DIFFERENTIABLE STRUCTURE IN A CERTAIN CLASS OF WHEELER'S SUPERSPACES

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It is shown that in a certain class of superspaces, namely in the set of all complete Riemannian metrics on a finite-dimensional manifold, one can introduce differentiable structure modelled on $\mathscr{F}$-S-space.

1. Introduction

During the last years a considerable growth of interest in the problem of quantization of gravitational theories may be observed. This interest was raised mainly by the work of Wheeler, who opened a new way of attacking the problem. This consist in considering a new kind of structures, which he called superspaces. The principle ideas may be found in Wheeler [5], Bergmann [1] and others. The weak point of those considerations was that they neglect the question of existence of a differentiable structure in the superspace. This is what we aim to do in this note. We shall take into account the simplest case where the superspace considered is the set of (all) complete Riemannian metrics (only the complete metrics are physically interesting) on a finite-dimensional manifold M. A structure of the differentiable manifold, modelled on a Fréchet–Schwartz space, is introduced on that superspace. Differentiation is understood in the sense of [3].

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2. Complete metrics on M

Let M be a paracompact C^∞-manifold. It is well known [4] that there exists a positive-definite complete Riemannian metric on it. Let us denote by $\mathscr{P}$ the set of all those metrics. We wish to introduce a differentiable structure in $\mathscr{P}$. At first let us recall some well-known facts.

LEMMA 1. *(M, g) is a complete manifold if and only if every bounded and closed set is compact.*

LEMMA 2. *(M, g) is a complete manifold if and only if (M, g) is geodesically complete.*

Assume now that g' and g are complete metrics on M. Since the set is compact and closed independently of the choice of metric, we have:

LEMMA 3. *The set $V \subset M$ is g-bounded if and only if it is g'-bounded.*

Thus if (M, g) is complete and we want (M, g') to be complete, then the implication

$$(V \text{ is } g'\text{-bounded}) \Rightarrow (V \text{ is } g\text{-bounded})$$

must hold, or equivalently, *for each $r'>0$ there exists $r=f(r')$ such that for a certain point $x_0 \in M$ we have the implication*

$$(x \in K'(x_0, r')) \Rightarrow (x \in K(x_0, r)).$$

K and K' denote balls in (M, g) and (M, g'), respectively. We can choose the function $r=f(r')>0$ to be strictly increasing.

The above condition can be written in one of the following forms:

a) *for each $r'>0$, there exists $r=f(r')>0$ such that*

$$d'(x, x_0) \leqslant r' \quad \text{implies} \quad d(x, x_0) \leqslant f(r');$$

b) *there exists a strictly increasing function $f: R^+ \to R^+$ such that $d(x, x_0) \leqslant f(d'(x, x_0))$;*

c) *there exists a strictly increasing function $\tilde{f}: R^+ \to R^+$ such that $\tilde{f} \xrightarrow[r\to\infty]{} \infty$ and $d'(x, x_0) \leqslant \tilde{f}(d(x, x_0))$;*

d and d' denote the distance in (M, g) and (M, g'), respectively; obviously one can choose $\tilde{f} = f^{-1}$.

THEOREM 1. *Let g be a complete metric on M and $g' \geqslant \varepsilon^2 g$ for certain $\varepsilon>0$, explicitly that means: for each vector field u satisfying $g'(u, u) \geqslant \varepsilon^2 g(u, u)$; then (M, g') is complete.*

Proof: We have $\sqrt{g'(u, u)} \geqslant \varepsilon \sqrt{g(u, u)}$ and there from $\|\gamma\|' \geqslant \varepsilon \|\gamma\|$. $\|\gamma\|'$ and $\|\gamma\|$ denote the length of the smooth curve γ in the metric corresponding to g' and g, respectively. In particular, if γ is a geodesics in (M, g') connecting x and x_0, then

$$d'(x, x_0) = \|\gamma\|' \geqslant \varepsilon \|\gamma\| \geqslant \varepsilon d(x, x_0).$$

The last inequality follows from the fact that in (M, g) and $(M, \varepsilon g)$ geodesics are identical. Taking into account point b) above, the proof follows.

We put $h := g' - g$; then for $h > \delta g$, $\delta > -1$, (M, g') is complete.

3. Differentiable structure in $\mathscr{P}$

For each $g \in \mathscr{P}$, we introduce the set T_g, consisting of all 2-covariant C^∞ tensor-fields h, satisfying the condition $|h| < \varepsilon g$ for certain $\varepsilon > 0$. T_g is obviously a vector space.

In T_g we introduce a topology of uniform convergence with respect to g and almost uniform convergence of all derivatives. It is easy to see that T_g equipped with this topology is a Fréchet–Schwartz space. On the sets

$$T_g \supset U_g^\varepsilon := \{h \in T_g : |h| < \varepsilon g\} \quad (\varepsilon < 1)$$

we define embeddings (see Theorem 1):

$$T_g \supset U_g^\varepsilon \ni h \to \kappa_g^{-1}(h) = g + h \in \mathscr{P}.$$

Those mappings determine a topology in $\mathscr{P}$.

LEMMA 4. *If* $g+h=\tilde{h}+f$, *where* $\tilde{h} \in U_g^\varepsilon \subset T_g$, $\tilde{h} \in U_f^\delta \subset T_f$ $(f, g \in \mathscr{P})$, *then* $g \in T_f$ *and* $f \in T_g$.

Proof: From the above, $g=f-h+\tilde{h}$, therefore

$$|g| \leqslant |f|+|h|+|\tilde{h}| \leqslant (1+\delta)|f|+\varepsilon|g| \quad \text{and} \quad g \leqslant \frac{1+\delta}{1-\varepsilon} f \quad (\varepsilon<1).$$

PROPOSITION. $T_f=T_g$ *as topological vector spaces.*

Let us take two maps κ_g and κ_f such that the mapping $\kappa_f \circ \kappa_g^{-1}: T_g \to T_f$ is well defined. From Lemma 4 it follows that $T_f=T_g$. Since $\kappa_f \circ \kappa_g^{-1}$ is a translation, it is differentiable. This completes the proof of the following theorem.

THEOREM 2. *The triplet* $(\mathscr{P}, T, K)$, *where* $T=\bigcup_{g \in \mathscr{P}} T_g$, $K=\bigcup_{g \in \mathscr{P}} \kappa_g$, *is a differentiable* ($\mathscr{F}$-S)-*manifold of the class* C^∞ *in the sense of* [3].

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