (π, R) is an admissible control mode.

Q.E.D.

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