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Differentiable structure in a conjugate vector bundle of infinite dimension

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CONTENTS

INTRODUCTION

This paper is a continuation of research devoted to the rigorous development of calculus of variation ([14], [15]). Owing to the studies begun by Eells and his school and continued by Kijowski and Komorowski many important families of functions have been equipped with the (natural) differentiable structure, e.g. the set $\mathscr{V}$ of smooth sections over compact domains of a bundle V is a C^∞ manifold modelled on the F-S space.

Such an approach has enabled Palais and Smale to work out the general Morse theory, and Kijowski and Komorowski to formulate the strict Lagrange formalism in classical field theory.

So far we have dealt with smooth fields and smooth local functionals. However, in classical field theories one is forced to deal with singularities of fields (e.g. an electromagnetic field of a single particle), singularities of Lagrangian and non-local effects (e.g. the scale microeffect [31], the memory effect [31], interacting fields [23]).

These needs force us to consider theories in which both singular fields and singular non-local functionals appear. We can do this in the case of a vector bundle V, taking section-distributions [24] instead of sections.

If V is a vector bundle, $\mathscr{V}$ is also a vector bundle and the family of section-distributions forms a conjugate bundle $\mathscr{V}'$. Hence, the natural problem is that of equipping $\mathscr{V}$ with a differentiable structure. Although $\mathscr{V}'$ cannot be equipped with a topology, it carries a natural structure of a manifold over Marinescu spaces ([11]). To this goal we solve a more general problem of existence of a differentiable structure in a conjugate bundle to a C^k-vector bundle modelled on F-S spaces (Chapter III). For this purpose we develop the differential calculus in the Marinescu spaces which are products of normed Marinescu spaces and infrabarrelled DF locally convex vector spaces (Chapters I and II).

As an example we consider in Chapter IV the bundle of section-distributions.

By the Lagrange density function we mean the function of bundles

$\mathscr{V} \ni v \rightarrow L(v) \in \mathscr{V}'$. A Lagrangian associated with L is a function $\mathscr{V} \ni v \rightarrow \mathscr{L}(v) := \langle L(v), v \rangle$. It is shown that if L is of class C^1 then $\mathscr{L}$ is of class C^1.

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CHAPTER I

DIFFERENTIATION IN CARTESIAN PRODUCTS OF NORMED AND DF-TYPE INFRABARRELLED SPACES

In Chapter III we shall deal with manifolds modeled on spaces which are in some sense "families" of products of normed and DF-type infrabarrelled locally convex vector spaces (l.c.v. spaces). In this chapter we shall outline the theory of differentiation in such products. This theory is based on W. Szczyrba's theory of differentiation in metrizable, quasi-normable and DF-S l.c.v. spaces.

§ 1. Preliminaries. We shall deal with locally convex vector spaces which are assumed to be Hausdorff. We shall denote them by block letters E, F, G. The field of scalars will be real or complex and fixed throughout the whole paper. $\mathcal{N}(E)$ will denote the base of absolutely convex neighbourhoods of zero in E. If $U \in \mathcal{N}(E)$ then $\| \ \|_U$ will denote the gauge function of U, and E_U will denote the normed space $E/_{N(U)}$ where $N(U) := \{e \in E: \|e\|_U = 0\}$ with the norm induced by $\| \ \|_U$. The symbol $\mathscr{B}(E)$ will stand for the family of all absolutely convex, closed and bounded subsets of E. If $B \in \mathscr{B}(E)$ then E_B will denote the linear span of B with the topology defined by B. The symbols E'_s and E'_b will denote the weak and the strong conjugate, respectively, to E spaces. The spaces of all continuous linear, bilinear, and n-linear mappings from $E, E \times E, E \times \ldots \times E$ into F will be denoted by $L(E, F)$, $L(E, E; F)$ and $L^n(E, F)$, respectively.

§ 2. Fundamental definitions. Let T be a mapping $T: E \supset \Omega \to F$, where Ω is open in E.

DEFINITION. We say that the mapping $T: E \supset \Omega \to F$ *has the Gâteaux (weak) derivative at a point* $e_0 \in \Omega$ if there exists a linear mapping $\nabla T(e_0) \in L(E, F)$ such that the mapping

$$E \supset U \ni h \to r_{e_0}(h) := T(e_0 + h) - T(e_0) - \nabla T(e_0) h \in F$$

for each $h \in U$ has the following property:

$$\lim_{t \to 0} \left(\frac{1}{t} r_{e_0}(th) \right) = 0, \quad \text{where } t \in \boldsymbol{K} \ (\boldsymbol{R} \text{ or } \boldsymbol{C}),\ U \in \mathcal{N}(E) \text{ and } e_0 + U \subset \Omega.$$

The definition above is identical with the definition of the Gâteaux derivative in normed spaces. The Fréchet derivative may be defined in many different ways, but we will use the definition given in Marinescu [20] and Keller [12].

DEFINITION. We say that the mapping $T\colon E \supset \Omega \to F$ is *differentiable in the sense of Fréchet at the point* $e \in \Omega$ if there exists $L \in L(E, F)$ such that the mapping

$$E \supset \Omega \ni h \to r_e(h) := T(e+h) - T(e) - L(h) \in F$$

has the following property: for each $V \in \mathcal{N}(F)$ there exists $U \in \mathcal{N}(E)$ such that for each M.-S. sequence $\{h_\lambda\}_{\lambda \in \Lambda}$, $h_\lambda \to 0$ in E, $\lim \frac{\|r_e(h_\lambda)\|_V}{\|h_\lambda\|_U} = 0$.

Remark. If $\|h\|_U = \|r_e(h)\|_V = 0$ we put $\frac{\|r_e(h)\|_V}{\|h\|_U} = 0$.

The mapping L is defined uniquely (if it exists) and is denoted by $T'(e)$ or $DT(e)$. We say that L is the *Fréchet derivative of T at the point e*. We say that the derivative T' is *continuous at e_0* if the mapping $e \to T'(e) \in L_b(E, F)$ is continuous at e_0. From the definition immediately follows

THEOREM 1.

a) *A mapping which is differentiable in the sense of Fréchet at the point e is continuous at e.*

b) *A mapping differentiable in the sense of Fréchet at the point e has the Gâteaux-derivative at this point and $T'(e) = \nabla T(e)$.*

c) *If T and T_1 are differentiable at the point e, then $aT + bT_1$ $(a, b \in K)$ is differentiable at e and*

$$(aT + bT_1)'(e) = aT'(e) + bT_1'(e).$$

d) *If the mapping $T_1\colon E \supset \Omega_1 \to F$ is differentiable at e and $T_2\colon F \supset \Omega_2 \to G$ is differentiable at $T_1(e)$, then $T_2 \circ T_1$ is differentiable at e and $(T_2 \circ T_1)'(e) = T_2'(T_1(e)) \circ T_1'(e)$.*

Remark. In what follows by differentiability we shall understand differentiability in the sense of Fréchet. J. Kijowski and W. Szczyrba in ([13], [28]) have constructed a theory of differentiation in a certain class of l.c.v. spaces.

In this paper we shall base ourselves on the theory of W. Szczyrba for two classes of l.c.v. spaces: metrizable, and quasi-normable or Schwartz of DF-type (DF-S) l.c.v. spaces.

Modifying the theorems given in [28] we shall obtain the theory of differentiation in the Cartesian product of normed and infrabarrelled DF-spaces.

DEFINITIONS.

a) We say that an l.c.v. space E is *quasi-normable* if for each equicontinuous set $A \subset E'$ there exists $V \subset \mathcal{N}(E)$ such that on A the topology induced by E'_b coincides with the topology of uniform convergence on V.

b) We say that an l.c.v. space E is *DF-space* if the following conditions are fulfilled:

1° E admits a countable basis of bounded sets.

2° Every bounded set in E'_b which is a countable sum of equicontinuous sets is also equicontinuous.

Property 2° is called *σ-infrabarrelledness.*

c) We say that an l.c.v. space E is *Schwartz* (S) if for each $U \in \mathcal{N}(E)$ there exists $V \in \mathcal{N}(E)$ such that V is precompact in E_U.

§ 3. Certain properties of mappings in some l.c.v. space. We shall give below some theorems corresponding to theorems due to W. Szczyrba (theorems 2 & 14 in [28]).

PROPOSITION 1. *Let E_2 be a normed l.c.v. space and E_1 and G—any l.c.v. space. Suppose we are given a mapping*

$$T\colon E_1 \times E_2 \supset \Omega \times E_2 \to G$$

satisfying the following conditions:

1° *for each $e \in \Omega$, $T(e, \cdot) \in L(E_2, G)$,*

2° *the mapping $\Omega \ni e \to T(e, \cdot) \in L_b(E_2, G)$ is continuous at e_0.*

Then T is continuous at (e_0, h) $(h \in E_2)$.

COROLLARY. *Let E_1 and F be any l.c.v. spaces, and E_2 a normed l.c.v. space. Then $L_b(E_1, L_b(E_2, F)) = L_b(E_1, E_2; F)$.*

Proof. We have $L(E_1, E_2; F) \subset L(E_1, L_b(E_2, F))$. Let $f \in L(E_1, L_b(E_2, F))$. From the proposition it follows that the mapping $(e_1, e_2) \to f(e_1)e_2$ is continuous, hence algebraic equality. The equivalence of topologies is obvious. ∎

If we want to change the roles of E_1 and E_2 in Proposition 1, we must assume more about the mapping T.

DEFINITION. We say that a mapping $T\colon E \supset \Omega \to F$ *is locally bounded* on Ω if for each $e \in \Omega$ there exists $V \in \mathcal{N}(E)$ such that, for each $B \in \mathscr{B}(E)$, $T((e+V) \cap B)$ is bounded in F.

DEFINITION. A mapping $T\colon E \supset \Omega \to F$ is called *quasi-locally bounded* at $e \in \Omega$ if for each $B \in \mathscr{B}(E)$ there exist $\varepsilon > 0$ and $B_1 \in \mathscr{B}(F)$ such that $T(e + \varepsilon B) \subset B_1 + T(e)$.

Lemma 1. *Let* E_1 *be normed,* E_2*—infrabarrelled, and* F*—any l.c.v. space. Suppose we are given a mapping*

$$T\colon E_1 \times E_2 \supset \Omega \times E_2 \to F$$

satisfying following properties:

1° *for each* $e \in \Omega$ $T(e, \cdot) \in L(E_2, F)$,

2° *for each* $W \in \mathcal{N}(F)$ *the mapping* $\Omega \ni e \to T(e, \cdot) \in L_b(E_2, F_W)$ *is continuous and quasi-locally bounded at* e_0.

Then T *is continuous at* (e_0, h) $(h \in E_2)$.

Proof. Because of infrabarrelledness and quasi-local boundedness, for each $W \in \mathcal{N}(F)$ there exists $U \in \mathcal{N}(E_1)$ such that the family $T(e_0 + U, \cdot) \subset L(E_2, E_W)$ is equicontinuous. Hence the continuity of T at $(e_0, 0)$. From the continuity the proof follows. ■

Remark. It is easy to see that in Lemma 1 it is sufficient to assume the continuity of the function

$$e \to T(e, \cdot) \in L_s(E_2, F).$$

Corollary. *Let* E_1 *be normed,* E_2*—infrabarrelled, and* F*—any l.c.v. space. Then*

$$L_b(E_1, E_2; F) = L_b\big(E_1, L_b(E_2, F)\big).$$

Proof. Let $f \in L_b\big(E_1, L_b(E_2, F)\big)$. This function is bounded, and so Lemma 1 implies that the mapping

$$E_1 \times E_2 \ni (e_1, e_2) \to f(e_1) e_2$$

is continuous. Further as in the Corollary to Proposition 1. ■

Now let us recall some well-known facts concerning l.c.v. space.

Proposition 2 ([9]). *Every infrabarrelled DF-space is quasi-normable.*

Lemma 2 ([9]). *An l.c.v. space* E *is quasi-normable if and only if for every* $U \in \mathcal{N}(E)$ *there exists* $V \in \mathcal{N}(E)$ *such that for every* $\lambda > 0$ *we can find* $B \in \mathcal{B}(E)$ *satisfying the relation* $V \subset \lambda U + B$.

We can modify the above lemma:

Lemma 2′ ([28]). *Let* E *be a quasi-normable l.c.v. space, and* Ω *a neighbourhood of zero in* E, $Y \in \mathcal{N}(E)$ *and* $Y + Y \subset \Omega$. *Then for every* $\lambda > 0$ *there exists a bounded set* $B \subset \Omega$ *with* $V \subset \lambda Y + B$.

Lemma 3 ([28]). *Let* E *be a DF-space,* $(U_i)_{i=1}^{\infty}$ *a sequence of neighbourhoods of zero in* E. *Then there exists* $V \in \mathcal{N}(E)$ *such that there exists a sequence of real numbers* $\lambda_i > 0$ *satisfying the relation* $\lambda_i U \subset V_i$.

With the above lemmas we can prove a theorem of fundamental importance for the following considerations. It will be a simple modification of W. Szczyrba's theorem (theorem 14′ in [28]).

THEOREM 2. *Let E_1 be normed, E_2 and E_3 infrabarrelled of DF-type, and F — any l.c.v. space. Let T be a mapping $T: E_1 \times E_2 \times E_3 \supset \Omega \times E_3 \to F$ satisfying the following conditions:*

1° for each $e = (e_1, e_2) \in \Omega$, $T(e, \cdot) \in L(E_3, F)$,

2° for each $W \in \mathcal{N}(F)$ there exists an open neighbourhood O_W of the point (e_{01}, e_{02}) such that the mapping $O_W \ni (e_1, e_2) = e \to T(e, \cdot) \in L_b(E_3, F)$ is continuous at the point $e_0 = (e_{01}, e_{02})$ and uniformly continuous with respect to the second variable on O_W,

3° for any bounded set $B \subset O_W$ the family $T(B, \cdot)$ is bounded in $L_b(E_3, F_W)$.

Then the mapping T is continuous at (e_0, h) $(h \in E_3)$.

Proof. Let the sequence $(B_i)_{i=1}^{\infty}$ form a basis of bounded sets in E_3. From the uniform continuity it follows that for any $W \in \mathcal{N}(F)$ and any B_i there exists $V_i \in \mathcal{N}(E_2)$ such that for $e \in O_W$ and $h \in B_i$

$$(*) \qquad \sup_{y \in V_i} \left\| T\big(e + (0, y), h\big) - T(e, h) \right\|_W \leqslant 1.$$

From Lemma 3 we infer that there exist $V \in \mathcal{N}(E)$ and a sequence $\lambda_i > 0$ such that $\lambda_i V \subset V_i$. Now we can choose open sets $\Omega_W = \Omega'_W \times \Omega''_W$ and V such that $V + V + e_{02} \subset \Omega''_W$. From Lemma 2′ there exist $\tilde{V} \in \mathcal{N}(E_2)$ and a bounded set A_a, for any $a > 0$, such that $\tilde{V} \subset aV + A_a$. So we have for a (bounded) $U \in \mathcal{N}(E_1)$ contained in Ω'_W the relation

$$e_0 + U \times V \subset e_0 + U \times A_a + \{0\} \times (aV).$$

Taking $A_i = A_{\lambda_i}$ from (*) we have:

$$\sup_{h \in B_i} \sup_{e \in U \times A} \|T(e_0 + e, h)\|_W$$
$$\leqslant \sup_{h \in B_i} \sup_{e \in U \times A_i} \sup_{y \in \lambda_i V} \left\| T\big(e_0 + e + (0, y), h\big) - T(e_0 + e, h) \right\|_W + \sup_{h \in B_i} \sup_{e \in U \times A_i} \|T(e_0 +$$
$$+ e, h)\|_W \leqslant 1 + \sup_{h \in B_i} \sup_{e \in U \times A_i} \|T(e_0 + e, h)\|_W < \infty.$$

The last inequality holds because of 3° and the boundedness of $U \times A_i \subset \Omega_W$.

In other words, the family $\{T(e_0 + e, \cdot)\}_{e \in U \times V}$ is bounded in $L_b(E_3, F_W)$ and, because of the infrabarrelledness, equicontinuous. Hence we have the continuity of T at $(e_0, 0)$. The continuity at (e_0, h) $(h \in E_3)$ is the result of the continuity of the mapping

$$e \to T(e, \cdot) \in L_b(E_3, F) \text{ at } e_0. \blacksquare$$

§ 4. Mean value theorems. The theorems below are based on similar results (cf. [28]) in the case of quasi-normable, metrizable or DF-S l.c.v. spaces.

THEOREM 3. *Let E_1 be a normed and E_2 an infrabarrelled DF-space, $E = E_1 \times E_2$, and F—any l.c.v. space. Let $T\colon E \supset \Omega \to F$ be a mapping continuously differentiable at the point $e_0 \in \Omega$. Let D_2T (i.e. the partial derivative in the direction of E_2) satisfy the assumptions of Theorem 2. Then for each $V \in \mathcal{N}(F)$ there exist $U, W \in \mathcal{N}(E)$ such that for every $e \in e_0 + U$, $h \in W$ the following is true:*

$$\|T(e+h)-T(e)\|_V \leqslant C\|h\|_W, \quad \text{where } C = \sup_{k,s\in W}\|T'(e+k)s\|_V < \infty.$$

Proof. Let us write $e_0 = (e_{01}, e_{02})$, $e = (e_1, e_2)$, $h = (h_1, h_2)$ etc. For any $W \in \mathcal{N}(E)$, $V \in \mathcal{N}(F)$, we have the following relation ([3], [5]):

$$\|T(e+h)-T(e)\|_V \leqslant \sup_{0<\theta<1}\|T'(e+\theta h)h\|_V \leqslant \sup_{k,s\in W}\|T'(e+s)k\|_V\|h\|_W$$

(the last inequality holds for $h \in W$). Also

$$\|T'(e+s)k\|_V \leqslant \|D_1T(e+s)k_1\|_V + \|D_2T(e+s)k_2\|_V.$$

From Proposition 1 and Theorem 2 it follows that there exist $\tilde{U} \in \mathcal{N}(E)$, $W_1 \in \mathcal{N}(E_1)$ and $W_2 \in \mathcal{N}(E_2)$ with

1° $\sup_{k_1\in W_1}\sup_{s\in\tilde{U}}\|T'(e_0+s)k_1\|_V < \infty$,

2° $\sup_{k_2\in W_2}\sup_{s\in\tilde{U}}\|T'(e_0+s)k_2\|_V < \infty$.

Hence, by taking $U+U \subset \tilde{U}$, $W \subset W_1 \times W_2$ and $W \subset U$, the proof follows. ■

Remark. In the proof of the above theorem we used the continuity of D_2T at the point $(e_0, 0)$ only.

The next theorem deals with the problem of estimating the remainder.

PROPOSITION 3 ([28]). *Let E be a quasi-normable and F a normed l.c.v. space. Let A be an equicontinuous subset of $L(E, F)$. Then there exists $V \in \mathcal{N}(E)$ such that on A the topology induced by $L_b(E, F)$ coincides with the topology of uniform convergence on V.*

THEOREM 4. *Let the assumptions of Theorem 3 be fulfilled. Let T' be continuous in a neighbourhood of e_0. Then for any $V \in \mathcal{N}(F)$ there exist $U, W \in \mathcal{N}(E)$ such that for each $e \in e_0 + U$, $Y \in \mathcal{N}(E)$ and $h \in Y$ the following relation holds:*

$$\|r_e(h)\|_V \leqslant C_e(Y)\|h\|_W,$$

where

$$C_e(Y) = \sup_{k\in Y}\sup_{s\in W}\|T'(e+k)s - T'(e)s\|_V$$

and

$$\lim_{Y\in\mathcal{N}(E)} C_e(Y) = 0.$$

Proof. Let us consider the mapping $T(e+h) = T(e+h) - T'(e)h$. As in the proof of Theorem 3, we have

$$\|T(e+h)-T(e)-T'(e)h\|_V \leqslant \sup_{0<\theta<1} \|T'(e+\theta h)h - T'(e)h\|_V$$
$$\leqslant \sup_{k\in Y}\sup_{s\in W} \|T'(e+k)s - T'(e)s\|_V \|h\|_W \qquad (h\in Y).$$

Because of Proposition 1 and Theorem 2 the following families of functions are equicontinuous for a certain $U\in\mathcal{N}(E)$:

$$A_1 = \{D_1T(e_0+e)\}_{e\in U} \subset L(E_1, F_V),$$
$$A_2 = \{D_2T(e_0+e)\}_{e\in U} \subset L(E_2, F_V).$$

But on A_i $(i = 1, 2)$ the topology induced from $L_b(E_i, F_V)$ coincides with the topology of uniform convergence on a certain $W_i\in\mathcal{N}(E_i)$ (Proposition 3). Taking $W = W_1\times W_2$, from the continuity it follows that

$$\sup_{s\in W}\|T'(e+k)s - T'(e)\|_V \xrightarrow[k\to 0]{} 0. \blacksquare$$

One can easily see that in the above theorem (and also in Theorem 3) we have taken advantage of the existence and continuity of the Gâteaux derivative only. Thus Theorem 4 may be interpreted as a theorem giving the existence of a Fréchet derivative of the continuously Gâteaux-differentiable function. Hence one can obtain the theorem on partially differentiable functions, but we are going to prove it independently in a more efficient way.

THEOREM 5. *Let E be normed, and F and G — any l.c.v. spaces. If a mapping*

$$T\colon E\times F\supset \Omega\to G$$

is partially differentiable at (e_0, f_0) and the derivative in the direction of E is continuous in a neighbourhood of (e_0, f_0), then T is differentiable at (e_0, f_0).

Proof. Let us write $h\in E$, $s\in F$. The remainder may be put in the following form:

$$(*)\qquad T(e_0+h, f_0+s) - T(e_0, f_0) - T'(e_0, f_0)(h, s)$$
$$= D_1T(e_0, f_0+s)h - D_1T(e_0, f_0)h + \overset{1}{r}_{(e_0,f_0+s)}(h) + \overset{2}{r}_{(e_0,f_0)}(s).$$

By the theorem on remainders in normed spaces there exist $U, W \in \mathcal{N}(E)$ such that

$$\|\overset{1}{r}_{(e_0,f_0+s)}(h)\|_V \leqslant \sup_{k\in V}\sup_{t\in W}\|D_1T(e_0+k, f_0+s)t - D_1T(e_0, f_0+s)\|_V\|h\|_W = a(s,t)\|h\|_W.$$

By the continuity of D_1T, $a(s,t)\xrightarrow[s,t\to 0]{}0$. Similarly,

$$\|D_1T(e_0, f_0+s)h - D_1T(e_0, f_0)h\|_V \leqslant \sup_{t\in W}\|D_1T(e_0, f_0+s)t - D_1(e_0, f_0)t\|_V\|h\|_W = \beta(s)\|h\|_W$$

and $\beta(s)\xrightarrow[s\to 0]{}0$.

Since $\overset{2}{r}_{(e_0,f_0)}(s)$ is a remainder, the whole expression (*) is a remainder. ∎

Remark. In the above theorem the normability of E and continuity of D_1 may be replaced by the assumptions of another "remainder theorem" (e.g. in the case of DF-infrabarrelled, DF-S, metrizable and quasi-normable spaces, etc. cf [28]).

Assuming the continuity of D_2 in Theorem 5 we obtain the necessary and sufficient conditions for the continuity of T'.

§ 5. Differentiation of a superposition. In the following we shall prove a theorem important in the differential calculus.

THEOREM 6. *Let E_1, F_1 be normed, and E_2 and F_2 infrabarrelled DF-l.c.v. spaces $E = E_1\times E_2$, $F = F_1\times F_2$.*

Let G be any l.c.v. space. Let the functions

$$T_1: E\supset \Omega_1\to F\,(e_0\in\Omega_1), \qquad T_2: F\supset\Omega_2\to G$$

$(f_0 = T_1(e_0)\in\Omega_2)$ *satisfy the following conditions:*

1° *T_1 and T_2 are continuously differentiable at e_0 and f_0 respectively,*

2° *for each $W\in\mathcal{N}(G)$ the mapping $O_W \ni e\to D_2T_2(e)$ is uniformly continuous with respect to the second variable on a neighbourhood O_W of f_0,*

3° *for each bounded subset $B\subset O_W$, $D_2T_2(B)$ is bounded in $L_b(F_2, G_W)$.*

Then the mapping $T_2\circ T_1$ is continuously differentiable at e_0.

Proof. By Theorem 1 it is enough to prove the continuity of the derivative. We have:

$$\begin{aligned}(T_2\circ T_1)'(e) &= T_2'(T_1(e))\circ T_1'(e)\\ &= D_1T_2(T_1(e))\circ D_1T_1(e) + D_1T_2(T_1(e))\circ D_2T_1(e)\\ &\quad + D_2T_2(T_1(e))\circ D_1T_1(e) + D_2T_2(T_1(e))\circ D_2T_1(e).\end{aligned}$$

Let us fix $W\in\mathcal{N}(G)$. By Proposition 1 and Theorem 2 there are $V\in\mathcal{N}(F)$, $V_1\in\mathcal{N}(F_1)$, $V_2\in\mathcal{N}(F_2)$ such that

$$\|D_1T_2(f_0+h)s_1\|_V < \tfrac{1}{4} \quad\text{and}\quad \|D_2T_2(f_0+h)s_2\|_V < \tfrac{1}{4}$$

for $h\in V$, $s_1\in V_1$, $s_2\in V_2$.

Because of the continuity of T_1 and T_1', for each bounded set $B \in \mathscr{B}(E)$ there exists $U \in \mathscr{N}(E)$ such that $T_1(e_0+U) \subset f_0+V$ and $T_1(e_0+U)B \subset V_1 \times V_2$. Hence the proof follows immediately. ∎

Remark. One can easily notice that by assuming in Theorem 6 the following additional conditions:

1° the mappings T_1 and D_2T_1 are locally bounded at e_0,

2° T_1 is uniformly continuous on a certain open set $\mathcal{O} \ni e_0$ with respect to the second variable,

3° for each $W \in \mathscr{N}(G)$ the mapping $O_W \ni e \to T_2'(e) \in L_b(F, G_W)$ is locally bounded at f_0 and uniformly continuous on some neighbourhood $O_W \ni f_0$,

4° for each $V \in \mathscr{N}(F)$ the mapping $O_W \ni e \to D_2T_1 \in L_b(E, F_V)$ is uniformly continuous with respect to the second variable on some neighbourhood $O_V \ni e_0$,

one infers that for each $W \in \mathscr{N}(G)$ the mapping $O_W \ni e \to D_2T_2 \circ T_1 \in L_b(E, G_W)$ is locally bounded at e_0 and uniformly continuous with respect to the second variable on some neighbourhood $O_W \ni e_0$.

§ 6. Higher order derivatives. By Proposition 1 and Theorem 2, $L_b(E, L_b(E, F)) = L_b(E, E; F)$, where $E = E_1 \times E_2$, E_1 is a normed and E_2 an infrabarelled DF-l.c.v. space, and F is any l.c.v. space. By induction we can prove an analogous equality in the case of n-linear mappings. Thus the nth derivative is a symmetric, n-linear function.

Using a generalization of Proposition 3 to the case of a multilinear mapping, one can prove theorems on "the higher order derivative", "higher order remainders", "the Taylor formula", etc.

We are not going to formulate and prove them because they can be obtained as a simple modification of the corresponding theorems in the case of metrizable, quasi-normable spaces (cf [28]).

CHAPTER II

DIFFERENTIAL CALCULUS IN MARINESCU SPACES

In this chapter we shall be dealing with the theory of differentiation in a certain class, important for applications, of pseudotopological spaces ("Limesräume" of Fischer [7]), the so-called Marinescu spaces or "Unions of topological spaces" ("reunion pseudotopologique" cf [21], [11]). A "Union of topological spaces" is a generalization of spaces investigated and applied by several authors: L. Waelbrock ("espaces à bornes" cf [30]), J. Sebastiao e Silva ("réunions d'espaces normés" cf [25], [26]), J. Mikusiński ("réunions d'espaces de Banach" cf [22]), B. H. Arnold [1], M. F. Suhynin ("politopological spaces" [27]) and others. Except for [27] all those concepts are special cases of the notion of "espace vectoriel bornologique" investigated by H. Hogbe-Nlend and his group in Bordeaux. The basic concepts of this theory can be found in [10].

The theory which we are going to present coincides with that of [10] and [4] (in the case of "espaces bornologiques convexes"). Close to ours is the concept of a differential function due to J. Sebastiao e Silva [25], E. Dubinsky [6] and M. F. Suhynin. On the other hand, it has no interesting analogy with the theories of Fröhlicher & Bucher [8] and A. Bastiani [2].

§ 1. Basic concepts and definitions.

DEFINITION ([11], [21]). A *Marinescu space* (M-space) is a vector space E with the family $\{E_\lambda\}_{\lambda\in\Lambda}$ of l.c.v. spaces with the following properties:

1° Λ is a directed set,

2° for each $\lambda\in\Lambda$, E_λ is a subspace of E and $E=\bigcup_{\lambda\in\Lambda}E_\lambda$,

3° for each pair $(\lambda_1,\lambda_2)\in\Lambda\times\Lambda$ there exists $\lambda\in\Lambda$ such that $E_\lambda\supset E_{\lambda_i}$ and the injections $E_{\lambda_i}\to E_\lambda$ are continuous.

$\{E_\lambda\}_{\lambda\in\Lambda}$ ($\{E_\lambda\}$ if it does not cause any misunderstandings) and E will be used simultaneusly as a symbol of an M-space. In the following we assume all E_λ to be Hausdorff.

EXAMPLES.

1. If $E_\lambda=(E,\tau_\lambda)$, we say that E is *politopological*.
2. If each E_λ is normed, then E is called a *bornological* (*convex*) *space*.

3. $E = (E, \tau)$. The M-space $\{E_B\}_{B\in\mathscr{B}(E)}$ is called a *bornological Von Neumann space associated with E* and will be denoted by $\boldsymbol{B}E$.

Remark. In the definition of an M-space the convexity assumptions are not necessary but we are not interested in such a generalization.

Definition. A Moore–Smith sequence $\{x_\nu\}_{\nu\in N}$ in $\{E_\lambda\}_{\lambda\in A}$ *converges* to $x_0 \in E$ if there exists $\lambda \in A$ such that for each $U \in \mathcal{N}(E_\lambda)$ there exists $\nu \in N$ such that $(\nu' > \nu) \Rightarrow (x_{\nu'} \in x_0 + U)$.

Remark. In the above definition we can put $x_0 + U \subset E_\lambda$.

Definition. A mapping $T: \{E_\lambda\} \to \{F_\nu\}$ is *continuous at* $x_0 \in E$ if $T(x_\lambda) \to (x_0)$ for every M.-S. sequence $x_\lambda \to x_0$.

Lemma 4. *A mapping $T: \{E_\lambda\} \to \{F_\nu\}$ is continuous at x_0 iff for every λ such that $x_0 \in E_\lambda$ there exist ν and a neighbourhood $U \subset E_\lambda$ $(x_0 \in U)$ such that $T(U) \subset F_\nu$ and the mapping $T|_U: E_\lambda \supset U \to F_\nu$ is continuous at x_0.*

Proof. 1° At first we shall demonstrate that there exist $U \subset E_\lambda$ and ν such that $T(U) \subset F_\nu$. Assume that there exists an E_λ such that for each F_ν and $U \subset E_\lambda$ $(x_0 \in U)$ there exists $x \in U$ with $T(x) \notin F_\nu$. Taking a basis $\{U_\beta\}$ of neighbourhoods of x_0 in E_λ, we obtain a sequence $x_{\beta\nu}$ converging to x_0. But $T(x_{\beta\nu})$ does not converge in F.

2° Similarly we prove continuity. If, for each U and ν, $T: E_\lambda \supset U \to F_\nu$ is discontinuous at x_0, then for each ν and β there exists $x \in U_\beta$ with $T(x) \notin U_\nu \subset F_\nu$ $(U_\nu \ni T(x_0))$. This gives a contradiction (as in 1°). ■

Remarks. 1° The assumption $x_0 \in E_\lambda$ may be omitted by taking the affine spaces $x_0 + E_\lambda$.

2° From the proof of Lemma 4 it follows that in the case of normed E_λ it is not sufficient to look upon continuity as sequence continuity. It is sufficient if the family $\{F_\nu\}$ is countable. A counterexample can be found in [10].

In the same manner as in the category of l.c.v. space one can define products, direct limits and other induced structures (cf. [11]).

Examples.

1. Let $\{E_\lambda\}_{\lambda\in A}$ and $\{F_\nu\}_{\nu\in N}$ be M-spaces. The set $E \times F$ equipped with the structure of an M-space defined by $\{E_\lambda \times F_\nu\}_{(\lambda,\nu)\in A\times N}$ is called the (Cartesian) product of M-spaces E and F.

2. One can easily notice that $\{E_\lambda\}_{\lambda\in A} = \varinjlim E_\lambda$ (direct limit of M-spaces).

3. $\{E_\lambda\} \times \{E_\lambda\} = \{E_\lambda \times E_\lambda\}$.

§ 2. Differentiation in Marinescu spaces. Let $E = \{E_\lambda\}$. The space E can be equipped in a natural way with two topologies:

1° The topology of a direct limit of l.c.v. spaces E_λ. E with this topology will be denoted by TE.

2° The topology of a direct limit of topological spaces E_λ. tE will stand for E with this topology. The topology of tE is a larger topology preserving the convergence of sequences which converge in E.

Remarks.

1. tE may not be a topological vector space (cf. [10], [1]).

2. The sequences converging in tE do not converge in general in $\{E_\lambda\}$ (cf [16]).

By a neighbourhood in E we mean one in tE.

PROPOSITION 4. *Let U be a neighbourhood of x_0 in $\{E_\lambda\}$. Then, for each λ, $U\cap(x_0+E_\lambda)$ is a neighbourhood in $E_\lambda+x_0$.*

DEFINITION. A mapping $r\colon \{E_\lambda\}\supset\Omega\to\{F_\nu\}$ is a *remainder* if for each λ there exists ν such that $r\colon E_\lambda\cap\Omega\to F_\nu$ is a remainder in the sense of an l.c.v. space.

DEFINITION. A mapping $T\colon E\supset\Omega\to F$ is *differentiable at $x\in\Omega$* if there exists $L\in L(E,F)$ (the collection of all linear and continuous mappings from E to F) such that the mapping

$$r_x(h) := T(x+h)-T(x)-Lh \quad \text{is a remainder.}$$

Notice that L is defined uniquely. The mapping L will be called the (Fréchet) *derivative of T at x* and denoted by $T'(x)$. The definition immediately implies

THEOREM 7. 1° *Every mapping differentiable at x is also continuous at x.*

2° *A linear combination of differentiable functions is also differentiable and its derivative is a linear combination of derivatives.*

3° *A superposition of differentiable functions is also differentiable and*

$$(T_2\circ T_1)'(x) = T_2'\big(T_1(x)\big)\circ T_1'(x).$$

DEFINITION. We say that a sequence of mappings $T_\sigma\colon E\to\{F_\nu\}$ *converges uniformly on $A\subset E$* if there exists ν such that $T_\sigma(A)\subset F_\nu$ and the sequence $T_\sigma\colon A\to F_\nu$ converges uniformly on A.

DEFINITION. A set $B\subset\{E_\lambda\}$ is called *bounded* if $B\subset E_\lambda$ and B is bounded in E_λ for a certain λ.

Now we can introduce the structure of an M-space in the space $L(E,F)$ (compare with [11]).

Let Λ and N be directed sets. Let us denote by Π the set of all monotonic functions $\pi\colon\Lambda\to N$ equipped with the relation $(\pi_1>\pi_2)\Leftrightarrow\big(\text{for each } \lambda\in\Lambda\ \pi_1(\lambda)>\pi_2(\lambda)\big)$.

Let $\{E_\lambda\}_{\lambda\in\Lambda}$ and $\{F_\nu\}_{\nu\in N}$ be M-spaces. For each $\pi\in\Pi$ we define the set

$$L_\pi(E,F) = \{f\in L(E,F)\colon f(E_\lambda)\subset F_{\pi(\lambda)} \text{ and } f\colon E_\lambda\to F_{\pi(\lambda)} \text{ is continuous}\}.$$

Of course $L_\pi(E, F)$ is a vector space, $\bigcup_{\pi\in\Pi} L_\pi(E, F) = L(E, F)$ and for each pair (π_1, π_2) there exists $\pi \succ \pi_i$ $(i = 1, 2)$ such that $\big(L_{\pi_1}(E, F) \cup L_{\pi_2}(E, F)\big) \subset L_\pi(E, F)$.

Let $B = \{B_\lambda\}_{\lambda\in\Lambda}$ be a family of sets such that B_λ is bounded in E_λ and $O_\nu \in \mathcal{N}(F_\nu)$. The sets

$$U_O^B := \{f\in L_\pi(E, F): f(B_\lambda) \subset O_{\pi(\lambda)}\}$$

form a basis for the locally convex topology in $L_\pi(E, F)$. One can easily examine that $L(E, F) = \{L_\pi(E, F)\}_{\pi\in\Pi}$ is an M-space. We shall denote it by $L_b(E, F)$.

In the same manner we define the M-space $L_s(E, F)$.

Proposition 5. *The convergence in $L_b(E, F)$ is uniform on bounded sets.*

Now we can introduce the concept of a continuously differentiable function and higher order derivatives.

Definition. We say that a mapping $T: \{E_\lambda\} \supset \Omega \to \{F_\nu\}$ defined on an open set Ω is *continuously differentiable at $e_0 \in \Omega$* if T is differentiable on a certain neighbourhood O of e_0 and the mapping $E \supset O \ni e \to T'(e) \in L_b(E, F)$ is continuous at e_0.

Definition. Let T be as above. We say that T is *differentiable twice at e_0* if the mapping

$$E \supset O \ni e \to T'(e) \in L_b(E, F)$$

is differentiable at e_0.

Higher order derivatives are defined by induction.

Theorem 8. *A mapping $T: \{E_\lambda\} \supset \Omega \to \{F_\nu\}$ is differentiable at $e_0 \in \Omega$ iff for each λ there exist ν and $O_\lambda \in \mathcal{N}(E_\lambda)$ such that*

$$T(e_0 + O_\lambda) \subset T(e_0) + F_\nu$$

and the mapping $T: e_0 + O_\lambda \to T(e_0) + F_\nu$ is differentiable at e_0.

Proof. $\Leftarrow$ Let $T(e_0 + O_\lambda) \subset T(e_0) + F_\nu$ and let the mapping $T: e_0 + O_\lambda \to T(e_0) + F_\nu$ be differentiable at e_0. This means that

$$T(e_0 + h) - T(e_0) = T'_{\lambda,\nu}(e_0)h + r_\lambda(h), \quad \text{where } T'_{\lambda,\nu} \in L(E_\lambda, F_\nu).$$

Now it is enough to prove that $T'_{\lambda',\nu'}(e_0)$ coincides with $T'_{\lambda,\nu}(e_0)$ on E_λ for $\lambda' \succ \lambda$ (we can assume that $\nu' \succ \nu$). Accordingly, notice that on $e_0 + O_\lambda$ the mapping T may be treated as a mapping into $F_{\nu'} + T(e_0)$ (differentiable of course). But $r_{\lambda'}|_{E_\lambda}: E_\lambda \to F_{\nu'}$ is a remainder, and so by the uniqueness of the derivative in l.c.v. space the proof follows.

$\Rightarrow$ The proof is obvious. ■

COROLLARY. *A mapping* $T\colon \{E_\lambda\} \supset \Omega \to \{F_\nu\}$ *has the continuous derivative at* $e_0 \in \Omega$ *iff for each* λ *there exist* ν *and* $O_\lambda \in \mathcal{N}(E_\lambda)$ *such that*

$$T\colon e_0 + O_\lambda \to F_\nu + T(e_0)$$

has a continuous derivative at e_0.

Proof. By Theorem 8 it suffices to prove that the mapping $\Omega \ni e \to T'(e) \in L_b(E, F)$ is continuous at e_0 iff for every λ there exist ν and $O_\lambda \in \mathcal{N}(E_\lambda)$ such that the mapping $E_\lambda \supset O_\lambda + e_0 \ni e \to T'_{\lambda,\nu} \in L_b(E_\lambda, F_\nu)$ is continuous.

$\Rightarrow$ Let the mapping $\Omega \ni e \to T'(e) \in L_b(E, F)$ be continuous at e_0. By Lemma 4 this means that for each λ there exist π and $O_\lambda \in \mathcal{N}(E_\lambda)$ such that $O_\lambda + e_0 \ni e \to T'(e) \in L_\pi(E, F)$ is continuous at e_0. But $T'(e)|_{E_\lambda} = T'_\lambda(e)$, and so we have continuity of the mapping $e \to T_{\lambda,\pi(\lambda)}(e) \in L_b(E_\lambda, F_{\pi(\lambda)})$.

$\Leftarrow$ Let the mapping $T\colon e_0 + O_\lambda \to T'_{\lambda,\nu(\lambda)}(e) \in L_b(E_\lambda, F_{\nu(\lambda)})$ be continuous. Let us fix λ_0 and put $\pi(\lambda)$ such that $\pi(\lambda) \succ \nu(\lambda)$ and $\pi(\lambda) \succ \pi(\lambda_0)$. As in the proof of Theorem 8, we have $T'_{\lambda,\nu}|_{E_{\lambda'}} = T'_{\lambda',\nu}$ for $\lambda \succ \lambda'$. By the definition of $L_\pi(E, F)$, the mapping $e_0 + O_{\lambda_0} \ni e \to T'(e) \in L_\pi(E, F)$ is continuous at e_0. ∎

In the same way we can prove a corresponding theorem for higher order derivatives. We may also prove a collection of theorems corresponding to theorems of differential calculus in l.c.v. space, according to the kind of E_λ, F_ν (see Chapter I). The procedure is similar in all cases and we are going to give a few examples only. At first let us introduce some concepts.

DEFINITION. A Marinescu space $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be *metrizable* if, for every $\lambda \in \Lambda$, E_λ is metrizable.

In the same manner we introduce the notions of DF, Schwartz, Fréchet, complete, ... — Marinescu spaces.

Remark. In the above definition the expression "for every $\lambda \in \Lambda$ there exists $\lambda' > \lambda$ such that" may be put instead of "for every $\lambda \in \Lambda$ ...".

Now, the examples:

LEMMA 5. *Let* E *be an infrabarrelled-DF-M-space. Let* F *be any M-space. Then*

$$L_b^n(E;\ F) = L_b\big(E, L_b(E, \ldots L_b(E, F) \ldots)\big).$$

Proof. Put $n = 2$. Of course $L\big(E, L_b(E, F)\big) \subset L(E, E;\ F)$. $f \in L\big(E, L_b(E, F)\big)$ generates the mapping $f_\lambda \in L\big(E_\lambda, L_\pi(E, F)\big)$ and $\tilde{f}_\lambda \in L\big(E_\lambda, L_b(E_\lambda, F_{\pi(\lambda)})\big)$. By Theorem 1 $L_b\big(E_\lambda, L_b(E_\lambda, F_{\pi(\lambda)})\big) = L_b(E_\lambda, E_\lambda; F_{\pi(\lambda)})$. Thus f defines an element of $L(E, E; F)$. The equivalence of M-structures is obvious.

For $n > 2$ the proof follows by induction. ∎

COROLLARY. *An n-th derivative is an n-linear, symmetric and continuous function.*

The definition of the Gâteaux derivative for l.c.v. space is valid also for M-spaces.

THEOREM 9. *Let E_1 be normed, E_2—quasibarrelled-DF, any F any M-spaces. $E = E_1 \times E_2$. Let a mapping $T\colon E \supset \Omega \to F$ be continuously Gâteaux-differentiable on a certain neighbourhood of e_0. If for each λ there exist O_λ and ν such that $E_\lambda \supset O_\lambda$ and $T(e_0+O_\lambda) \subset T(e_0)+F_\nu$, then T is continuously differentiable at e_0.*

Proof. By the assumptions, for each λ there exist O_λ and ν such that the mapping $T\colon e_0+O_\lambda \to F_\nu + T(e_0)$ is continuously Gâteaux-differentiable at e_0. Thus it is differentiable at e_0 (Theorem 4). By Theorem 8 the proof follows. ∎

§ 3. Differential calculus in bornological-Von Neumann spaces. In this part we shall be dealing with differentiation in the special case of M-spaces (bornological Von Neumann spaces) and its connections with differentiation in l.c.v.s.

We start with some lemmas concerning continuous functions.

LEMMA 6. *Let E be a bornological l.c.v. space, and F—any l.c.v. space. Then $L_b(\boldsymbol{B}E, \boldsymbol{B}F) = \boldsymbol{B}L_b(E, F)$.*

Proof. Let $f \in L(E, F)$. For each $B \in \mathscr{B}(E)$ there exists $B_1 \in \mathscr{B}(F)$ such that $f\colon E_B \to F_{B_1}$ is bounded and continuous. Thus $f \in L(\boldsymbol{B}E, \boldsymbol{B}F)$. Now let $f \in L(\boldsymbol{B}E, \boldsymbol{B}F)$; then $f\colon E \to F$ is bounded. Hence we obtain algebraic equality. The equivalence of M-structures follows from the fact that in $L_b(E, F)$ sets $\{f \in L(E, F)\colon f(B) \subset B_1\}$ (B and B_1 are fixed) form a basis of bounded sets. ∎

In further considerations the concept of "Mackey convergence condition" will be useful.

DEFINITION ([9]). We say that an l.c.v. space E satisfies the *strict Mackey convergence condition* (M.c.c.) if for every bounded set A there exists $B \in \mathscr{B}(E)$ such that the topology induced by E on A coincides with the topology induced on A by the normed space E_B.

LEMMA 7 ([9]). *A metrizable l.c.v. space satisfies the strict Mackey convergence condition.*

$C_B(E, F)$ will stand for the space of all continuous, quasilocally bounded functions $f\colon E \to F$.

PROPOSITION 6. *Let E be a metrizable l.c.v. space and let F satisfy M.c.c. Then $C_B(E, F) = C(\boldsymbol{B}E, \boldsymbol{B}F)$.*

Proof. $C_B(E, F) \subset C(\boldsymbol{B}E, \boldsymbol{B}F)$ because F satisfies M.c.c. $C_B(E, F) \supset C(\boldsymbol{B}E, \boldsymbol{B}F)$ because the converging sequence in E convergs also in $\boldsymbol{B}E$ (Lemma 7). ∎

COROLLARY. *If, in Proposition 6, E is an F-S (Fréchet-Schwartz) space, then*

$$C(E,F) = C(\boldsymbol{B}E, \boldsymbol{B}F).$$

Proof. In an F-S space every closed bounded set is compact. Thus $C_B(E,F) = C(E,F)$. ■

LEMMA 8. *Let E be a quasinormable and infrabarrelled and F a normed l.c.v. space. Then $L_b(E,F)$ satisfies M.c.c.*

Proof. By the infrabarrelledness of E it follows that a set $B \in \mathscr{B}(L_b(E,F))$ is equicontinuous. Thus the topology in $[L_b(E,F)]_B$ is the topology of uniform convergence on a certain $U \in \mathscr{N}(E)$. By quasinormability the topology induced by $L_b(E,F)$ on $[L_b(E,F)]_B$ is the topology of uniform convergence on a certain $V \in \mathscr{N}(E)$. By taking $B_1 = \{f\colon \|f(e)\| \leqslant a, e \in V\}$ where a is such that $B_1 \supset B$ the proof follows. ■

With Lemma 8 we can prove the following important

THEOREM 10. *Let E be a quasi-normable and infrabarrelled, and F any metrizable l.c.v. space. Then $L_b(E,F)$ satisfies M.c.c.*

Proof. Let B be a bounded subset of $L_b(E,F)$. It is enough to prove that there exists a bounded set $B_1 \subset L_b(E,F)$ such that for each $\lambda > 0$ there exists $V \in \mathscr{N}(L_b(E,F))$ with $V \cap B \subset \lambda B_1 \cap B$ (because the topology induced by a bounded set is finer than that induced by $L_b(E,F)$). Let $\{U_i\}_{i=1}^{\infty}$ form a basis of absolutely convex neighbourhoods of zero in F.

B is bounded iff it is bounded in $L_b(E,F_{U_i})$ $i = 1, 2, \ldots$ Thus by Lemma 8 the topology induced on B by $L_b(E,F_{U_i})$ coincides with the topology induced on B by $[L_b(E,F_{U_i})]_{B_i}$, where $B \subset B_i \subset L_b(E,F_{U_i})$ and B_i are bounded.

Hence the topology induced by $L_b(E,F)$ on B is given by sets $B \cap (\varepsilon B_i)$ where $\varepsilon > 0$ $i = 1, 2, \ldots$ Let $\lambda_i > 0$ form a sequence $\lambda_i \to \infty$. This means that for each $\lambda > 0$ there exists $n > 0$ such that, for $i > n$, $\lambda_i > \dfrac{1}{\lambda}$ $(\lambda\lambda_i > 1)$.

The set $B_1 = \bigcap_{i=1}^{\infty} \lambda_i B_i$ is bounded in $L_b(E,F)$, and if $x \in B$, $x \in \bigcap_{i=1}^{n} \lambda_i B_i$, then $x \in \lambda B_1 \cap B$. But

$$\Big\{B \cap \Big(\bigcap_{i=1}^{n} \lambda\lambda_i B_i\Big)\Big\} \supset \{V \cap B\}, \quad \text{where} \quad V \in \mathscr{N}\big(L_b(E,F)\big). \ ■$$

Using the following facts:

1° if E is a quasi-normable, and F a normed l.c.v. space, then the topology induced by $L_b^n(E,F)$ on a equicontinuous set $A \subset L^n(E,F)$ coincides with the topology of uniform convergence on $V \times V \times \ldots \times V$, where $V \in \mathscr{N}(E)$ (cf [28]), 2° if E is an infrabarrelled, and F a normed l.c.v. space, then a bounded set $A \subset L^n(E, F)$ is equicontinuous,

one can also prove in a similar way

THEOREM 10′. *Let E be a quasi-normable and infrabarrelled, and F a metrizable l.c.v. space. Then $L_b^n(E, F)$ satisfies M.c.c.*

Now we can prove some relations between differentiable functions on E and $\boldsymbol{B}E$.

THEOREM 11. *Let E be an F-S l.c.v. space. Let F satisfy the strict Mackey convergence condition. If a mapping $T\colon E \supset \Omega \to F$ is differentiable at $e_0 \in \Omega$ and continuous on $O \ni e_0$, then the mapping $T\colon \boldsymbol{B}E \supset \Omega \to \boldsymbol{B}F$ is differentiable at e_0.*

Proof. By Lemma 6 it suffices to prove that if a mapping $r\colon E \supset \Omega \to F$ is a remainder, then a remainder is also the mapping $r\colon \boldsymbol{B}E \supset \Omega \to \boldsymbol{B}F$. Let us fix $B \in \mathscr{B}(E)$. For each $U \in \mathscr{N}(E)$ there exists $c_U > 0$ such that $\|x\|_U \leqslant c_U \|x\|_B$. Moreover, there exist $B_1 \in \mathscr{B}(F)$ and $\varepsilon > 0$ such that $r(\varepsilon B) \subset B_1$ (by the continuity of r and Proposition 6). Thus

$$\frac{\|r(x)\|_V}{\|x\|_U} \geqslant \frac{\|r(x)\|_V}{c_U \|x\|_B} \quad \text{and} \quad \frac{r(x)}{\|x\|_B} \in F_{B_1}$$

if $x \in \varepsilon B$. This means that the mapping $E_B \cap O \ni x \to \dfrac{r(x)}{\|x\|_B} \in F$ is continuous and there exists a $B_2 \in \mathscr{B}(F)$ such that the mapping $E_B \supset \varepsilon B \ni x \to \dfrac{r(x)}{\|x\|_B} \in F_{B_2}$ is continuous. ■

COROLLARY. *Let E be F-S, and F a metrizable l.c.v. space. If a mapping $T\colon E \supset \Omega \to F$ has the continuous n-th derivative at $e \in \Omega$, then also the mapping $T\colon \boldsymbol{B}E \supset \Omega \to \boldsymbol{B}F$ has the continuous n-th derivative at e.*

Proof. Differentiability follows from Theorems 11 and 10′ and from the fact that an F-S space is barrelled and quasi-normable. The continuity of the derivatives follows from Theorem 10′ and Proposition 6. ■

CHAPTER III

DIFFERENTIABLE STRUCTURE IN A CONJUGATE BUNDLE

In this Chapter we shall discuss definitions of differentiable manifolds (§ 1) and vector bundles (§ 2) modelled on infinite-dimensional, non-banachian l.c.v. space.

Then we shall prove the existence of a differentiable (pseudotopological) structure in the bundle conjugate to the F-S bundle (i.e. the basis and the fiber are F-S spaces).

§ 1. Non-banachian differentiable manifolds. In the differential calculus in Banach spaces the class of continuously differentiable functions plays a fundamental role. The theorem on continuous differentiability of superpositions of C^1 functions is also valid in it.

In view of this fact one can define a differentiable manifold modelled on Banach spaces ([14], [18]).

DEFINITION. A differentiable manifold of class C^k is a *triplet* $(\mathscr{P}, T, K)$, where

1° $\mathscr{P}$ is a certain set,

2° T: $T(\mathscr{P}) := \bigcup_{p \in \mathscr{P}} T_p(\mathscr{P})$, where $T_p(\mathscr{P})$ are Banach spaces,

3° $K := \bigcup_{p \in \mathscr{P}} K_p$ where K_p is a non-empty set of bijections which map subsets containing p on neighbourhoods of zero in $T_p(\mathscr{P})$,

4° the following axioms are satisfied:

a) if $\varkappa \in K_p$, then $\varkappa(p) = 0$,

b) if $\varkappa_1, \varkappa_2 \in K$, then the mapping $\varkappa_2 \circ \varkappa_1^{-1}$ is a C^k-diffeomorphism defined on an open set,

c) the set K is complete in the sense that every larger set does not satisfy the axioms (a) and (b).

The set K is called a (*complete*) *atlas* and its elements—*maps*. With this definition one can introduce the notion of a continuously differentiable function on a manifold ([18]) and develop differential calculus on it.

We have a similar situation in the case of metrizable, quasi-normable l.c.v. space ([28]). For the same reasons we can adopt the definition of the manifold given above. However, in the case of DF-S spaces (and the spaces we have considered in Chapter I) some complications appear. From Chapter I it is known that in the calculus the most important role in such spaces is played by continuously differentiable functions with additional assumptions as to the kind of boundedness and uniform continuity. Also in the theorem on derivative of superposition something more than continuouity is needed.

Taking into account the above-mentioned facts, we shall define a manifold modelled on spaces which are Cartesian products of a normed space and a DF-infrabarrelled one in the following manner.

Definition. Let E_1 be a normed, E_2 a DF-infrabarrelled, and F any l.c.v. space $E = E_1 \times E_2$. We say that a mapping $T: E \supset \Omega \to F$ is of *class* $\bar{C}^k$ *on* Ω if the following conditions are satisfied:

1° T is of class C^k on Ω,

2° $T^{(i)}$ is locally bounded on Ω, $0 \leqslant i \leqslant k$,

3° for each $W \in \mathcal{N}(F)$ the mapping $\Omega \ni e \to T^{(i)}(e) \in L_b^i(E, F_W)$ is locally uniformly continuous on Ω.

Now, we define a manifold modelled on such Cartesian products as in the case of Banach spaces, puting $\bar{C}^k$ instead of C^k.

Remarks.

1. $\bar{C}^k$ is the smallest class of functions closed under superposition and containing all functions important in calculus.

2. We have made no use of the normability of E_1. The same definition is valid in the case of general DF-infrabarrelled spaces.

All these considerations are valid for M-spaces. In other words, we can define a manifold modelled on M-spaces of types considered above by changing the expression "open in T_p" into "open in tT_p".

§ 2. Infinite-dimensional vector bundles. By a "vector bundle" we shall understand a triple $(\mathscr{P}, X, \pi)$ where $\pi: \mathscr{P} \to X$ is a surjection and, for each $x \in X$, $\pi^{-1}(x)$ is equipped with the structure of a vector space. We shall assume that X and $\mathscr{P}$ are topological spaces and $\mathscr{P}$ is locally trivial. If X is a (banachian) C^k-manifold and $\pi^{-1}(x)$, $x \in X$, are Banach spaces, we can introduce the notion of C^k-bundle. Let $\{O_a\}_{a \in A}$ be a covering of X with a corresponding trivialization $t_a: \pi^{-1}(O_a) \to P_a \times O_a$. P_a is a Banach space. We can choose $\{O_a\}$ such that each O_a is contained in the domain of a certain map $\varkappa_a$

Let us define the mapping $\eta_a: \pi^{-1}(O_a) \to \varkappa_a(O_a) \times P_a$ by the following: if $t_a(p) = (\pi(p), e)$, $e \in P_a$, then $\eta_a(p) = (\varkappa_a \circ \pi(p), e)$.

Definition. A vector bundle $(\mathscr{P}, X, \pi)$ *is of class* C^k if X is a banachian C^k-manifold and, for each a_1, a_2, $\eta_{a_2} \circ \eta_{a_1}^{-1}$ is of class C^k (on the set on which it is defined) and an isomorphism of an l.c.v. space with respect to the second variable.

Of course $\{\eta_a\}_{a\in A}$ is an atlas on $\mathscr{P}$. Hence it defines the structure of differentiable C^k-manifold. The definition above is compatible with that given by Lang [18]. Namely: each C^k-bundle in our sense is a C^{k-1}-bundle in the sense of Lang. We can treat non-banachian bundles similarly, puting a corresponding class (e.g. $\bar{C}^k$) instead of C^k. Let us concentrate on the most interesting case for us, where X is a banachian manifold and the fibre is a DF-infrabarrelled l.c.v. space. Taking into account the fact that η_a has a special form, we can change the class $\bar{C}^k$ into a larger one.

Definition. Let E_1 be a metrizable and E_2 a DF-infrabarrelled l.c.v. space $E = E_1 \times E_2$. Let F be any l.c.v. space. We say that a mapping $T\colon E \supset \Omega \to F$ is of *class* $\tilde{C}^k$ *on* Ω if the following conditions are fulfilled:

1° T is of class C^k on Ω,

2° T is locally bounded and locally uniformly continuous with respect to the second variable on Ω,

3° for each $W \in \mathscr{N}(F)$ the mapping $\Omega \ni e \to D_1^i D_2^j T(e)$ is locally uniformly continuous and locally bounded on Ω for $1 \leqslant i+j \leqslant k$, $j \neq 0$.

Now, let $T a_1 a_2$ be the mapping defined by

$$\eta_{a_2} \circ \eta_{a_1}^{-1}(x, e) = \big(\varkappa_{a_2} \circ \varkappa_{a_1}^{-1}(x), T_{a_1 a_2}(x, e)\big).$$

Definition. We say that a vector bundle $(\mathscr{P}, X, \pi)$ *is of class* $\tilde{C}^k$ if the following conditions are satisfied:

1° X is a manifold of class C^k,

2° $\eta_{a_2} \circ \eta_{a_1}^{-1}$ is of class C^k and an isomorphism of the l.c.v.s. in the second variable,

3° $T_{a_1 a_2}$ is of class $\tilde{C}^k$ on its domain of existence.

Remark. From 2° it follows from the definition of $\tilde{C}^k$ that the second part of 2° (uniform continuity) is satisfied.

With this definitions we can consider the differential calculus of morphisms of bundles. In general, η_a do not form an atlas for a manifold of class $\tilde{C}^k$. Thus a $\tilde{C}^k$-bundle is not necessarily a $\tilde{C}^k$-manifold. The above construction may be use if X is a banachian M-space.

§ 3. Conjugate bundle. The aim of this section is to introduce the differentiable structure in a certain class of conjugate bundles.

Definition. A conjugate bundle to a vector bundle $(\mathscr{P}, X, \pi)$ is a bundle $(\mathscr{P}', X, \pi')$ where $\mathscr{P}' = \bigcup_{x\in X} \mathscr{P}_x'$ and $\mathscr{P}_x' = \big(\pi^{-1}(x)\big)'$ is the strong conjugate to $\mathscr{P}_x$ in the sense of l.c.v. spaces and π' is the natural projection.

Let X be a differentiable manifold of class C^k, modelled on an F-S space. By the corollary to Theorem 11, X is a C^k-manifold modelled on the spaces $\boldsymbol{B}T_p$. X equipped with this structure will be denoted by $\boldsymbol{B}X$. Moreover, it is a $\bar{C}^k$-manifold.

Caution. From now on F' will denote F'_b. If a mapping $T\colon E\times G \supset \Omega\times G\to F$ is linear and continuous in the second variable, then $T^*\colon E\times F'\to G'$ will stand for the mapping conjugate to T in the second variable.

Now we can prove the main theorem.

THEOREM 12. *Let E, F, G be an F-S l.c.v. space. Let $T\colon E\times G\supset\Omega\times G\to F$ be of class C^{n+1} and, for each $e\in\Omega$, let $T(e,\cdot)$ be linear and continuous.*

Then the mapping $T^*\colon \boldsymbol{B}E\times F'\supset\Omega\times F'\to G'$ is of class $\tilde{C}^n$.

Proof. I. Continuity and boundedness of T. Let $(x, e)\in\Omega\times F'$. By continuity, for each $B\in\mathscr{B}(E)$ and $B_1\in\mathscr{B}(G)$ there exist $\varepsilon>0$ and $B_2\in\mathscr{B}(F)$ such that, for $h\in\varepsilon B$, $g\in B_1$ and $f\in B_2^0$ (the polar in F'), we have

$$|\langle T(x+h, g), e+f\rangle-\langle T(x, g), e\rangle| = |\langle T(x+h, g), f\rangle+\langle T(x+h, g)-T(x, g), e\rangle|\leqslant 1.$$

Let us take ε and B_2 such that $x+\varepsilon B\subset\Omega$ and $T(x+\varepsilon B, B_1)\subset\frac{1}{2}B_2$ (εB and B_1 are compact in the F-S space). Thus $|\langle T(x+h, g), f\rangle|\leqslant\frac{1}{2}$. On the other hand the mapping $(x, e)\to D_2T(x, e) = T(x, .)\in L_b(G, F)$ is continuous. Hence there exists $\varepsilon>0$ with $|\langle T(x+h, g)-T(x, g), e\rangle|\leqslant\frac{1}{2}$. This means that T^* is continuous.

Remark. It is easy to see that the mapping $T^*\colon E\times F'\to G'$ is not necessarily continuouss (εB cannot be replaced by a neighbourhood in E).

Now we shall prove the boundedness of T^*. By the barrelledness of F and G it is enough to prove that for each $U\in\mathscr{N}(F)$, $\varepsilon>0$ and $B\in\mathscr{B}(E)$ (such that $x+\varepsilon B\subset\Omega$) there exist $V\in\mathscr{N}(G)$ with $T(x+\varepsilon B, V)\subset U$.

But F and G are bornological, and so the sets $U_{\{\lambda_a\}} := \operatorname{conv}\big(\bigcup_a \lambda_a B_a\big)$, where $\{B_a\}$ is a base of bounded sets and $\{\lambda_a\}$ ranges over all sequences of positive numbers, forms a basis of neighbourhoods of zero ($\operatorname{conv}A$ denotes an absolutely convex hull of A). Let $U = \operatorname{conv}\big(\bigcup_a \lambda_a B_a\big)$. From previous considerations we know that $T(x+\varepsilon B, B_\gamma)\subset B_{a(\gamma)}$ ($\{B_\gamma\}$ and $\{B_a\}$ form bases of bounded sets in G and F, respectively). Taking $V = \operatorname{conv}\big(\bigcup_\gamma \lambda_{a(\gamma)}B_\gamma\big)$, we have $T(x+\varepsilon B, V)\subset U$.

II. Differentiability of T. $n=1$. We have $D_2T^*(e, f)(\cdot) = T^*(e,\cdot)$. D_2T^* is continuous on $\Omega\times F'\subset \boldsymbol{B}E\times F'$ if for each $U\in\mathscr{N}(F)$, $B\in\mathscr{B}(G)$, $B_1\in\mathscr{B}(E)$ and $x\in\Omega$ there exists $\varepsilon>0$ such that $T^*(x+\varepsilon B_1, U^0)\subset B^0$, i.e. $T(x+\varepsilon B_1, B)\subset U$. But this is implied by the continuity of D_2T. Uniform continuity follows from the compactness of $x+\varepsilon B$. We prove boundedness as in I. Now let us turn to D_1T. We notice that the mapping L,

$$\Omega\times G\times E\ni(e, g, h)\to D_1T(e, g)h =: L(e, g, h)\in F,$$

is linear and continuous in the second variable, and so the following mappings are well-defined: $D_1^* T\colon \Omega \times F' \times E \to G'$ and $r^* = (T - D_1 T)^*$. We shall show that $D_1 T^* = D_1^* T$. For metrizable spaces Proposition 1 is valid ([28]). Thus L is continuous. The following fact is true:

$$(*) \qquad D_1^* T \in L(E, G').$$

Indeed, it is enough to show that for each $(e, f') \in \Omega \times F'$, $B \in \mathscr{B}(G)$ there exists $V \in \mathscr{N}(E)$ such that $D_1^* T(e, f')(V) \subset B^0$, i.e. $D_1 T(e, B)(V) \subset (f')^0$. But $(f')^0$ is a neighbourhood, by continuity we have the required fact. From this we have also $D_1^* T \in L(\boldsymbol{B}E, G')$.

By $(*)$ it is enough to show that r^* is a remainder. At first we shall prove the following lemma.

LEMMA 10. *Let E be a normable l.c.v. space and let a mapping $E \times G \times E \supset \Omega_1 \times G \times \Omega_2 \ni (e, g, h) \to r_{(e,g)}(h) \in F$ be linear and continuous in the second variable. For each $U \in \mathscr{N}(F)$ and $h_\lambda \to 0$ let $\frac{\|r_{(e,g)}(h_\lambda)\|_U}{\|h\|} \to 0$ uniformly on bounded sets in G. Then $r^*_{(e,f')}$ is a remainder.*

Proof. We have to show that the mapping $h \to \frac{r^*_{(e,f')}(h)}{\|h\|}$ is continuous at zero, i.e., for each $B \in \mathscr{B}(G)$,

$$\frac{1}{\|h_\lambda\|} \sup_{g \in B} |\langle r^*_{(e,f')}(h_\lambda), g\rangle| = \frac{1}{\|h_\lambda\|} \sup_{g \in B} |\langle f', r_{(e,g)}(h_\lambda)\rangle| \xrightarrow[h_\lambda \to 0]{} 0 \ (f' \in F'). \ \blacksquare$$

Let us fix $B \in \mathscr{B}(E)$ and $x \in \Omega$ and consider the mapping $T_B\colon = T\colon (E_B + x) \times G \supset (U_B + x) \times G \to F$ $(U_B + x \subset \Omega)$. T_B is continuously differentiable, and so, by the mean value theorem, in normed spaces for each $V \in \mathscr{N}(F)$

$$\|r_{(e,g)}(h)\|_V \leqslant \sup_{s \in B} \sup_{k \in Y} \|D_1 T(e+k, g)s - D_1 T(e, g)s\|_V \|h\|_B.$$

Since $D_2 D_1 T$ is continuous, we have for each $B_2 \in \mathscr{B}(G)$

$$\sup_{s \in B} \sup_{g \in B_1} \|D_1 T(e+k, g)s - D_1 T(e, g)s\|_V \xrightarrow[k \to 0]{} 0.$$

Thus, by Lemma 10, r^* is a remainder. The continuity of the derivative can be proved in the same way as the continuity of T.

III. $n > 1$. By induction as for $n = 1$. $\blacksquare$

Remark. T^* from the above theorem is not in general of class $\bar{C}^n$ (even for bilinear T^*).

Summing up the results of this section we obtain

THEOREM 13. *Let $(\mathscr{P}, X, \pi)$ be a C^{k+1}-vector bundle. Let X and the fibres be F-S. Then the bundle $(\mathscr{P}', \boldsymbol{B}X, \pi')$ is of class $\tilde{C}^k$.*

CHAPTER IV

THE BUNDLE OF SECTION-DISTRIBUTIONS

In this chapter we shall equip the bundle of section-distributions with a differentiable structure by applying the construction developed in Chapter III. These bundles appear in a natural way in the field theories dealing with singular fields or singular functionals. As an example we shall verify the differentiability of some functionals describing interacting fields.

§ 1. The bundle of section-distributions. Let X denote a C^∞-manifold of dimension $n+1$.

DEFINITION. An n-dimensional (imbedded) C^∞-submanifold of X is called a *border* if it is the boundary of a relatively compact domain in X.

$\mathscr{P}$ denotes the set of all borders in X.

If $\pi\colon V\to X$ is smooth vector bundle, then by $\mathscr{V}$ we denote the bundle over $\mathscr{P}$ whose fibre $\mathscr{V}_P$ $(P = \partial D)$ is formed by all C^∞-sections over D, prolongable to an open neighbourhood of D. The bundle $\mathscr{V}'$ is called the *bundle of section-distributions*. By the theorem of Whitney (cf. [19]) $\mathscr{V}_P$ with the topology of uniform convergence of all derivatives on D is complete. Thus it is an F-S space. Hence in order to equip $\mathscr{V}'$ (which is the object of our interest) with a differentiable structure it is enough to prove the existence of a differentiable structure (with an F-S base) on $\mathscr{V}$ (Theorem 13). But this was done by Kijowski and Komorowski [15] (see also [14] and [17]).

Now, let us recall the main ideas of their construction.

A. Differentiable structure in $\mathscr{P}$.

DEFINITION. By a *transversal homotopy* H through a border $P\in\mathscr{P}$ we mean a C^∞-diffeomorphism

$$P\times]-r, r[\ni(p, t)\to H(p, t)\in X,\quad H(p, 0) = p,\ r>0.$$

$\mathfrak{H}(P)$ will denote the set of all transversal homotopies through P. Let $\mathscr{E}(P)$ denote the set of C^∞-functions on P with the topology of uniform convergence of all derivatives. It is an F-S space.

$H \in \mathfrak{H}(P)$ defines the mapping

$$\mathscr{E}(P) \supset U \ni \varphi \to \varkappa^{-1}(\varphi) := \{H(p, \varphi(p)) \in X : p \in P\} \in \mathscr{P},$$

where U is a neighbourhood of zero.

The family $\{\varkappa\}$ forms an atlas of the C^∞-manifold $\mathscr{P}$.

B. Differentiable structure in $\mathscr{V}$. From now on by "domain" we mean a compact submanifold of X with the boundary which is a border. Let $\mathscr{E}(X, X)$ denote the set of C^∞-mappings $X \to X$ with the usual topology.

DEFINITION. By a *dragging of a domain* $D \subset X$ along a transversal homotopy $H \in \mathfrak{H}(\partial D)$ we mean a continuous mapping $\mathscr{E}(\partial D) \supset U \ni \varphi \to \sigma_\varphi \in \mathscr{E}(X, X)$ where σ satisfies the following conditions:

1° σ_φ is C^∞-diffeomorphism,

2° $\sigma_0(x) = x$, where $\sigma_0 := \sigma_\varphi$ for $\varphi \equiv 0$,

3° if $x \in \partial D$ then $\sigma_\varphi(x) = H(x, \varphi(x))$.

Example of a dragging. Let $H \in \mathfrak{H}(\partial D)$ be defined on $\partial D \times]-r, r[$, $r > 0$. If $0 \leqslant \xi \in C_0(R^1)$ with $\operatorname{supp} \xi \subset]-r, r[$ and $\xi(0) = 1$; then the function

$$\sigma_\varphi(x) = \begin{cases} H(p, t + \xi(t)\varphi(p)) & \text{if } x = H(p, t), (p, t) \in \partial D \times]-r, r[, \\ x & \text{at other points} \end{cases}$$

is a dragging.

Now, let us fix a homotopy $H \in \mathfrak{H}(\partial D)$, a dragging σ along H and a smooth linear connection K in the bundle V. The parallel displacement of a fibre over $x \in X$ along the curve $[0, 1] \ni t \to \sigma_{t\varphi}(x) X$ to the fibre over the point $\sigma_\varphi(x)$ defines the bundle-mapping

$$K_\varphi : V|_D \to V|_{\sigma_\varphi(D)}$$

and thus the mapping

$$\tilde{K}_\varphi : \mathscr{V}_{\partial D} \to \mathscr{V}_{\sigma_\varphi(\partial D)}.$$

It is easy to see that $\tilde{K}_\varphi$ is an isomorphism of an l.c.v. space. We can define, for a certain $U \in \mathscr{N}(\mathscr{E}(\partial D))$, the mapping

$$\eta(\varphi, v) := K_\varphi \circ v \circ \sigma_\varphi^{-1} \in \mathscr{V}_{\sigma_\varphi(\partial D)}.$$

Kijowski and Komorowski have proved that the mappings (or rather their inverses) form an atlas of a C^∞-bundle. Thus $\mathscr{V}$ is a C^∞-bundle in the sense of Chapter III.

Remark. We have assumed that P is a C^∞-submanifold. One can generalize the construction presented above by admitting "piecewise smooth borders".

In the following we shall need another generalization. Let us consider in the space $X^m := X \times X \times \ldots \times X$ (m times) the family of sets $D^m := D \times D \times \ldots \times D$ (m times), where D is a domain in X. We shall denote it by $\mathscr{P}^m$. There is a natural bijection $\mathscr{P}^m \leftrightarrow \mathscr{P}$ which enables us to endow $\mathscr{P}^m$ with a differentiable structure transported from $\mathscr{P}$.

Let $V^m := V \otimes V \otimes \ldots \otimes V$ (m times) be the exterior tensor product (cf. [29]). The bundle of smooth sections of V^m over domains D^m will be denoted by $\mathscr{V}^m$. It is the bundle over $\mathscr{P}^m$. A dragging of D defines the dragging of D^m by the formula

$$\sigma_\varphi^m(x_1, \ldots, x_m) := \big(\sigma_\varphi(x_1), \ldots, \sigma_\varphi(x_m)\big),$$

and hence a differentiable structure in $\mathscr{V}^m$ (connection K in V induces the connection K^m in V^m) and in $\mathscr{V}^{m'}$.

§ 2. An application in the field theory. As was mentioned in the introduction, the bundle of section-distributions is useful in the strict formulation of variational problems with a non-local and singular Lagrangian.

DEFINITIONS. 1° By a *Lagrangian-density function* we mean the mapping of bundles

$$\mathscr{V} \ni v \to L(v) \in \mathscr{V}^{m'},$$

where the induced mapping of bases is the natural bijection $\mathscr{P} \leftrightarrow \boldsymbol{B}\mathscr{P}^m$.

As was pointed in Chapter III, the bijection $\mathscr{P} \leftrightarrow \boldsymbol{B}\mathscr{P}^m$ is not continuous. We can omit the complications originating in that fact by treating $\mathscr{V}$ as a bundle over $\boldsymbol{B}\mathscr{P}$. It is easy to see (Chapter III) that the bundle $(\mathscr{V}, \boldsymbol{B}\mathscr{P}, \pi)$ is of class C^∞. Now, we say that Lagrange density is of class C^k if it is of class C^k as a mapping between the bundles $(\mathscr{V}, \boldsymbol{B}\mathscr{P}, \pi)$ and $(\mathscr{V}^{m'}, \boldsymbol{B}\mathscr{P}, \pi')$.

2° The *Lagrangian function associated with* L is the function

$$\mathscr{V} \ni v \to \mathscr{L}(v) := \langle L(v), v \otimes \ldots \otimes v \rangle \in R^1.$$

Of course the introduction of $\mathscr{V}^{m'}$ is not necessary (we can obtain all $\mathscr{L}$ taking $m = 1$) but it is very useful (cf § 3).

Since we are interested in $\mathscr{L}$ of class C^1, the following theorem may be useful.

THEOREM 14. *If a Lagrangian-density function L is of class C^1, then the associated Lagrangian is also of class C^1.*

Proof. Let us choose a map η related to a domain D $\eta\colon \mathscr{V} \to \mathscr{E}(\partial D) \times \mathscr{V}_{\partial D}$. η_m and $\tilde{\eta}_m$ denote induced maps of bundles $\mathscr{V}^m$ and $\mathscr{V}^{m'}$, respectively. Using the notation of Chapter 3, we have in these coordinate systems $\eta_m(\bar{v} \otimes \ldots \otimes \bar{v}) = (\varphi, \bar{v} \otimes \ldots \otimes \bar{v})$ and $\tilde{\eta}_m\big(L(\bar{v})\big) = \big(\varphi, \tilde{L}(\varphi, \bar{v})\big)$ for $\bar{v} = \eta^{-1}(\varphi, \bar{v})$. Thus $\mathscr{L} \circ \eta^{-1}(\varphi, \bar{v}) = \langle \tilde{\eta}^{-1}\big(\varphi, \tilde{L}(\varphi; \bar{v})\big), \eta_m^{-1}(\varphi, \bar{v}^m)\rangle$ ($\bar{v}^m := \bar{v} \otimes \ldots \otimes \bar{v}$), and since $\tilde{\eta} = \eta_1^*$ (η_1 is the inverse function to η in the second variable), we have

$$\mathscr{L}\circ\eta^{-1}(\varphi, \bar{v}) = \langle \tilde{L}(\varphi, \bar{v}), \bar{v}^m \rangle.$$

The mapping $(v', v)\to\langle v', v\rangle$ is not continuous, but it is hypocontinuous and hence the mapping $\mathscr{V}^{m'}_{\partial D}\times \boldsymbol{B}\mathscr{V}^{m}_{\partial D}\ni(v', v)\to\langle v', v\rangle$ is continuous (and thus of class C^∞). On the other hand, the mappings

$$\boldsymbol{B}\mathscr{V}_{\partial D}\ni v\to v^m\in\boldsymbol{B}\mathscr{V}^m_{\partial D}$$

and $\boldsymbol{B}\mathscr{E}(\partial D)\times\boldsymbol{B}\mathscr{V}_{\partial D}\ni(\varphi, \bar{v})\to\tilde{L}(\varphi, \bar{v})\in\mathscr{V}^{m'}_{\partial D}$ are of class C^1 (Chapter II § 3). Hence the mapping $\mathscr{L}\circ\eta^{-1}$ is of class C^1 on $\boldsymbol{B}\mathscr{E}(\partial D)\times\boldsymbol{B}\mathscr{V}_{\partial D}$.

Now, the proof follows from the lemma below.

LEMMA 11. *Let E be a metrizable, quasi-normable l.c.v. space. If the mapping $T\colon \boldsymbol{B}E\supset\Omega\to R^1$ is of class C^1, then the mapping $T\colon E\supset\Omega\to R^1$ is also of class C^1.*

Proof. Notice that the notion of the directional derivative does not depend on the topological structure of E. But $E' = (\boldsymbol{B}E)'$ (E is bornological), and so the Gâteaux-derivatives in E and $\boldsymbol{B}E$ are equal: $\nabla T = T'_{\boldsymbol{B}}$ ($T'_{\boldsymbol{B}}$ denotes the derivative of T in $\boldsymbol{B}E$). The mapping $\boldsymbol{B}E\supset\Omega\ni x\to\nabla T(x)$ is continuous, and so by the metrizability of E we have ($x_n\to x$ in E) $\Rightarrow$($x_n\to x$ in $\boldsymbol{B}E$) $\Rightarrow\big(\nabla T(x_n)\to\nabla T(x)$ in $\boldsymbol{B}E'_b\big)\Rightarrow\big(\nabla T(x_n)\to\nabla T(x)$ in $E'_b\big)$. Since in metrizable, quasi-normable spaces any function which is continuously Gâteaux-differentiable is also of class C^1, the proof follows. ■

§ 3. Example of a Lagrangian. Let M be a smooth, oriented, imbedded k-dimensional submanifold of X^n. Let ω be a $(\overset{m}{\otimes} V)^*|_M$-valued k-form on M. For a locally integrable ω we can define a Lagrangian-density L_ω

$$\mathscr{V}_{\partial D}\ni v\to\int_{D^m\cap M}\langle\omega, \cdot\rangle\in\mathscr{V}^{m'}.$$

For simplicity of calculations and notations let us concentrate on the case of $m = 2$.

DEFINITION. A submanifold $M\subset X\times X$ is said to be *transversal* to ∂D^2 ($M\psi\partial D^2$) if $M\psi(\partial D\times D)$ and $M\psi(D\times\partial D)$.

THEOREM 15 ([29]). *If ω is continuous and M is transversal to ∂D^2, then L_ω is continuously differentiable in a certain neighbourhood (in $\mathscr{V}$)* of $v\in\mathscr{V}_{\partial D}$.

Such Lagrangians are useful in describing non-local interactions which may appear as a consequence of internal structure of particles (cf. [23] and [31]). Under certain assumptions (cf. [29]) one can interprete m as the number of interacting particles.

REFERENCES

[1] B. H. Arnold, *Topologies defined by bounded sets*, Duke Math. J. 18 (1951), pp. 631–642.

[2] A. Bastiani, *Applications différentiables et variétés différentiables de dimension infinie*, J. Anal. Math. 13 (1964), pp. 1–114.

[3] N. Bourbaki, *Fonctions d'une variable rèele*, Paris 1958.

[4] I. Colombeau, *L'inversion d'une application différentiable entre espaces bornologiques*, C.R.A.S. 270 (1970), pp. 1962–1964.

[5] J. Dieudonné, *Foundations of modern analysis*, New York 1960.

[6] E. Dubinsky, *Differential calculus in Montel spaces*, Trans. Amer. Math. Soc. 110 (1964), pp. 1–24.

[7] H. R. Fischer, *Limesräume*, Math. Ann. 137 (1959), pp. 269–303.

[8] A. Frölicher and W. Bucher, *Calculus in vector spaces without norm*, Lecture Notes in Math. 30 (1966).

[9] A. Grothendieck, *Sur les espaces (F) et (DF)*, Summa Bras. Math. 3 (1954), pp. 57–123.

[10] H. Hogbe-Nlend, *Théorie des Bornologies et Applications*, Lecture Notes in Math. 213 (1971).

[11] H. Jarchow, *Marinescu Räume*, Comm. Math. Helv. 44 (1969), pp. 138–163.

[12] H. Keller, *Differenzierbarkeit in topologischen Vectorräumen*, Comm. Math. Helv. 38 (1964), pp. 308–320.

[13] J. Kijowski and J. Komorowski, *A differentiable structure in the set of all bundle sections over compact subsets*, Studia Math. 32 (1969), pp. 191–207.

[14] J. Kijowski, *Existence of differentiable structure in the set of submanifolds*, Studia Math. 33 (1969), pp. 93–108.

[15] — and W. Szczyrba, *On differentiability in an important class of locally convex spaces*, Studia Math. 30 (1968), pp 247–257.

[16] J. Kisyński, *Convergence du type L*, Coll. Math. 72(1960), pp. 205–211.

[17] J. Komorowski, *A geometrical formulation of the general free boundary problems*, Reports Math. Phys. 1 (1970), pp. 105–133.

[18] S. Lang, *Introduction to differentiable manifold*, New York 1962.

[19] B. Malgrange, *Ideals of differentiable functions*, Oxford 1966.

[20] G. Marinescu, *Différentielles de Gâteaux et Fréchet dans les espaces localement convexes*, Bull. Math. Soc. Sci. Math.-Phys. R.P.R. 1 (1957), pp. 77–86.

[21] — *Espaces vectoriels pseudotopologiques et théorie des distributions*, Berlin 1963.

[22] J. Mikusiński, *Distributions à valeurs dans les reunions d'espaces de Banach*, Studia Math. 29 (1960), pp. 249–285.

[23] J. Rzewuski, *Field Theory* I, Warszawa 1958.

[24] L. Schwartz, *Théorie des distributions*, Paris 1966.

[25] J. Sebastião e Silva, *Les espaces à bornes et la notion de fonction différentiable*, Atti Acc. Naz. dei Lincei, Serie VIII, 34 (1963), pp. 57–61.

[26] — *Les espaces à bornes et les reunions d'espaces normes*, ibidem, pp. 134–137.

[27] М. Ф. Сухинин, *О локальной обратимости дифференцируемого отображения*, У.М.Н. 15, 5 (1970), pp. 249–250.
[28] W. Szczyrba, *Differentiation in locally convex spaces*, Studia Math. 39 (1971), pp. 289–306.
[29] P. Urbański, *Dissertation for the Ph. D. degree.*
[30] L. Waelbrock, *Les espaces à bornes completes*, Colleque sur l'Analyse Fonctionelle CBRM 1961, pp. 51–57.
[31] K. Wilmański, *Dynamics of bodies with microstructure*, Archiwum Mechaniki Stosowanej 6, 20 (1968), pp. 705–743.