Direct methods in the generalized potential theory

Nota di PAWEL URBANSKI (*)
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Riassunto. *Si veda l'introduzione.*

Introduction

The generalized potential theory is a generalization of electro-and-magnetostatics and is closely connected with the theory of harmonic forms on Riemannian manifolds. The method of integral equations and the method of orthogonal projections are main tools in this theory ([1]-[3], [5], [12]). In the present paper we propose an approach which is related to direct (variational) methods in the theory of elliptic boundary value problems ([5], [6]). The main idea of this method is to treat different boundary value problems as continuous, selfadjoint mappings from certain hilbertizable spaces to their dual spaces. In contrast to the method of orthogonal projection no use of Riemannian structure is made and difficulties typical for methods based on the analysis of unbounded operators are avoided (compare with [7], [8]). This results in a clearer conceptual structure and simpler proofs of fundamental theorems ([7], [8]). The method presented here is based on the general approach to linear field theories outlined in references [9], [10].

This work is a contribution to programme of symplectic formulations of field theories conducted jointly with Professor Tulczyjew.

1. A geometric framework for the generalized potential theory

Let us consider the de Rham complex on a smooth, real manifold M of dimension n:

(*) Division of Mathematical Methods in Physics, University of Warsaw - Hoza 74, 00-682 Warszawa, Poland.

$$0 \to R \to C^0(M) \xrightarrow{d} C^1(M) \xrightarrow{d} \dots \to C^n(M) \to 0$$

and the dual complex of densities ([4]):

$$0 \leftarrow \Gamma^0(M) \xleftarrow{\delta} \Gamma^1(M) \xleftarrow{\delta} \dots \leftarrow \Gamma^n(M) \leftarrow 0.$$

Let Ω be a compact, n-dimensional submanifold of M with a boundary and let k be an integer, $0 \leqslant k \leqslant n$. $\hat{X}(\hat{Y})$ will denote the hilbertizable space of L^2 k-forms (k-densities) on M. There is a canonical duality between $\hat{X}$ and $\hat{Y}$: for $B \in \hat{X}$ and $H \in \hat{Y}$ we define

$$\langle B, H \rangle = \int_{\Omega} B \lrcorner H.$$

Thus $\hat{Y}$ can be identified with the space dual to $\hat{X}$.

Let $X(Y)$ be a hilbertizable space of $(k-1)$-forms ($(k+1)$-densities) on Ω such that smooth forms (densities) are dense in $X(Y)$ and the operator $d: X \to \hat{X}$ ($\delta: Y \to \hat{Y}$) is continuous. The closure in $X(Y)$ of the subspace of smooth forms (densities) which vanish at the boundary will be denoted by $X_0(Y_0)$. The de Rham complex and the complex of densities induce sequences of continuous operators

$$X \xrightarrow{d} \hat{X} \xrightarrow{\delta_0^*} Y_0^*, \quad X_0 \xrightarrow{d_0} \hat{X} \xrightarrow{\delta^*} Y^*$$

and

$$X^* \xleftarrow{d^*} \hat{Y} \xleftarrow{\delta_0} Y_0, \quad X_0^* \xleftarrow{d_0^*} \hat{Y} \xleftarrow{\delta} Y$$

where stars denote dual spaces and conjugate operators, $d_0(\delta_0)$ is the restriction of $d(\delta)$ to $X_0(Y_0)$. It is evident that $\delta^* d_0 = 0$ and $d^* \delta_0 = 0$.

Definition 1.1 A space X of $(k-1)$-forms is *admissible* if subspaces $d(X)$ and $d(X_0)$ are closed in $\hat{X}$. A space Y of $(k+1)$-densities is admissible if subspaces $\delta(Y)$ and $\delta(Y_0)$ are closed in $\hat{Y}$.

Examples:

i) If Ω is a domain with a smooth boundary then Sobolev spaces of forms and densities with the index 1 are admissible.

ii) If Ω is a Lipshitz domain then for any Riemannian structure on M the space $H(d, \delta)$ of L^2-forms such that their differential and codifferential are L^2-forms is admissible ([7]).

Proposition 1.2 Let $X_i\,(i=1,2)$ be admissible spaces of $(k-1)$-forms and let d_i denotes the operator d with the domain X_i. There is a canonical identification of $d_1^*(\hat{Y})$ and $d_2^*(\hat{Y})$ as topological vector spaces.

Proof. We have $\operatorname{im} d_1 = \operatorname{im} d_2$ and, consequently, $\ker d_1^* = (\operatorname{im} d_1)^\circ = (\operatorname{im} d_2)^\circ = \ker d_2^*$. It follows that $d_1^*(\hat{Y}) = d_2^*(\hat{Y})$. Since $d_i^*(\hat{Y})$ is closed in X_i^* induced mappings

$$d_i^* : \hat{Y}/\ker d_i^*(\hat{Y}) \to d_i^*(\hat{Y})$$

are isomorphisms and the needed equality follows.

Q.E.D.

Similar arguments show that images of d_0^*, δ^*, δ_0^* do not depend on the choice of X and Y.

2. Boundary value problems

Let $0 \leqslant k \leqslant n$ remains fixed. The generalized potential theory deals with linear, continuous and selfadjoint mappings

$$\Lambda : X \to X^*$$

where X is as in Section 1. These mappings have form

$$\Lambda = d^* \underline{\Lambda} d$$

where $\underline{\Lambda} : \hat{X} \to \hat{Y}$ is a linear, continuous and strictly positive mapping. The mapping $\underline{\Lambda}$ is the differential of the action function

$$\hat{L} : \hat{X} \to R : B \to \int \mu \circ B$$

μ is a fibre preserving quadratic mapping of $\Lambda^k T^* M$ into the bundle of scalar densities.

Theorem 2.1. i) $\ker \Lambda = \ker d$, ii) $\operatorname{im} \Lambda = (\ker d)^\circ$ if and only if $\operatorname{im} d$ is closed in $\hat{X}$.

Proof. Since $\underline{\Lambda}$ is positive $\ker \Lambda = \ker d$. Since Λ is selfadjoint $(\ker d)^\circ$ is the closure of $\operatorname{im} \Lambda$. But $\operatorname{im} \Lambda$ is closed if and only if the norm induced by Λ on $X/\ker d$ is equivalent to the norm of a quotient space. Because

$\underline{\Lambda}$ is strictly positive and

$$d : X/\ker d \to \hat{X}$$

is an injection, these norms are equivalent if and only if $d(X)$ is closed in $\hat{X}$. Q.E.D.

Corollary 2.2 If X is admissible, then the equation $\Lambda A = J$ has a solution if and only if $J \in (\ker d)^{\circ}$.

A similar theorem is true if we replace Λ by $d_0^* \underline{\Lambda} d_0$. Frequently the following system of equations is considered:

$$(*) \qquad \begin{aligned} d^* \underline{\Lambda} B &= J \\ \delta_0^* B &= m \ . \end{aligned}$$

If X and Y are admissible, then for $m \in (\ker \delta_0)^{\circ}$ and $J \in (\ker d)^{\circ}$ there exist $B_1 \in \hat{X}$ and $A \in X$ such that $\delta_0^* B_1 = m$ and $A = J - d^* \hat{\Lambda} B_1$. Since $\delta_0^* d = 0$ we have, that $b = dA + B_1$ is a solution of (*). Thus we have

Theorem 2.2. The problem (*) has a solution if and only if $J \in (\ker d)^{\circ}$, $m \in (\ker \delta_0)^{\circ}$. The kernel of the problem is isomorphic to the quotient space $\ker \delta_0^* / \operatorname{im} d$.

The problem (*) corresponds to the Neumann boundary value problem in the generalized potential theory. The space $\ker \delta_0^* / \operatorname{im} d$ is isomorphic to the space of the Rham *k-th* cohomology ([11]). The Dirichlet boundary value problem

$$\begin{aligned} d_0^* \underline{\Lambda} \ B &= j \\ \delta^* B &= M \end{aligned}$$

can be considered in a similar way.

3. Application to magnetostatics ($n = 3$, $k = 1$).

Let Ω be a domain in R^3 with a smooth boundary. We identify forms and densities with vector fields and functions. For smooth fields J can be represented by a pair (j, j_s) where j is a vector field on Ω and j_s is a vector field on $\partial\Omega$, tangent to $\partial\Omega$. The Neumann boundary value problem is the following

$$\operatorname{curl} H = j$$
$$\operatorname{div} B = m$$
$$H = \hat{\mu} B \quad \text{in } \Omega$$
$$H \times n = j_s \quad \text{on} \quad \partial\Omega .$$

n is the unit vector normal to the boundary and μ is the symmetric mapping corresponding to the quadratic form μ. The Dirichlet boundary value problem is the following

$$\operatorname{curl} H = j$$
$$\operatorname{div} B = m$$
$$H = \hat{\mu} B \quad \text{in } \Omega$$
$$B \times n = m_s \quad \text{on} \quad \partial\Omega .$$

Theorem 2.2 gives criteria for the existence of solutions of the problems. In particular, since the Sobolev space $H^1(\Omega; R^3)$ is admissible, $j \in (\ker d)$ and $(m, m_s) \in (\ker \delta^*)$ if the following conditions are satisfied

$j \in L^2(\Omega; R^3)'$, $\operatorname{div} j = 0$ and j is orthogonal to the generators of the de Rham first cohomology group,

$(m, m_s) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ and $\int_\Omega m - \int_{\partial\Omega} m_s = 0.$

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