A SYMPLETIC APPROACH TO FIELD THEORY OF ELLIPTIC TYPE

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INTRODUCTION

Until recently symplectic methods have been used almost exclusively in connection with the study of time evolution of physical systems. Symplectic formulations of Hamiltonian mechanics have been well popularized ([1], [2] , [3]). Hamiltonian formulations of field theories are also well known ([4] , [5]).Recent work of W.M.Tulczyjew unifies Hamiltonian and Lagrangian formulations of mechanics within one symplectic framework (see ref. [1] , [6] - [12]) and extends the use of symplectic methods to the study of both statics and dynamics of reciprocal systems ([10] , [13] , [14]). In particular the general framework has been applied to field theory ([15]).

In the present paper we study the analysis of linear elliptic partial differential equations within the symplectic framework. The conceptual structure of this analysis was outlined in an unpublished paper of W.M.Tulczyjew ([16]).

The paper consists of six chapters. The first two contain an extract of W.M.Tulczyjew's paper [16] . Chapter III contains the proofs of isomorphism theorems. These theorems are well known for smooth fields (see ref. [17] , [18]).

In Chapter IV a symplectic interpretation of the isomorphism theorems is given.

Reductions of field dynamics are described in Chapter V.

A symplectic interpretation of the Green's function is given in Chapter VI.

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CONTENTS

CHAPTER I

A SYMPLECTIC FRAMEWORK FOR LINEAR FIELD THEORY (*)

In this and the subsequent chapters we present the principal elements of the symplectic framework for linear field theories.

Only smooth fields are considered. Proofs of most statements are omitted. Topological aspects of the framework are discussed in later chapters.

1. The field bundle and the state bundle.

Throughout this paper we use the following notation:

M – an open domain in R^m.

$M \ni t$ – a point in M, coordinate representation of t – (t^α).

Field bundle X – a vector bundle of rank n over M.

x – an element of X, coordinates of x – (t^α, x^A),

$\underset{\sim}{x}$ – a section of X, coordinate representation $x^A(t)$,

$\underset{\sim}{X}$ – the vector space of sections of X over M.

Source bundle $F: = \mathrm{Hom}(X; \Lambda^m T^*M)$

f – an element of F, coordinate representation of

$$f - (t^\alpha, f_A). \quad fx = f_A x^A \, dt^1 \ldots dt^m.$$

$\underset{\sim}{f}$ – a section of F, coordinate representation $f_A(t)$,

$\underset{\sim}{F}$ – the vector space of sections of F over M.

Stress bundle $P: = \mathrm{Hom}(X; \Lambda^{m-1} T^*M)$

p – an element of P, coordinate representation of

$$p - (t^\alpha, p^\lambda_A). \; px = (-1)^{\lambda-1} p_A x^A \, dt^1 \ldots \lambda \ldots dt^m$$

(*) After W.M. Tulczyjew [16].

$\underset{\sim}{p}$ - a section of P, coordinate representation $\underset{\sim}{p}^{\lambda}_{A}(t)$

The bundle Y: = P ⊕ F

y - an element of Y,

$\underset{\sim}{y}$ - a section of Y,

$\underset{\sim}{Y}$ - the space of section of Y

State bundle S = (X ⊕ X) ⊕ Y

s - an element of S,

$\underset{\sim}{s}$ - a section of S,

$\underset{\sim}{S}$ - the space of sections of S over M.

2. Symplectic spaces associated with currents.

Let T be an m-current (see e.g. [19] or [20]) with compact support contained in M.

Let $\underset{\sim}{\bar{x}} \oplus \underset{\sim}{x}$, $\underset{\sim}{y}$ be sections of X ⊕ X and Y over M.

We introduce the bilinear form

$$(\underset{\sim}{X} \oplus \underset{\sim}{X}) \times \underset{\sim}{Y} \ni (\underset{\sim}{\bar{x}} \oplus \underset{\sim}{x}, \underset{\sim}{y}) \longrightarrow \langle \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x}, \underset{\sim}{y} \rangle_T := \langle T, d(\underset{\sim}{p}\underset{\sim}{x}) - \underset{\sim}{f}\underset{\sim}{x} \rangle \in R. \quad (1.2.1)$$

In terms of this mapping we define spaces

$$0_T(X) := \{\underset{\sim}{X} \oplus \underset{\sim}{X} \ni \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x} : \langle \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x}, \underset{\sim}{y} \rangle_T = 0 \text{ for each } \underset{\sim}{y} \in \underset{\sim}{Y}\}$$

$$0_T(Y) := \{\underset{\sim}{Y} \ni \underset{\sim}{y} : \langle \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x}, \underset{\sim}{y} \rangle_T = 0 \text{ for each } \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x} \in \underset{\sim}{X} \oplus \underset{\sim}{X}\}$$

Quotient spaces $(X \oplus X)_T := (\underset{\sim}{X} \oplus \underset{\sim}{X})/_{0_T(X)}$

and $Y_T := Y/_{0_T(Y)}$ together with the bilinear mapping

$\langle\, ,\, \rangle_T : (X \oplus X)_T \times Y_T \to R$ induced by (1.2.1) form a dual pair.

Let $\bar{X}_T$ be the image of $\underset{\sim}{X} \oplus \{0\}$ in $(X \oplus X)_T$ under the canonical projection.

In the same way we introduce spaces X_T, P_T and F_T.

1.2.2. Proposition

There are canonical identifications

$$(X \oplus X)_T = \bar{X}_T \oplus X_T, \qquad Y_T = P_T \oplus F_T. \blacklozenge$$

By $\underset{\sim}{\bar{x}}_T$, $\underset{\sim}{x}_T$, $\underset{\sim}{p}_T$, $\underset{\sim}{f}_T$ e.t.c. we shall denote equivalence classes in $\bar{X}_T$, X_T, P_T, F_T e.t.c. of sections $\bar{x}, x, p, f$ e.t.c.

The space $S_T = (\bar{X}_T \oplus X_T) \oplus (P_T \oplus F_T)$ and the bilinear mapping

$$\omega_T : S_T \times S_T \ni (\bar{x}_T \oplus x_T \oplus y_T, \bar{x}'_T \oplus x'_T \oplus y'_T) \longrightarrow < \bar{x}'_T \oplus x'_T, y_T >_T -$$

$$- < \bar{x}_T + x_T, y'_T >_T \in R \tag{1.2.3}$$

form a symplectic vector space.

It is obvious that

$$< \bar{x}_T \oplus x_T, p_T \oplus f_T >_T = < \bar{x}_T \oplus 0 , \; p_T \oplus 0 >_T + < 0 \oplus x_T, 0 \oplus f_T >_T$$

We set

$$< \bar{x}_T, p_T >_T \; : = < \bar{x}_T \oplus 0, \; p_T \oplus 0 >_T$$
$$< x_T, f_T >_T \; : = < 0 \oplus x_T, \; 0 \oplus f_T >_T \tag{1.2.4}$$

3. Example. Dirac currents.

Let $v \varepsilon \Lambda^m T_t(M)$ be a non-zero m-vector at $t \varepsilon M$.
T will denote the current defined by v.
We have

$$< \bar{x}_T \oplus x_T, y_T >_T = < v, \; d(p\, x) - f\, x >$$

Since $v \neq 0$ it follows that spaces $\bar{X}_T$, X_T, P_T and F_T depend on t only. Consequently we denote these spaces by $\bar{X}_t, X_t, P_t, F_t$ and their elements by $\bar{x}_t, x_t, p_t, f_t$.

1.3.1. Proposition. At each point $t \varepsilon M, \bar{X}_t$ is the jet space $J^1_t(X)$, $X_t = J^o_t(X)$, $F_t = J^o_t(F)$ and P_t is a quotient space of $J^1_t(P)$. ◆

In a coordinate system $\bar{x}_T$ can be represented by $\bar{x}^A(t)$ and $\bar{x}^A_\lambda(t) \; : = \partial_\lambda \bar{x}^A(t)$.

Similarly, we represent x_T by $x^A(t)$, f_T by $f_A(t)$, p_T by $p_A(t)$ and $\partial_\lambda p^\lambda_A(t) = : \bar{p}_A(t)$.

If $v = \partial_1 \Lambda \ldots \Lambda \partial_m$ then

$$< \bar{x}_T, p_T >_T = \bar{x}^A(t) p_A(t) + \bar{x}^A_\lambda(t) p^\lambda_A(t)$$

$$< x_t, f_T >_T = - x^A(t) f_A(t).$$

4. Example. Bounded domains.

In this section T will denote a current defined by an oriented, compact domain Ω with smooth boundary $\partial\Omega$.

1.4.1. Proposition. $\bar{X}_T$ is the space of sections of X over $\partial\Omega$,
X_T is the space of sections of X over Ω,
P_T is the space of restrictions of sections of P over $\partial\Omega$ to the tangent bundle $T(\partial\Omega)$,
F_T is the space of sections of F over Ω.◆

We have

$$< \bar{x}_T \oplus x_T, p_T \oplus f_T >_T = \int_{\partial\Omega} p\, \bar{x} - \int_{\Omega} f\, x \qquad (1.4.2)$$

If we assume the existence of a Riemannian structure on M we can write (1.4.2) in the form (see [19])

$$< \bar{x}_T \oplus x_T, p_T \oplus f_T >_T = \int_{\partial\Omega} p_A x^A \, ds - \int_{\Omega} f_A x^A \, dv$$

where p_A is the normal to $\partial\Omega$ component of p_A^λ. ds is the surface element on $\partial\Omega$.

5. Special symplectic structures (s.s.s.)

Let (Σ,ω) be a symplectic space. By a special symplectic structure in Σ we understand a dual pair $(V, V^*, < , >)$ and projections

$\Pi : \Sigma \longrightarrow V$, $\rho : \Sigma \longrightarrow V^*$ such that the mapping

$\Pi \oplus \rho : \Sigma \longrightarrow V \oplus V^*$ is a symplectomorphism onto $V \oplus V^*$ with the canonical symplectic structure.

We consider the following special symplectic structures in S_T for each current T

(i) field s.s.s.

$$V = \bar{X}_T \oplus X_T , \qquad V^* = P_T \oplus F_T$$

$$< \bar{x}_T \oplus x_T, p_T \oplus f_T > := < \bar{x}_T \oplus x_T, p_T \oplus f_T >_T$$

and Π, ρ are canonical projections.

(ii) stress-source or Neumann-source s.s.s.

$$V = P_T \oplus F_T \quad , \quad V^* = \bar{X}_T \oplus X_T \quad ,$$

$$< p_T \oplus f_T, \bar{x}_T \oplus x_T >: = - < \bar{x}_T \oplus x_T, p_T \oplus f_T >_T \; ,$$

and Π, ρ are canonical projections.

(iii) stress-field or Neumann-field s.s.s.

$$V = X_T \oplus P_T \quad , \quad V^* = \bar{X}_T \oplus F_T$$

$$< x_T \oplus p_T \; , \; \bar{x}_T \oplus f_T >: = \sqrt{- < \bar{x}_T \; , \; p_T >_T + < x_T \; , \; f_T >_T} \quad ,$$

and Π, ρ are canonical projections.

(iv) field- source or Dirichlet-source s.s.s.

$$V = \bar{X}_T \oplus F_T, \quad V^* = X_T \oplus P_T$$

$$< \bar{x}_T \oplus f_T, x_T \oplus p_T > = < \bar{x}_T \; , \; p_T >_T - < x_T \; , \; f_T >_T \quad .$$

6. Lagrangian subspaces.

Let (Σ, ω) be a symplectic space and let $W \subset \Sigma$. We define a subspace $W^{\S} : = \{ s \in \Sigma : \omega(s,w) = 0 \quad \forall w \in W \}$.

We say that

(i) W is isotropic if $W^{\S} \supset W$

(ii) W is coisotropic if $W^{\S} \subset W$

(iii) W is Lagrangian if $W = W^{\S}$

1.6.1. Proposition. W is Lagrangian iff it is maximal (minimal) isotropic (coisotropic).◆

1.6.2. Proposition. Let be $\Sigma = V \oplus V^*$, $W \subset \Sigma$ an isotropic subspace and $\Pi(W) = V$ where Π is the canonical projection $\Pi : \Sigma \longrightarrow V$. Then W is Lagrangian.

Proof. Suppose W is not maximal. Then there exists $v_o \oplus v_o^* \in W$ such that $W \cup \{v_o + v_o^*\}$ is isotropic. But, since $\Pi(W) = V$, for each $v \in V$ there exists $\lambda(v) \in V$, $\lambda(v_o) \neq v_o^*$, such that $v \oplus \lambda(v) \in W$.

Hence

$$\omega(v \oplus \lambda(v), v_o \oplus \lambda(v_o)) = 0 = \langle v, \lambda(v_o) \rangle - \langle v_o, \lambda(v) \rangle \text{ and}$$

$$\omega(v \oplus \lambda(v), v_o \oplus v_o^*) = 0 = \langle v, v_o \rangle - \langle v_o, \lambda(v) \rangle \text{ for each } v \in V.$$

It follows that $\lambda(v_o) = v_o^*$.◆

7. Generating functions.

Let $\Sigma = V \oplus V^*$ be a symplectic space with the canonical symplectic structure.

Let $W \subset \Sigma$ be a Lagrangian subspace and let $V_1 \subset V$ be the image of W under the canonical projection. V_1 is called the constraint subspace. The function

$\bar{L} : W \; w = v \oplus v^* \longrightarrow \langle v, v^* \rangle$ defined on W does not depend on v^* for fixed v:
if $v \oplus v^*$, $v \oplus v_1^* \in W$ then $\langle v, v_1^* \rangle - \langle v, v^* \rangle = 0$ since W is isotropic.

We define a function L on V_1.

$$L(v) = \frac{1}{2} \langle v, v^* \rangle \quad \text{where } v \oplus v^* \in W.$$

Since L is a quadratic function, we can define its differential dL through the polarization formula:

$$\langle v', dL(v) \rangle = L(v+v') - L(v) - L(v').$$

We obtain the following proposition.

1.7.1. Proposition

$$W = \{\Sigma\; v \oplus v^* : v \in V_1, \; \langle v', dL(v) \rangle = \langle v', v^* \rangle \;\; \forall v' \in V_1\}$$ ◆

We say that L is the generating function of the Lagrangian subspace W.

If the space V^* is sufficiently large a quadratic function is a generating function of a Lagrangian subspace (see e.g. [21]).

8. Field dynamics.

Let us denote by $\underline{S}$ the set $\bigcup_{t\in M} \underline{S}_t$ where $\underline{S}_t = \bar{\underline{X}}_t \oplus \underline{X}_t \oplus \underline{P}_t \oplus \underline{F}_t$ is the symplectic space associated with a Dirac current.

It can be shown that $\underline{S}$ is a vector bundle called the infinitesimal state bundle. The dynamics is defined by the Lagrangian subbundle (see [21]) $\underline{D}$ of $\underline{S}$. This subbundle can be interpreted as defining a system of differential equations called field equations.

A section $\underset{\sim}{s}$ of S over M is said to be a solution of the field equations if for each point $t \in M$ the equivalence class of $\underset{\sim}{s}$ in $\underline{S}_t$ belongs to $\underline{D}_t$.

Solutions of the field equations form the vector subspace $\underset{\sim}{D}$ of $\underset{\sim}{S}$. In the following we shall assume that $\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x} \oplus \underset{\sim}{p} \oplus \underset{\sim}{f} \in D$ implies $\bar{\underset{\sim}{x}} = \underset{\sim}{x}$. (1.8.1)

For each current T equivalence classes of elements of $\underset{\sim}{D}$ form the vector space D_T.

1.8.2. Proposition

D_T is the isotropic subspace of S_T.

In the following we shall assume that D_T is the Lagrangian subspace of S_T. We shall refer to this subspace as the field dynamics associated with the current T.

9. Generating functions of dynamics.

Since $\underline{D}_t$ is the Lagrangian subspace of $\underline{S}_t$ it has a generating function with respect to each s.s.s. (see section 7).

As an example we shall consider field s.s.s.

For each point $t \in M$ there is a canonical composition

$$(\bar{\underline{X}}_t \oplus \underline{X}_t) \times (\underline{P}_t \oplus \underline{F}_t) \quad (\bar{\underline{x}}_t \oplus \underline{x}_t, \underline{y}_t) \to \underline{y}_t \cdot (\bar{\underline{x}}_t \oplus \underline{x}_t) \in \Lambda^m T_t^* (M)$$

such that for each non-zero vector v at t we have

$$< v, \underline{y}_t \cdot (\bar{\underline{x}}_t \oplus \underline{x}_t) > = < \bar{\underline{x}}_t \oplus \underline{x}_t, \underline{y}_t >_T,$$ where T is the Dirac current

defined by v.

For $\underline{\bar{x}}_t \oplus \underline{x}_t \oplus \underline{p}_t \oplus \underline{f}_t \varepsilon \underline{D}_t$ we define the mapping

$L_t^N(\underline{\bar{x}} \oplus \underline{x}_t) = \frac{1}{2}(\underline{p}_t \oplus \underline{f}_t).(\underline{\bar{x}}_t \oplus \underline{x}_t)$ called the Lagrangian density. Arguments used in section 7 show that L_t^N is well defined on the constaint subspace.

1.9.1. Proposition

Generating function L_T^N of D_T (with respect to the field s.s.s.) is given by the formula

$$L_T^N(\bar{x}_T \oplus x_T) = <T, L^N(\bar{x} \oplus x)>$$

where $L^N(\bar{x} \oplus x)$ is a section of $\Lambda^m T^*(M)$ defined by $L^N(\bar{x} \oplus x)(t) = L_t^N(\bar{x}_t \oplus x_t)$.

Remark: A similar formula is valid for the Neumann-field s.s.s.

CHAPTER II

INFINITESIMAL DYNAMICS

In this chapter we discuss the infinitesimal dynamics with respect to the field and Neumann-field s.s.s. . Relations between field and Neumann-field approaches will be established. At the end of the chapter Green's formulae will be presented.

1. Infinitesimal dynamics in the field s.s.s.

The assumption 1.8.1 implies the existence of the constraint in $\underline{\bar{X}}_t \oplus \underline{X}_t$. We may express this constraint by the equality $\Pi^o\, \underline{\bar{x}}_t = \underline{x}_t$ where Π^o is the canonical projection from the 1-jet space to the 0-jet space. We say that the dynamics is regular with respect to the field s.s.s. if this is the only constraint.

It means that the dynamics is generated by a quadratic function on this constraint subspace. We shall parametrize the constraint subspace by $\underline{\bar{x}}_t$. Using a local coordinate representation (see 1.3.) we have that L^N is a quadratic function of $(\bar{x}^A, \bar{x}^A)$. The most general form of L^N is then

$$L^N(\bar{x}^A,\bar{x}^A) = \frac{1}{2}\lambda_{AB}\bar{x}^A\bar{x}^B + \lambda^\mu{}_{AB}\bar{x}^A_\mu\bar{x}^B + \frac{1}{2}\lambda^{\mu\nu}{}_{AB}\bar{x}^A_\mu\bar{x}^B_\nu \qquad 2.1.1.$$

where λ_{AB}, $\lambda^\mu{}_{AB}$ and $\lambda^{\mu\nu}{}_{AB}$ are real-valued functions on M which satisfy symmetry conditions

$$\lambda_{AB} = \lambda_{BA} \quad , \; \lambda^{\mu\nu}{}_{AB} = \lambda^{\nu\mu}{}_{BA} \; .$$

According to 1.7.1 and 1.3. the infinitesimal dynamics is described by

$$p^{\mu}_{A} = \lambda^{\mu\nu}_{AB}\, \bar{x}^{B}_{\nu} + \lambda^{\mu}_{AB}\, \bar{x}^{B}$$

$$\bar{p}_{A} - f_{A} = \lambda^{\mu}_{BA}\, \bar{x}^{B} + \lambda_{AB}\, \bar{x}^{B} \qquad (2.1.2.)$$

The space of solutions of the field equations is then described by the relations

$$\underset{\sim}{x}^{A} = \bar{\underset{\sim}{x}}^{A} \qquad \text{(the constraint condition)}$$

and

$$\underset{\sim}{p}^{\mu}_{A} = \lambda^{\mu\nu}_{AB}\, \partial_{\nu} \underset{\sim}{x}^{B} + \lambda^{\mu}_{AB}\, \underset{\sim}{x}^{B}$$

$$\partial_{\nu}\, \underset{\sim}{p}^{\nu}_{A} - \underset{\sim}{f}_{A} = \lambda^{\mu}_{BA} \partial_{\mu} \underset{\sim}{x}^{B} + \lambda_{AB}\, \underset{\sim}{x}^{B} \qquad (2.1.3)$$

It is the Lagrange form of the field equations.

Equivalently, we write

$$\bar{\underset{\sim}{x}} = \underset{\sim}{x}$$

and

$$\underset{\sim}{p}^{\mu}_{A} = \mathcal{H}^{\mu}_{A}(t;\underset{\sim}{x}, \partial_{\nu} \underset{\sim}{x}) := \lambda^{\mu\nu}_{AB}\, \partial_{\nu} \underset{\sim}{x}^{B} + \lambda^{\mu}_{AB}\, \underset{\sim}{x}^{B}$$

$$\underset{\sim}{f}_{A} = \varepsilon_{A}(t;\underset{\sim}{x}, \partial_{\nu}\underset{\sim}{x}, \partial_{\mu\lambda}\underset{\sim}{x}) = \lambda^{\mu\nu}_{AB} \partial_{\mu}\partial_{\nu} \underset{\sim}{x}^{B} + (\partial_{\mu}\, \lambda^{\mu\nu}_{AB} + \lambda^{\nu}_{AB} - \lambda^{\nu}_{BA}) \partial_{\nu}\, \underset{\sim}{x}^{B} + (\partial_{\mu}\, \lambda^{\mu}_{AB} - \lambda_{AB})\, \underset{\sim}{x}^{B} .$$

This is the Euler form of the field equations. The second order differential operator $\varepsilon : \underset{\sim}{X} \longrightarrow \underset{\sim}{F}$ is called the Euler–Lagrange operator.

2. Infinitesimal dynamics in the Neumann-field s.s.s.

Now, suppose that for each $t \in M$ the infinitesimal dynamics projects onto $\underline{X}_{t} \oplus \underline{P}_{t}$. It is then generated by a quadratic function L^{D}_{t} on $\underline{X}_{t} \oplus \underline{P}_{t}$. The local expression of the most general form of L^{D} is

$$L^D(x^A, p_B^\nu, \bar{p}_C) = \frac{1}{2}\,\bar{\lambda}_{AB}\,x^A x^B + \frac{1}{2}\,\bar{\lambda}^{AB}_{\mu\nu}\,p_A^\mu p_B^\nu + \bar{\lambda}^{AB}_\nu\,p_A^\nu\,\bar{p}_B +$$
$$+\bar{\lambda}_\nu{}^A{}_B\,p_A^\nu\,x^B + \bar{\lambda}^A{}_B\,\bar{p}_A\,x^B + \frac{1}{2}\,\bar{\lambda}^{AB}\,\bar{p}_A\,\bar{p}_B \qquad (2.2.1)$$

where all coefficients $\bar{\lambda}$ are real-valued functions on M which satify symmetry conditions:

$$\bar{\lambda}_{AB} = \bar{\lambda}_{BA}\,, \qquad \bar{\lambda}^{AB}_{\mu\nu} = \bar{\lambda}^{BA}_{\nu\mu}\,, \qquad \bar{\lambda}^{AB} = \bar{\lambda}^{BA}\,.$$

The infinitesimal dynamics is then described by

$$-\bar{x}^A = \bar{\lambda}^{AB}\,\bar{p}_B + \bar{\lambda}^A_B\,x^B + \bar{\lambda}^{BA}_\nu\,p_B^\nu$$
$$-\bar{x}^A_\mu = \bar{\lambda}^{AB}_{\mu\nu}\,p_B^\nu + \bar{\lambda}^{AB}_\mu\,\bar{p}_B + \bar{\lambda}^A_\mu{}_B\,x^B \qquad (2.2.2)$$
$$-f_A = \bar{\lambda}_{AB}\,x^B + \bar{\lambda}^B_\nu{}_A\,p_B^\nu + \bar{\lambda}^B{}_A\,\bar{p}_B$$

The assumption (1.8.1) does not induce constraints but limits the choice of generating functions. The admissible generating function must satisfy the following conditions:

$$\bar{\lambda}^{AB} = 0\,, \qquad \bar{\lambda}^A_B = -\,\delta^A_B\,, \qquad \bar{\lambda}^{BA} = 0 \qquad \text{where } \delta^A_B$$

is the Kronecker symbol.

Hence the general form of the admissible generating function is

$$L^D(x^A, p_B, \bar{p}_C) = \frac{1}{2}\,\bar{\lambda}_{AB}x^A x^B + \frac{1}{2}\,\bar{\lambda}^{AB}_{\mu\nu}\,p_A^\mu\,p_B^\nu + \bar{\lambda}^A_\mu{}_B\,p_A^\mu x^B - \bar{p}_A x^A \quad (2.2.3)$$

The infinitesimal dynamics is then described by relations

$$\bar{x}^A = x^A$$
$$\bar{x}^A_\mu = -\,\bar{\lambda}^{AB}_{\mu\nu}\,p_B^\nu - \bar{\lambda}^A_{\mu B}\,x^B \qquad (2.2.4)$$
$$f_A = -\,\bar{\lambda}_{AB}\,x^B - \bar{\lambda}^B_{\nu A}\,p_B^\nu + \bar{p}_A$$

3. The characterization of $\underset{\sim}{D}$. Neumann-field approach.

According to (1.8) $\underset{\sim}{D}$ is the set of solution of the field equations (see 2.2.4.):

$$\begin{aligned} \bar{x}^A &= x^A \\ \partial_\mu \bar{x}^A &= -\bar{\lambda}^{AB}_{\mu\nu} p^\nu_B - \bar{\lambda}^A_{\mu B} x^B \\ f_A &= -\bar{\lambda}_{AB} x^B - \bar{\lambda}^B_{\nu A} p^\nu_B + \partial_\nu p^\nu_A \end{aligned} \qquad (2.3.1)$$

This is the local expression of the field equations in the Lagrange form.

We see that not for each section $x \oplus p$ the system 2.3.1. can be integrated.

The integrability conditions are:

$$\partial_\mu x^A = -\bar{\lambda}^{AB}_{\mu\nu} p_B - \bar{\lambda}^A_{\mu B} x^B \qquad (2.3.2)$$

Remark. If $\bar{\lambda}^{AB}_{\mu\nu}$ is invertible, 2.3.2 means that

$$p^\nu_A = \mathcal{M}^\nu{}_A(x^B, \partial_\mu x^C).$$

4. A Legendre transformation.

In 2.1. and 2.2. we described infinitesimal dynamics in two different ways. It is easy to see that the dynamics is regular with respect to the field s.s.s. (2.1) and there is no constraint in Neumann-field s.s.s. iff the matrix of rank n.m $\underline{\lambda} = \left[\lambda^{\mu\nu}_{AB}\right]$ is invertible.

In that case

$$\begin{aligned} \bar{\lambda}^{BA}_{\nu\mu} &= -(\lambda^{-1})^{BA}_{\nu\mu} \\ \bar{\lambda}^B_{\nu C} &= (\lambda^{-1})^{BA}_{\nu\mu} \lambda^\mu{}_{AC} \\ \bar{\lambda}_{AB} &= \lambda_{AB} - \lambda^\nu{}_{DA} (\lambda^{-1})^{DC}_{\nu\mu} \lambda^\mu{}_{CB} \end{aligned} \qquad (2.4.1)$$

Formulae 2.4.1. are valid for example when the Euler-Lagrange operator ε is elliptic.

5. Green's formulae.

Let $x \oplus x \oplus p \oplus f$ and $x' \oplus x' \oplus p' \oplus f'$ be in D. Since D_T is isotropic for each current T, we have that

$$< \partial T, \underset{\sim}{p}.\underset{\sim}{x}' - \underset{\sim}{p}'.\underset{\sim}{x} > - < T, \underset{\sim}{f}.\underset{\sim}{x}' - \underset{\sim}{f}'.\underset{\sim}{x} > = 0 \qquad (2.5.1)$$

But D is described by 2.1.4, so 2.5.1. is equivalent to the following

2.5.2. Proposition

If $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are elements of X then

$$< \partial T, \pi(\underset{\sim}{x})\underset{\sim}{x}' - \pi(\underset{\sim}{x}')\underset{\sim}{x} > - < T, \varepsilon(\underset{\sim}{x})\underset{\sim}{x}' - \varepsilon(\underset{\sim}{x}')\underset{\sim}{x} > = 0$$

for each current T. Consequently,

$$d(\pi(x)x' - \pi(x')x - \varepsilon(x)x' + \varepsilon(x')x = 0. \blacklozenge$$

The first equality in the proposition is easily recognized as an abstract version of the second Green's formula.

In the case when T is an oriented domain Ω we get, using coordinate systems,

$$\begin{aligned} &\int_{\partial\Omega} \underset{\sim}{x}'^A \pi^\mu{}_A(x)\, dS_\mu - \int_{\partial\Omega} \underset{\sim}{x}^A \pi^\mu{}_A(\underset{\sim}{x}')dS_\mu = \\ &= \int_\Omega \underset{\sim}{x}'^A \varepsilon_A(\underset{\sim}{x})dv - \int_\Omega \underset{\sim}{x}^A \varepsilon_A(\underset{\sim}{x}')dv \end{aligned} \qquad (2.5.3)$$

According to 1.7.1., for each current T the dynamics D_T is described as follows

$$\begin{aligned} D_T = \{ &S_T \ni \bar{\underset{\sim}{x}}_T \oplus \underset{\sim}{x}_T \oplus \underset{\sim}{p}_T \oplus \underset{\sim}{f}_T : \bar{\underset{\sim}{x}} = \underset{\sim}{x}, < \bar{\underset{\sim}{x}}'_T, \underset{\sim}{p}_T >_T \oplus < \underset{\sim}{x}'_T, \underset{\sim}{f}_T >_T = \\ &= dL^N_T(\bar{\underset{\sim}{x}}_T \oplus \underset{\sim}{x}_T, \underset{\sim}{x}'_T \oplus \underset{\sim}{x}'_T) \quad \forall \bar{\underset{\sim}{x}}'_T \oplus \underset{\sim}{x}'_T \quad \text{with } \bar{\underset{\sim}{x}}' = \underset{\sim}{x}' \} \end{aligned} \qquad (2.5.4)$$

or equivalently

$$\begin{aligned} D_T = \{ &S_T \ni \bar{\underset{\sim}{x}}_T \oplus \underset{\sim}{x}_T \oplus \underset{\sim}{p}_T \oplus \underset{\sim}{f}_T : - < \bar{\underset{\sim}{x}}_T, \underset{\sim}{p}'_T >_T + < \underset{\sim}{x}'_T, \underset{\sim}{f}_T >_T = \\ &= dL^D_T(\underset{\sim}{x}_T \oplus \underset{\sim}{p}_T, \underset{\sim}{x}'_T \oplus \underset{\sim}{p}'_T) \quad \forall \underset{\sim}{x}'_T \oplus \underset{\sim}{p}'_T \in X_T \oplus P_T \} \end{aligned} \qquad (2.5.5)$$

Using 1.9.1. and 2.1.4. it follows from 2.5.4. that

$$< \partial T, \pi(\underset{\sim}{x})\underset{\sim}{x}' > - < T, \varepsilon(\underset{\sim}{x})\underset{\sim}{x}' > = < T, dL^N(\underset{\sim}{x}, \underset{\sim}{x}') > \qquad (2.5.6)$$

This is an abstract form of the first Geen's formula.

An analogue of this formula for the Neumann-field s.s.s. is the following equation. We used the fact, that $L_T^D = L_T^N - < x_T, p_T >_T$ (2.5.7) which follows directly from 1.7. and 1.5.

$$-<\partial T, \underset{\sim}{p}' \bar{\underset{\sim}{x}}> - <T, \underset{\sim}{f}\underset{\sim}{x}'> = <T, dL^N(\underset{\sim}{x}, \underset{\sim}{x}')> - <\sqrt{T}, \underset{\sim}{p}\ \underset{\sim}{x}'> -$$

$$- <\sqrt{T}, \underset{\sim}{p}'\ \underset{\sim}{x}> \quad \text{for each} \quad \underset{\sim}{x}' \varepsilon \underset{\sim}{X}, \quad \underset{\sim}{p}' \varepsilon \underset{\sim}{P}.$$

CHAPTER III

ISOMORPHISMS THEOREMS

From now on we assume that the Euler-Lagrange operator ε is elliptic with smooth coefficients λ. This is the situation in static systems. For elliptic boundary problems there are well known isomorphisms theorems (e.g. [17] , [18]). In fact there are series of theorems parametrized by the Sobolev-space index s ($s \geq 2$). In this chapter we shall complete these series to each real value s. We use transposition and interpolation methods following ideas of Lions and Magenes ([22]).

Since our transposition procedure differs from that of Lions and Magenes, our isomorphisms theorems are different as well.

We should also mention the work of Rojtberg ([23] - [26]), who proved a complet series of isomorphisms, although his theorems are much weaker than ours and have no symplectic interpretation.

Since we are interested in field theory, we shall deal with selfadjoint problems only. In fact we shall consider Neumann and Dirichlet problems only. But the methods used in the paper may be applied to other boundary problems as well as to nonself-adjoint dynamics.

Corresponding results will be presented elsewhere.

In the following $\Omega \subset M$ will denote a bounded, closed, oriented domain with the smooth boundary $\partial\Omega$.

1. Sobolev spaces. Strong symplectic structures.

We saw in 1.4. that for a current T which is an oriented, bounded domain Ω, $\bar{X}_T$ is the space of sections of X over $\partial\Omega$. Let $\underset{\sim}{\bar{X}}^s$ denote the Sobolev space $H^s(\partial\Omega;X)$ (see [27]) of sections over $\partial\Omega$ of the bundle X, $s\in R$. For $s<0$, $\bar{X}^s$ is the corresponding space of sections-distributions (see e.g. [28]).

An element $p\in \mathrm{Hom}(X, \Lambda^{m-1}T^*M)$ in a fiber over $t\in\partial\Omega$ defines in a natural way an element of $\mathrm{Hom}(X_{|\partial\Omega}; \Lambda^{m-1}T^*(\partial\Omega))$. We shall denote this bundle by $P_{|\partial\Omega}$.

Let P^s denote the space $H^s(\partial\Omega; P_{|\partial\Omega})$. It is obvious that P^s is the dual space to X^{-s} with the duality 1.2.4. Now we introduce topological vector spaces

$$X^s = H^s(\Omega;X), \quad F^s = H^s(\Omega;F) \quad \text{for} \quad s\geq 0.$$

For $s\leq 0$ we define X^s as a strong dual space to F^{-s} and vice-versa. Again the duality is that of 1.2.4.

In the following we shall deal with state spaces

$$S^{s,s'} := (\bar{X}^{s'} \oplus X^s) \oplus (P^{-s'} \oplus F^{-s}) \qquad s,s'\in R \qquad (3.1.1)$$

The reason of the choice of these spaces is that we want to deal with strong symplectic structures only.

The assumption 1.8.1. implies that the dynamics is in the subspace

$$\bar{x}_T = x|_{\partial\Omega} \qquad (3.1.2)$$

providing this equation is meaningful. It is well known (see e.g. [22]) that for $s>\frac{1}{2}$ the mapping $X\ni x \longrightarrow x|_{\partial\Omega}$ defines by continuity the surjection

$$X^s \longrightarrow \bar{X}^{s-\frac{1}{2}}$$

This is not true for $s\leq\frac{1}{2}$. Hence the condition 3.1.2, is meaningful for $s>\frac{1}{2}$ only.

On the other hand we see, that the most natural choice for s' (assuming that we shall deal with field s.s.s.) is $s' = s-\frac{1}{2}$. (1.8.1) implies the existence of the constraint in $\bar{X}^{s-\frac{1}{2}} \oplus X^s$ for $s>\frac{1}{2}$. The constraint subspace we denote by X_N^s.

The reduction (see [29] and Chapter V) of $S^{s,s-\frac{1}{2}}$ with respect to this constraint leads to the symplectic space $S_N^s = X_N^s \oplus Y_N^{-s}$ where

$$\underset{\sim}{Y}_N^{-s} = (\underset{\sim}{P}^{-s+\frac{1}{2}} \oplus \underset{\sim}{F}^{-s})/_K \tag{3.1.3}$$

with

$$K = \{\underset{\sim}{p} \oplus \underset{\sim}{f} \in \underset{\sim}{P}^{-s+\frac{1}{2}} \oplus \underset{\sim}{F}^{-s} : \langle \underset{\sim}{x}|_{\partial\Omega}, \underset{\sim}{p} \rangle = \langle \underset{\sim}{x}, \underset{\sim}{f} \rangle \quad \forall \underset{\sim}{x} \in \underset{\sim}{X}\}$$

(we write $\langle \bar{\underset{\sim}{x}}, \underset{\sim}{p} \rangle$ instead of $\langle \bar{\underset{\sim}{x}}, \underset{\sim}{p} \rangle_T$ and $-\langle \underset{\sim}{x}, \underset{\sim}{f} \rangle$ instead of $\langle \underset{\sim}{x}, \underset{\sim}{f} \rangle_T$).

In order to unify the notation we put $\underset{\sim}{X}_N^s$ and $\underset{\sim}{Y}_N^s$ for $\underset{\sim}{X}^{s-\frac{1}{2}} \oplus \underset{\sim}{X}^s$ and $\underset{\sim}{P}^{-s+\frac{1}{2}} \oplus F^{-s}$ when $s < \frac{1}{2}$.

The case $s = \frac{1}{2}$ will be considered later. One can easily see that if $s > \frac{1}{2}$ then $\underset{\sim}{X}_N^s$ is isomorphic to $\underset{\sim}{X}^s$ and $\underset{\sim}{Y}_N^{-s}$ is isomorphic to $\underset{\sim}{F}^{-s}$.

The most convenient choice of symplectic spaces with distinguished field-Neumann s.s.s. is different than in the case of the field s.s.s. . In this case a constraint appears as a consequence of the relation $\underset{\sim}{p}_A^{\nu} = \mathcal{H}_A^{\nu}(\underset{\sim}{x})$ in $\underset{\sim}{D}$.

First, we note that the most natural choice of the index s' is $s' = -s + \frac{3}{2}$. For fixed dynamics, which we denote by the symbol λ, we define

$$\begin{aligned} \underset{\sim}{X}_{D,\lambda}^s &:= \{\underset{\sim}{x} \oplus \underset{\sim}{p} \in \underset{\sim}{X}^s \oplus \underset{\sim}{P}^{s-\frac{3}{2}} : \underset{\sim}{p} = \mathcal{M}(\underset{\sim}{x})|_{\partial\Omega}\} \quad \text{for} \quad s > \frac{3}{2} \\ &= \underset{\sim}{X}^s \oplus \underset{\sim}{P}^{s-\frac{3}{2}} \quad \text{for} \quad s < \frac{3}{2} \end{aligned} \tag{3.1.4}$$

and

$$\begin{aligned} \underset{\sim}{Y}_{D,\lambda}^{-s} &= \underset{\sim}{X}^{-s+\frac{3}{2}} \oplus \underset{\sim}{F}^{-s}/K_\lambda \quad \text{for} \quad s < \frac{3}{2} \\ &= \bar{\underset{\sim}{X}}^{-s+\frac{3}{2}} \oplus \underset{\sim}{F}^{-s} \quad \text{for} \quad s > \frac{3}{2}, \end{aligned}$$

where

$$K_\lambda = \{\bar{\underset{\sim}{x}} \oplus \underset{\sim}{f} \in \underset{\sim}{X}^{-s+\frac{3}{2}} \oplus \underset{\sim}{F}^{-s} : \langle \bar{\underset{\sim}{x}}, \mathcal{H}(\underset{\sim}{x}) \rangle = -\langle \underset{\sim}{x}, \underset{\sim}{f} \rangle \quad \forall \underset{\sim}{x} \in \underset{\sim}{X}\}$$

$$\underset{\sim}{S}_{D,\lambda}^s := \underset{\sim}{X}_{D,\lambda}^s \oplus \underset{\sim}{Y}_{D,\lambda}^{-s}$$

As before we note that $\underset{\sim}{X}_{D,\lambda}^s$ is isomorphic to $\underset{\sim}{X}^s$ and $\underset{\sim}{Y}_{D,\lambda}^{-s}$ is isomorphic to $\underset{\sim}{F}^{-s}$ for $s > \frac{3}{2}$.

The case $s = \frac{3}{2}$ will be considered later.

The following Proposition is an obvious consequence of the Rellich's lemma.

3.1.5. Proposition

For $s > s'$ we have the following canonical injections which are compact:

$$\underset{\sim}{X}_N^s \longrightarrow \underset{\sim}{X}_N^{s'}, \quad \underset{\sim}{X}_{D,\lambda}^s \longrightarrow \underset{\sim}{X}_{D,\lambda}^{s'}$$

$$\underset{\sim}{Y}_N^s \longrightarrow \underset{\sim}{Y}_N^{s'}, \quad \underset{\sim}{Y}_{D,\lambda}^s \longrightarrow \underset{\sim}{Y}_{D,\lambda}^{s'} \quad ◆$$

2. Interpolation theorems.

From 3.1.5. it follows that the question on the interpolation properties of families $\{\underset{\sim}{X}_N^s\}_{s\neq\frac{1}{2}}$, $\{\underset{\sim}{X}_{D,\lambda}^s\}_{s\neq\frac{3}{2}}$ is well posed. This properties play fundamental role in our concept of isomorphism theorems.

In our proofs of interpolation theorems we follow methods used in [22]. We also refer to this work for basic concepts of the interpolation of Hilbert spaces.

3.2.1. Theorem. Let be $s, s' \neq \frac{1}{2}$, $s > s', 0<\theta<1$, $\theta s' + (1-\theta)s \neq \frac{1}{2}$. Then

$$\left[\underset{\sim}{X}_N^s, \underset{\sim}{X}_N^{s'}\right] = \underset{\sim}{X}_N^{(1-\theta)s+\theta s'}.$$

Proof.

Let us note that it is sufficient to prove the relations

$$\left[\underset{\sim}{X}_N^k, \underset{\sim}{X}_N^o\right]_\theta = \underset{\sim}{X}_N^{(1-\theta)k} \qquad (1-\theta)k \neq \frac{1}{2}$$

$$\left[\underset{\sim}{X}_N^k, \underset{\sim}{X}_N^{-k}\right]_{\frac{1}{2}} = \underset{\sim}{X}_N^o \quad \text{where } k \text{ is an integer.} \tag{2.2.2}$$

In fact, by the second interpolation theorem, for $0<s'<s<k$ it follows from 3.2.2. that

$$\left[\underset{\sim}{X}_N^s, \underset{\sim}{X}_N^{s'}\right]_\theta = \left[\underset{\sim}{X}_N^k, \underset{\sim}{X}_N^o\right]_{\theta_1} \quad \text{where} \quad \theta_1 = (1-\theta)(1-\frac{s}{k}) + \theta(1-\frac{s'}{k}).$$

For $0>s>s'>-k$ the theorem is obvious.

For $k>s>0>s'$ we have

$$\left[\underset{\sim}{X}_N^s , \underset{\sim}{X}_N^{s'}\right]_\theta = \left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{-k}\right]_{\theta_1} \quad \text{where} \quad \theta_1 = \frac{1}{2}(1-\theta)(1-\frac{s}{k}) + \frac{1}{2}\theta(1-\frac{s'}{k}),$$

but

$$\left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{-k}\right]_\theta = \left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{o}\right]_{2\theta} \quad \text{for} \quad \theta<\frac{1}{2}$$

and

$$\left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{-k}\right]_\theta = \left[\underset{\sim}{X}_N^o , \underset{\sim}{X}_N^{-k}\right]_{2\theta-1} \quad \text{for} \quad \theta>\frac{1}{2} \ .$$

The proof of both properties 3.2.2. will follow simultaneously.

A. The inclusion $\left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{o}\right]_\theta \subset \underset{\sim}{X}_N^{(1-\theta)k}$. (The inclusion $\left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^{-k}\right]_{\frac{1}{2}} \subset \underset{\sim}{X}_N^o$ is obvious).

We notice that with local trivalization and coordinate system in M we can reduce the problem to $\underset{\sim}{X}_N^o = H^{-\frac{1}{2}}(R_o^m) \oplus H^o(R_+^m)$ where

$$R_o^m = \{t\epsilon R^m\colon t^m = 0\}, \quad R_+^m = \{t\epsilon R^m\colon t^m \geq 0\}, \quad R_-^m = \{t\epsilon R^m\colon t^m \leq 0\},$$

and

$$\underset{\sim}{X}_N^k = \left\{\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x} \epsilon H^{k-\frac{1}{2}}(R_o^m) \oplus H^k(R_+^m)\colon \underset{\sim}{x}\Big|_{R_o^m} = \bar{\underset{\sim}{x}}\right\}.$$

Now, suppose we have constructed a mapping

$$\rho\colon H^{-\frac{1}{2}}(R_o^m) \oplus H^o(R_+^m) \longrightarrow H^o(R^m) \quad \text{with the properties:}$$

(i) ρ is a continuous injection and $\rho(\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x})\Big|_{R_+^m} = \underset{\sim}{x}$

(ii) for $\underset{\sim}{x}\epsilon H^k(R_+^m)$ $\quad \rho(\underset{\sim}{x}\big|_{R_o^m} \oplus \underset{\sim}{x}) \epsilon H^k(R^m)$

(iii) ρ defines the continuous mapping

$$\rho : H^{k-\frac{1}{2}}(R_o^m) \oplus H^k(R_+^m) \supset \underset{\sim}{X}_N^k \longrightarrow H^k(R^m)$$

(iv) if $\rho(\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x}) \epsilon H^s(R^m)$, $s>\frac{1}{2}$, then $\bar{\underset{\sim}{x}} = \underset{\sim}{x}\Big|_{R_o^m}$.

By the interpolation theorem we have then that the mapping

$$\rho : \left[\underset{\sim}{X}_N^k ,\ \underset{\sim}{X}_N^o \right]_\theta \longrightarrow \left[H^k(R^m), H^o(R^m) \right]_\theta = H^s(R^m) \qquad \theta = 1 - \frac{s}{k}$$

is continuous.

From (iv) it follows that the induced mapping

$$\underset{\sim}{X}_N^o \supset \left[\underset{\sim}{X}_N^k ,\ \underset{\sim}{X}_N^o \right]_\theta \ni \underset{\sim}{\bar{x}} \oplus \underset{\sim}{x} \longrightarrow \rho(\underset{\sim}{\bar{x}} \oplus \underset{\sim}{x})\Big|_{R_o^m} \oplus \rho(\bar{x} \oplus x) \in \underset{\sim}{X}_N^s$$

is the inclusion.

We pass to the construction of ρ.

1st step.

Let $\underset{\sim}{\bar{x}} \in H^{-\frac{1}{2}}(R_o^m)$. We define $\underset{\sim}{\tilde{x}} \in H^o(R^m)$ by its Fourier transform with respect to the variables $(t^1 \ldots t^{m-1}) =: t'$

$$\hat{\tilde{x}}(\xi', t^m) := \hat{\underset{\sim}{\bar{x}}}(\xi')\ \phi((1+|\xi'|^2)^{1/2}.t^m) \quad \text{where}$$

$\phi \in C_o^\infty(R)$ is such that $\phi(0) = 1$.

We see that $\underset{\sim}{\tilde{x}}(0) = \bar{x}$. Moreover, if $\underset{\sim}{\bar{x}} \in H^{s-\frac{1}{2}}(R_o^m)$, then

$$\|\underset{\sim}{\tilde{x}}\|_s^2 = \int (1+|\xi'|^2+|\xi^m|^2)^s\ |\hat{\underset{\sim}{\bar{x}}}|^2 \left| \int e^{it^m\xi^m}((1+|\xi'|^2)^{\frac{1}{2}}t^m)dt^m \right|^2 d\xi' d\xi^m \leq$$

$$\leq \int (1+|\xi'|^2+|\xi^m|^2)^s\ |\hat{\underset{\sim}{\bar{x}}}|^2 |\ \hat{\phi}((1+|\xi'|^2)^{-\frac{1}{2}}\xi^m)|^2 (1+|\xi'|^2)^{-1} d\xi' d\xi^m =$$

$$= \int (1+|\xi'|^2)^{-1} |\hat{\underset{\sim}{\bar{x}}}(\xi')|^2 \{ \int (1+|\xi'|^2+|\xi^m|^2)^s\ |\hat{\phi}((1+|\xi'|^2)^{-\frac{1}{2}}\xi^m)^2 . d\xi^m \} d\xi' =$$

$$= \int (1+|\xi'|^2)^{-1} |\hat{\underset{\sim}{\bar{x}}}(\xi')|^2 \{ \int (1+|\xi'|^2+(1+|\xi'|^2)|\xi^m|^2)^s .$$

$$.(1+|\xi'|^2)^{\frac{1}{2}}\ d\,\xi^m \}\, d\,\xi' \leq$$

$$\leq c \int (1+|\xi'|^2)^{s-\frac{1}{2}}\ |\hat{\underset{\sim}{\bar{x}}}(\xi')|^2 \leq c\, \|\underset{\sim}{\bar{x}}\|_{s-\frac{1}{2}} .$$

2nd step

Let $\{c_i\}_{i=1}^k$ be a set of numbers such that

$$\sum_{i=1}^{k} (i)^j c_i = 1 \quad \text{for} \quad j = 0$$
$$= 0 \quad \text{for} \quad j = 1,\dots,k-1$$

We see that the function

$$\tilde{\tilde{x}}(t) = \sum_{i=1}^{k} c_i x(t', it^m) \quad \text{has properties:}$$

$$\tilde{\tilde{x}}(t)\Big|_{t^m=0} = \bar{x}(t'), \quad D^j\tilde{\tilde{x}}\Big|_{t^m=0} = 0 \text{ for } j = 1,\dots,k-1,$$

where $D = \dfrac{\partial}{\partial t^m}$.

Moreover, the mapping $H^{s-\frac{1}{2}}(R_o^m) \ni \bar{x} \longrightarrow x \in H^s(R^m)$ is continuous for $0 \le s \le m$.

3rd step

For $x \in H^o(R_+^m)$ let us define $x_1 \in H^o(R_-^m)$ by

$$x_1(t', t^m) = \sum_{i=1}^{k} a_i x(t', -it^m) \qquad (t^m \le 0), \quad \text{where}$$

$$\sum_{i=1}^{k} (-i)^j a_i = \begin{matrix} 0 & \text{for} & j = 0 \\ 1 & \text{for} & j = 1,\dots,m-1 \end{matrix} \quad .$$

We see, that the mapping

$$\rho: H^{-\frac{1}{2}}(R_o^m) \oplus H^o(R_+^m) \longrightarrow H^o(R^m)$$

defined by

$$\rho(\bar{x} \oplus x) = \begin{cases} x & \text{on} \quad R_+^m \\ \bar{x}\big|_{R^m} + x_1 & \text{on} \quad R_-^m \end{cases}$$

is continuous and if $x \in H^k(R_+^m)$, $\bar{x} = x\big|_{R_o^m}$ then $\rho(\bar{x} \oplus x) \in H^k(R^m)$ and the mapping

$$\rho : X_N^k \longrightarrow H^k(R^m) \quad \text{is continuous.}$$

Hence, we see immediately that ρ is the mapping with properties (i) - (iv).

B. The inclusion $\left[\underset{\sim}{X}_N^k , \underset{\sim}{X}_N^o\right]_\theta \supset \underset{\sim}{X}_N^{(1-\theta)k}$

We shall use the following characterization of an interpolation space (see e.g. [22]):
for two Hilbert spaces $H \subset H'$, an element $h \in H'$ belongs to $[H,H']_\theta$ iff there exists a function $\tilde{h} \in L^2(R_+;H) \cap H^s(R_+;H')$ such that $\tilde{h}(0) = h$, $s = \frac{1}{2\theta}$.

Now, let $\bar{x} \oplus x \in \underset{\sim}{X}_N^{(1-\theta)k}$. Since

$\underset{\sim}{x} \in H^{(1-\theta)k}(R_+^m) = \left[H^k(R_+^m),\ H^o(R^m)\right]_\theta$, there exist

$$\tilde{\underset{\sim}{x}} \in L^2(R;H^k(R_+^m)) \cap H^s(R;H^o(R_+^m)) \qquad (s = \frac{1}{2\theta})$$

such that $\tilde{\underset{\sim}{x}}(0) = x$.

By the same argument there exists

$$\tilde{\bar{\underset{\sim}{x}}} \in L^2(R;H^{k-\frac{1}{2}}(R_o^m)) \cap H^s(R;H^{-\frac{1}{2}}(R_o^m)) \qquad (s = \frac{1}{2\theta})$$

with $\tilde{\bar{\underset{\sim}{x}}}(0) = \bar{\underset{\sim}{x}}$.

We shall construct $\tilde{\underset{\sim}{x}}$ and $\tilde{\bar{\underset{\sim}{x}}}$ such that

$$\tilde{\underset{\sim}{x}} \oplus \tilde{\bar{\underset{\sim}{x}}} \in L^2(R;\underset{\sim}{X}_N^k) \cap H^s(R;\underset{\sim}{X}_N^o) \tag{3.2.3}$$

First, we define the function $\tilde{\bar{\underset{\sim}{x}}}_1$ by its Fourier transform with respect to t':

$$\hat{\tilde{\bar{\underset{\sim}{x}}}}_1(t^o;\xi') = \hat{\bar{\underset{\sim}{x}}}(\xi')\ \phi((1+|\xi'|^2)^{\frac{k}{2s}}\ t^o) \qquad (\phi \text{ as before})$$

We have

$$\|\tilde{\bar{\underset{\sim}{x}}}_1\|^2_{0,k-\frac{1}{2}} = \int |\hat{\bar{\underset{\sim}{x}}}(\xi')|^2 |\phi((1+|\xi'|^2)^{\frac{k}{2s}} t^o)|^2 (1+|\xi'|^2)^{k-\frac{1}{2}} d\xi' dt^o =$$

$$= c \int (1+|\xi'|^2)^{k(1-\theta)-\frac{1}{2}} |\hat{\bar{\underset{\sim}{x}}}(\xi')|^2\, d\xi' < \infty$$

($\|\ \|_{\rho,s}$ stands for the norm in $H^\rho(R;H^s(R_o^m))$) and

$$\|\tilde{\tilde{x}}\|^2_{s,-\frac{1}{2}} = \int(1+|\xi^o|^2)^s|\hat{\tilde{\tilde{x}}}(\xi^o;\xi')|^2(1+|\xi'|^2)^{-\frac{1}{2}}\, d\xi'\, d\xi^o .$$

But

$$\hat{\tilde{\tilde{x}}}_1(\xi^o;\xi') = \int e^{it^o\xi^o}\,\hat{\tilde{\tilde{x}}}(\xi^o;\xi')dt^o = \hat{\tilde{x}}(\xi')\,(1+|\xi'|^2)^{-\frac{k}{2s}} .$$

$$.\ \hat{\phi}((1+|\xi'|^2)^{-\frac{k}{2s}}\xi^o),$$

hence

$$\|\tilde{\tilde{x}}_1\|^2_{s,-\frac{1}{2}} \le \int|\hat{\tilde{x}}(\xi')^2(1+|\xi'|^2)^{k(1-\theta)-\frac{1}{2}}d\xi' \int(1+\tau^2)^s|\,\hat{\phi}(\tau)|^2d\tau < \infty$$

Thus $\tilde{\tilde{x}}_1 \in L^2(R;H^{k-\frac{1}{2}}(R^m_o)) \cap H^s(R;H^{-\frac{1}{2}}(R^m_o))$.

Now, let us consider a function $\tilde{x}_1$ defined by its Fourier transform with respect to t^o and t':

$$\hat{\tilde{x}}_1(\xi^o;\xi',t^m) = \hat{\tilde{\tilde{x}}}_1(\xi^o;\xi')\ \phi((1+|\xi'|^2)^{\frac{1}{2}}t^m)$$

It is easy to check (calculations as in the first part of the proof) that

$$\tilde{x}_1 \in L^2(R;H^k(R^m)) \cap H^s(R;H^o(R^m)):$$

$$\|\tilde{x}_1\|^2_{0,k} = \int(1+|\xi'|^2+|\xi^m|^2)^k\,|\hat{\tilde{x}}(\xi^o;\xi)|^2\,d\xi^o d\xi \leqslant$$

$$\leqslant c\int(1+|\xi'|^2)^{k-\frac{1}{2}}\hat{\tilde{\tilde{x}}}_1(\xi^o;\xi')\,d\xi^o\,d\xi' < \infty$$

and

$$\|\tilde{x}_1\|^2_{s,0} = \int(1+|\xi^o|^2)^s\,|\hat{\tilde{x}}_1|^2\,d\xi^o d\xi \le$$

$$\le \int(1+|\xi'|^2)^{-\frac{1}{2}}|\hat{\tilde{\tilde{x}}}_1(\xi^o;\xi')|^2(1+|\xi^o|^2)^s d\xi^o d\xi' < \infty$$

By the trace theorem (see e.g. [22]):

$$\tilde{x}_1\Big|_{t^o=0} \in H^{k(1-\theta)}(R^m).$$

Let us consider the function on R_+^m :

$$\underset{\sim}{x}_2 = \underset{\sim}{x} - \tilde{\underset{\sim}{x}}_1\Big|_{t^o = 0} .$$

Of course $\underset{\sim}{x}_2 \in H^{k(1-\theta)}(R_+^m)$ and for $k(1-\theta) > \frac{1}{2}$ we have $\underset{\sim}{x}_2\big|_{R_o^m} = 0$.
Let us denote the space of such functions by $H_o^{k(1-\theta)}(R_+^m)$.

By Grisvard's theorem ([30]):

$$\left[H_o^k(R_+^m), H^o(R_+^m)\right]_\theta = H_o^{k(1-\theta)}(R_+^m) \quad \text{for} \quad k(1-\theta) > \frac{1}{2}$$
$$= H^{k(1-\theta)}(R_+^m) \quad \text{for} \quad k(1-\theta) < \frac{1}{2} .$$

Hence, there exist $\tilde{\underset{\sim}{x}}_2 \in L^2(R; H_o^k(R_+^m)) \cap H^s(R; H^o(R_+^m))$ 3.2.4.

such that $\tilde{\underset{\sim}{x}}_2(0) = \underset{\sim}{x}_2$.

Now, we put $\tilde{\underset{\sim}{x}} = \tilde{\underset{\sim}{x}}_1 + \tilde{\underset{\sim}{x}}_2$. Of course

$$\tilde{\underset{\sim}{x}} \in L^2(R; H^k(R_+^m)) \cap H^s(R; H^o(R_+^m)) \quad \text{and} \quad \tilde{\underset{\sim}{x}}(0) = \underset{\sim}{x}.$$

Because of 3.2.4 we have that

$$\tilde{\underset{\sim}{x}} \oplus \tilde{\underset{\sim}{x}} \in L^2(R; \underset{\sim}{X}^k) \cap H^s(R; \underset{\sim}{X}^o).$$

In order to prove that $\left[\underset{\sim}{X}^k, \underset{\sim}{X}^{-k}\right]_{\frac{1}{2}} = X^o$ we use the same method.

It is sufficient to notice that in the construction of $\tilde{\underset{\sim}{x}}_1$ we can limit ourselves to $t^m \geq 0$ and we get an element of

$$L^2(R; H^k(R_+^m)) \cap H^s(R; H^{-k}(R_+^m)).$$

In fact, the norm of $\tilde{\underset{\sim}{x}}_1\Big|_{t^m > 0}$ in $H^s(R; H^{-k}(R_+^m))$ is equivalent to the norm in $H^s(R; H^{-k}(R^m))$. Now, we get continuous injection which is a surjection. Hence it is an isomorphism. ◆

Applying interpolation theorem for conjugate spaces we get

3.2.5. Corollary

Let be $s > s'$, $s, s' \neq -\frac{1}{2}$, $0 < \theta < 1$, $(1-\theta)s + \theta s' \neq -\frac{1}{2}$.

Then

$$\left[\underset{\sim}{Y}_N^s \ , \ \underset{\sim}{Y}_N^{s'}\right]_\theta = \underset{\sim}{Y}_N^{(1-\theta)s+\theta s'}$$ ◆

In order to prove a similar theorem for the family $\{\underset{\sim}{X}^s_{D,\lambda}\}_{s\neq\frac{3}{2}}$ we use the same technique, however constructions are a little bit more complicated, e.g. we have to make use of the elipticity of an Euler-Lagrange operator.

3.2.6. Theorem

Let be $s, s' \neq \frac{3}{2}$, $s > s'$, $0<\theta<1$, $\theta s' + (1-\theta)s \neq \frac{3}{2}$, λ-elliptic (i.e. the corresponding Euler-Lagrange operator is elliptic). Then

$$\left[\underset{\sim}{X}^s_{D,\lambda} \ , \ \underset{\sim}{X}^{s'}_{D,\lambda}\right]_\theta = \underset{\sim}{X}^{(1-\theta)s+\theta s'}_{D,\lambda}$$

Proof.

As in 3.2.1. we reduce the problem to the case $\Omega = R^m_+$, $\partial\Omega = R^m_o$, $s=k$, $s'=1$ and $s' = k+2$ $(\theta = \frac{1}{2})$. We have (see 2.1.)

$$\mathcal{H}^\mu{}_A(\underset{\sim}{x}) = \lambda^{\mu\nu}_{AB}\partial_\nu \underset{\sim}{x}^B + \lambda_{AB}\underset{\sim}{x}^B \ .$$

But we are interested in $\mathcal{H}^m_A$ only:

$$\mathcal{H}_A(\underset{\sim}{x}) \ := \mathcal{H}^m_A(\underset{\sim}{x}) = \lambda^{mm}_{AB}\,\partial_m \underset{\sim}{x}^B + (\lambda^{m\nu'}_{AB}\,\partial_{\nu'}\underset{\sim}{x}^B + \lambda^m_{AB}\,\underset{\sim}{x}^B), \ \nu'=1,\dots,m-1.$$

We chose a trivialization in the bundle X in a such way that $\lambda^{mm}_{AB} = \lambda_B\,\delta^B_A$, $\lambda_B \neq 0$. It is possible because of elipticity of λ.

We put

$$\mathcal{H}'_A(\underset{\sim}{x}) \ := \lambda_A\,\partial_m \underset{\sim}{x}^A \qquad A = 1,\dots,n$$

$$\mathcal{H}''_A(\underset{\sim}{x}) \ := \lambda^{m\nu'}_{AB}\,\partial_{\nu'}\underset{\sim}{x}^B + \lambda^m_{AB}\,\underset{\sim}{x}^B \ .$$

In order to prove the inclusion

$$\left[\underset{\sim}{X}^s_{D,\lambda} \ , \ \underset{\sim}{X}^{s'}_{D,\lambda}\right]_\theta \subset \underset{\sim}{X}^{(1-\theta)s+\theta s'}_{D,\lambda} \qquad (3.2.7)$$

we shall construct a mapping

$$\rho: H^1(R^m_+ ; R^n) \oplus H^{-\frac{1}{2}}(R^m_o;R^n) \longrightarrow H^1(R^m;R^n)$$

as follows:

Let be $\underset{\sim}{x} \oplus \underset{\sim}{p} \in H^1(R^m_+;R^n) \oplus H^{-\frac{1}{2}}(R^m_o;R^n)$, we put

$$\bar{\underset{\sim}{p}}_A = (\underset{\sim}{p}_A - \mathcal{X}''_A(\underset{\sim}{x}|_{R^m_o}))\, \lambda_A^{-1} \tag{3.2.8}$$

where

$$\mathcal{X}''(\underset{\sim}{x}|_{R^m_o}) \in H^{-\frac{1}{2}}(R^m_o;R^n).$$

Now, we define a function $\underset{\sim}{x}_1$ on R^m by its Fourier transform with respect to t':

$$\hat{\underset{\sim}{x}}_1(\xi',t^m) = (1+|\xi'|^2)^{-\frac{1}{2}} \ ((1+|\xi'|^2)^{\frac{1}{2}} t^m)\ \hat{\bar{\underset{\sim}{p}}}_A \quad \text{where} \quad \phi\in C_o^\infty(R)$$

with $\phi'(0) = 1$, $\phi(o) = 0$.

Simple calculations show that $\underset{\sim}{x}_1 \in H^1(R;R^n)$; moreover, if $\underset{\sim}{p} \in H^{s-\frac{3}{2}}(R^m_o;R^n)$, $\underset{\sim}{x} \in H^s(R^m_o;R^n)$, then $\underset{\sim}{x}_1 \in H^s(R^m;R^n)$ and the mapping

$$H^s(R^m;R^n) \oplus H^{s-\frac{3}{2}}(R^m_o;R^n) \ni \underset{\sim}{x} \oplus \underset{\sim}{p} \longrightarrow \underset{\sim}{x}_1 \in H^s(R^m;R^n)$$

is continuous.

We take sets of real numbers $\{C_i\}_{i=1}^k$ and $\{a_i\}_{i=1}^k$ such that

$$\sum_{i=1}^k (i)^j\, C_i = \begin{cases} 0 & \text{for } j=0,2,\dots,k-1 \\ 1 & \text{for } j=1 \end{cases}$$

and

$$\sum_{i=1}^k (-i)^j\, a_i = \begin{cases} 0 & j=1 \\ 1 & j=0,2,\dots,k-1. \end{cases}$$

It is evident that the function $\underset{\sim}{x}_2$ on R^n_- :

$$\underset{\sim}{x}_2(t',t^m) := \sum_{i=1}^{k} C_i \underset{\sim}{x}_1(t',it^m) + \sum_{i=1}^{k} a_i \underset{\sim}{x}(t',-it^m), \quad t^m \le 0$$

is in $H^1(R^m_-;R^n)$.

Let us define ρ:

$$\rho(\underset{\sim}{x} \oplus \underset{\sim}{p}) := \begin{cases} \underset{\sim}{x} & \text{on } R^m_+ \\ \underset{\sim}{x}_2 & \text{on } R^m_- \end{cases}$$

The following properties of ρ are easy to verify

(i) ρ is continuous as the mapping

$$\rho : H^1(R^m_+;R^n) \oplus H^{-\frac{1}{2}}(R^m_o;R^n) \longrightarrow H^1(R^m;R^n)$$

(ii) ρ is continuous as the mapping

$$\rho : H^k(R^m_+;R^n) \oplus H^{k-\frac{3}{2}}(R^m_o;R^n) \supset \underset{\sim}{X}^k_{D,\lambda} \longrightarrow H^k(R^m;R^n)$$

(iii) if $\rho(\underset{\sim}{x} \oplus \underset{\sim}{p}) \in H^s(R^m;R^n)$, $s > \frac{3}{2}$ then

$$\underset{\sim}{p} = \mathcal{H}(x)\Big|_{R^m_o} .$$

With this properties we have, as in 3.2.1.,the inclusion 3.2.7.

The relation $\left[\underset{\sim}{X}^s_{D,\lambda}, \underset{\sim}{X}^s_{D,\lambda}\right]_\theta \supset \underset{\sim}{X}^{(1-\theta)s+\theta s'}_{D,\lambda}$ can be proved as in 3.2.1. be the construction of an element $\tilde{\underset{\sim}{x}} \oplus \tilde{\underset{\sim}{p}} \in L^2(R;\underset{\sim}{X}^k_{D,\lambda})$ $H^s(R;\underset{\sim}{X}^1_{D,\lambda})$ such that $\tilde{x}(o) + \tilde{p}(o)$ is a given element of $X^{(1-\theta)s+\theta s'}_{D,}$. This construction is analogous to that of 3.2.1. and follows in steps (we shall consider the case $s' = 1$).

Let be $\underset{\sim}{x} \oplus \underset{\sim}{p} \in \underset{\sim}{X}^{(1-\theta)s+\theta s'} \subset H^{(1-\theta)k+\theta}(R^m_t;R^n) \oplus H^{(1-\theta)k+\theta-\frac{3}{2}}(R^m_o;R^n)$.

1st step. $\underset{\sim}{p}$ defines $\bar{\underset{\sim}{p}}$ as in 3.2.8. which is in $H^{(1-\theta)+\theta-\frac{3}{2}}(R^m_o;R^n)$.
Hence there exists

$$\tilde{\bar{\underset{\sim}{p}}}_1 \in L^2(R;H^{k-\frac{3}{2}}(R^m_o;R^n)) \cap H^\sigma(R;H^{-\frac{1}{2}}(R^m_o;R^n)), \quad \sigma = \frac{1}{2\theta} ,$$

such that $\tilde{\bar{\underset{\sim}{p}}}_1(o) = \bar{\underset{\sim}{p}}$.

2nd step. We introduce

$$\tilde{x}_i \in L^2(R;H^k(R^m;R^n)) \cap H^\sigma(R;H^1(R^m_o;R^n)) \qquad i=1,2$$

putting

$$\hat{\tilde{x}}_1 = \hat{\bar{p}}_1 \, \phi_1((1+|\xi'|^2)^{\frac{1}{2}} t^m)(1+|\xi'|^2)^{-\frac{1}{2}} \quad \text{where} \quad \phi_1 \in C^\infty_o(R)$$

$$\phi_1(o) = 0, \ \phi_1'(o) = 1$$

and

$$\hat{\tilde{x}}_2 = \hat{\bar{x}} \, \phi_2((1+|\xi'|^2)^{\frac{1}{2}} t^m) \quad \text{where} \quad \phi_2 \in C^\infty_o(R), \ \phi_2(o)=1, \ \phi_2(o)=0$$

$$\bar{x} = x|_{R^m_o}$$

3rd step. We introduce $\tilde{x}_3$ in the following way:

Let us take $x_3 := x - x_1\big|_{t^o=0} - x_2\big|_{t^o=0}$.

We see that

$$x_3\big|_{t^m=0} = \partial_m x_3\big|_{t^m=0} = 0$$

(providing θ is such that these conditions are meaningfull) and $x_3 \in H^{(1-\theta)k+\theta}(R^m_+;R^n)$.

By Grisvard's theorem ([30]) there exists $\tilde{x}_3 \in L^2(R;H^k_o(R^m_+,R^n)) \cap H^\sigma(R;H^1_o(R^m_+,R^n))$ such that $\tilde{x}_3(o) = x_3$.

Here we use the notation:

$$H^k_o(R^m_+,R^n) = \{x \in H^k(R^m_+,R^n) : x\big|_{t^m=0} = 0, \ \partial_m x\big|_{t^m=0} = 0\},$$

$$H^1_o(R^m_+,R^n) = \{x \in H^1(R^m_+,R^n) : x\big|_{t^n=0} = 0\}.$$

4th step. We observe, that functions $\tilde{x} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$

$$\tilde{p} = \tilde{\bar{p}}_1 + \mathcal{H}''(\tilde{x}\big|_{t^m=0})$$

form an element $\tilde{x} \oplus \tilde{p}$ of $L^2(R;X^k_{D,\lambda}) \cap H^\sigma(R;X^1_{D,\lambda})$

such that $\tilde{p}(o) = p$, $\tilde{x}(o) = x$.

This completes the proof for $s' = 1$.

The proof for $s' = -k+2$, $\theta = \frac{1}{2}$ follows in the same way. ◆

3.2.9. Corollary.

Let $s, s' \neq -\frac{3}{2}$, $s > s'$, $0 < \theta < 1$, $\theta s' + (1-\theta)s \neq -\frac{3}{2}$, λ-elliptic. Then

$$\left[Y^{s}_{D}, \; Y^{s'}_{D,\lambda}\right] = Y^{(1-\theta)s+\theta s'}_{D,\lambda} \quad . ◆$$

Now, we complete families $\{X^{s}_{N}\}_{s \neq \frac{1}{2}}$, $\{X^{s}_{D,\lambda}\}_{s \neq \frac{3}{2}}$ putting

$$X^{\frac{1}{2}}_{N} := \left[X^{s_1}_{N}, X^{s_2}_{N}\right]_{\theta} \quad \text{where} \quad (1-\theta)s_1 + \theta s_2 = \frac{1}{2} .$$

$$X^{\frac{3}{2}}_{D,\lambda} := \left[X^{s_1}_{D,\lambda} , \; X^{s_2}_{D,\lambda}\right]_{\theta} \quad \text{where} \quad (1-\theta)s_1 + \theta s_2 = \frac{3}{2}$$

We define

$$Y^{-\frac{1}{2}}_{N} \; , \; Y^{-\frac{3}{2}}_{D,\lambda} \quad \text{as their dual space.}$$

3.2.10. Proposition – $X^{\frac{1}{2}}_{N}$ and $X^{\frac{3}{2}}_{D,\lambda}$ do not depend on the choice of s, s'.

Proof.

By 3.2.1. we have $\left[X^{s}_{N} = X^{k}_{N} , X^{-k}_{N}\right]_{\theta_i}$, $k > |s_i|$ $i=1,2$.

By the second interpolation theorem

But $(1-\theta)s_1+\theta s_2 = \frac{1}{2}$, so $(1-\theta)\theta_1+\theta\theta_2 = \frac{1}{2}(1 - \frac{1}{2k}) = \frac{1}{2}(\frac{1}{2} - \frac{1}{2k}) + \frac{1}{2}.\frac{1}{2}$.

Hence,

$$\left[X^{s_1}_{N} , \; X^{s_2}_{N}\right] = \left[X^{1}_{N} , \; X^{o}_{N}\right]_{\frac{1}{2}} \; .$$

The same can be done for $X_{D,\lambda}$. ◆

Unfortunately $X^{\frac{1}{2}}_{N}$ $(X^{\frac{3}{2}}_{D,\lambda})$ is not a closed subspace of $X^{o} \oplus X^{\frac{1}{2}}$

$(\underset{\sim}{X}^{\frac{3}{2}} \oplus \underset{\sim}{P}^{o})$.

3.2.11. Proposition – $\underset{\sim}{X}_N^{\frac{1}{2}}$ $(\underset{\sim}{X}_{D,\lambda}^{\frac{3}{2}})$ is a proper, dense subspace of

$\underset{\sim}{X}^{o} \oplus \underset{\sim}{X}^{\frac{1}{2}}$ $(X^{\frac{3}{2}} \oplus P^{o})$.

Proof. – We present the proof for $\underset{\sim}{X}_N$ only.

Suppose that $\left[\underset{\sim}{X}_N^1, \underset{\sim}{X}_N^o\right]_{\frac{1}{2}} = \bar{\underset{\sim}{X}}^{o} \oplus \underset{\sim}{X}^{\frac{1}{2}}$. It means that for every $\underset{\sim}{x} \in X^{\frac{1}{2}}$, $\bar{\underset{\sim}{x}} \in \bar{X}^{o}$ there exists $\tilde{\bar{\underset{\sim}{x}}} \oplus \tilde{\underset{\sim}{x}} \in L^2(R;\underset{\sim}{X}_N^1) \cap H^1(R;\underset{\sim}{X}_N^o)$ such that $\tilde{\bar{\underset{\sim}{x}}}(o) = \bar{\underset{\sim}{x}}$, $\underset{\sim}{x}(o) = \underset{\sim}{x}$. Now, let us fix $\underset{\sim}{x}_1 \in \underset{\sim}{X}^{\frac{1}{2}}$, ${}_1\bar{\underset{\sim}{x}} \in \underset{\sim}{X}^{o}$. Using the standard procedure (see proofs of theorems) we get $\underset{\sim}{x}_2 \in X^{\frac{1}{2}}$ such that $\underset{\sim}{x}_2(t^m = 0) = \bar{\underset{\sim}{x}}$. We put $\underset{\sim}{x} = \underset{\sim}{x}_1 + \underset{\sim}{x}_2$ and take $\tilde{\bar{\underset{\sim}{x}}} \oplus \tilde{\underset{\sim}{x}}$ described above. But we can spread out $\tilde{\bar{\underset{\sim}{x}}}$ to $\tilde{\underset{\sim}{x}}_2$ that $\tilde{\bar{x}} \oplus \tilde{x}_2 \in L^2(R;\underset{\sim}{X}_N^1) \cap H^1(R;\underset{\sim}{X}_N^o)$ where $\tilde{\underset{\sim}{x}}_2\big|_{t^o = 0} = \underset{\sim}{x}_2$, $\tilde{\underset{\sim}{x}}_2\big|_{t^m = 0} = \tilde{\bar{\underset{\sim}{x}}}$. Hence $\tilde{\underset{\sim}{x}}_1 := \tilde{\underset{\sim}{x}} - \tilde{\underset{\sim}{x}}_2$ is such that

$$\tilde{\underset{\sim}{x}}_1(o) = \underset{\sim}{x}_1 \text{ , } 0 \oplus \tilde{x}_1 \in L^2(R;\underset{\sim}{X}_N^1) \cap H^1(R;\underset{\sim}{X}_N^o) \quad \text{i.e.}$$

$$\tilde{\underset{\sim}{x}}_1 \in L^2(R;H_o^1(\Omega;X)) \cap H^1(R;H^o(\Omega;X)).$$

Since $\underset{\sim}{x}_1$ is an arbitrary element of $H^{\frac{1}{2}}(\Omega;X)$ it means that

$$\left[H_o^1(\Omega;X), H^o(\Omega;X)\right]_{\frac{1}{2}} = H^{\frac{1}{2}}(\Omega;X)$$

But it is not true (see [22]). ◆

Since $\underset{\sim}{X}_N^{\frac{1}{2}} \subsetneqq \bar{\underset{\sim}{X}}^{o} \oplus \underset{\sim}{X}^{\frac{1}{2}}$, $\underset{\sim}{X}_{D,\lambda}^{\frac{3}{2}} \subsetneqq \underset{\sim}{X}^{\frac{3}{2}} \oplus \underset{\sim}{P}^{o}$ we have that

$$\underset{\sim}{Y}_N^{-\frac{1}{2}} \supsetneqq \underset{\sim}{P}^{o} \oplus F^{-\frac{1}{2}} \text{ , } \quad \underset{\sim}{Y}_{D,\lambda}^{-\frac{3}{2}} \supsetneqq \bar{\underset{\sim}{X}}^{o} \oplus \underset{\sim}{F}^{-\frac{3}{2}}$$

3. Elipticity of the Neumann boundary problem

It is known ([31]) that the Dirichlet problem for elliptic dynamics is of the elliptic type.

The operator $\mathcal{K}^{\mu}_{A}(x)$ gives rise to the boundary problem which we call the Neumann problem.

3.3.1. Lemma - Suppose that the Euler-Lagrange operator ε is of elliptic type. Then the boundary Neumann problem is elliptic.

For the proof of the Lemma see Appendix. ◆

4. Isomorphism theorems

In this section we make use of the interpolation theorems in order to prove isomorphism theorems for Dirichlet and Neumann boundary problems. From now on we assume that the Euler-Lagrange operator ε is elliptic.

As we have seen in Chapter II, for smooth sections the dynamics in the phase space associated with Ω is represented by graphs of the operators

$$\Lambda_N : \bar{\underset{\sim}{X}} \oplus \underset{\sim}{X} \supset \underset{\sim}{X}_N \longrightarrow \underset{\sim}{P} \oplus \underset{\sim}{F} :$$

$$: \underset{\sim}{x}\big|_{\partial\Omega} \oplus \underset{\sim}{x} \longrightarrow \mathcal{K}^t(\underset{\sim}{x}) \oplus \varepsilon(\underset{\sim}{x}) \qquad (\mathcal{K}^t(\underset{\sim}{x}) = \mathcal{K}(\underset{\sim}{x})\big|_{\partial\Omega})$$

or

$$\Lambda_D : \underset{\sim}{P} \oplus \underset{\sim}{X} \supset \underset{\sim}{X}_{D,\lambda} \longrightarrow \bar{\underset{\sim}{X}} \oplus \underset{\sim}{F} :$$

$$: \mathcal{K}^t(\underset{\sim}{x}) \oplus x \longrightarrow \underset{\sim}{x}\big|_{\partial\Omega} \oplus \varepsilon(\underset{\sim}{x}) .$$

Moreover, since $\mathcal{K}$ is a first order operator and ε is of second order, we have families of operators

$$\Lambda_N^s: \underset{\sim}{X}_N^s \to \underset{\sim}{Y}_N^{s-2} \subset \underset{\sim}{Y}_N^{-s} \qquad s \geq 2 \qquad (3.4.1)$$

$$\Lambda_D^s: \underset{\sim}{X}_{D,\lambda}^s \longrightarrow \underset{\sim}{Y}_{D,\lambda}^{s-2} \subset Y_{D,\lambda}^{-s}$$

From 1.6.2 follows that the graph of $\Lambda_N^s(\Lambda_D^s)$ is a Lagrangian subspace of $\underset{\sim}{S}_N^s$ $(\underset{\sim}{S}_{D,\lambda}^s)$. It means that $\Lambda_N^s(\Lambda_D^s)$ is a selfadjoint operator (see 4.1.1.) as an operator

$$\Lambda_N^s : \underset{\sim}{X}_N^s \longrightarrow \underset{\sim}{Y}_N^{-s} \quad (\Lambda_D^s: \underset{\sim}{X}_{D,\lambda}^s \longrightarrow \underset{\sim}{Y}_{D,\lambda}^{-s}).$$

It follows that the conjugate to 3.4.1. mappings

$$^t\Lambda_N^s : \underset{\sim}{X}_N^{-s+2} \longrightarrow \underset{\sim}{Y}_N^{-s} \qquad s \geq 2$$

$$^t\Lambda_D^s : \underset{\sim}{X}_{D,}^{-s+2} \longrightarrow \underset{\sim}{Y}_D^{-s}$$

are extensions of Λ_N^s , Λ_D^s . Hence, we shall denote them by Λ_N^{-s+2} , Λ_D^{-s+2} respectively.

By the isomorphism theorem for smooth sections ([17]) Λ_N^s , Λ_D^s $(s \geq 2)$ have finite-dimensional kernels $\underset{\sim}{K}_N, \underset{\sim}{K}_D$ which do not depend on s and their images are closed in $\underset{\sim}{Y}_N^{s-2}$, $\underset{\sim}{Y}_{D,}^{s-2}$. Of course the images are anihilators of the kernels.

We see, that the conjugate operators Λ_N^{-s+2} , Λ_D^{-s+2} have the same kernels $\underset{\sim}{K}_N$, $\underset{\sim}{K}_D$ and their images are closed with finite-dimensional co--kernels.

From the interpolation theorem immediately follows.

3.4.2. Theorem - Let us denote by $\tilde{\underset{\sim}{Y}}_N^s$ and $\tilde{\underset{\sim}{Y}}_{D,\lambda}^s$ anihilators of $\underset{\sim}{K}_N$ and

$\underset{\sim}{K}_D$ in $\underset{\sim}{Y}_N^s$, $\underset{\sim}{Y}_{D,\lambda}^s$ respectively. Let the dynamics be elliptic, then we have the complete families of isomorphisms

$$\begin{aligned} \Lambda_N^s &: \underset{\sim}{X}_N^s / \underset{\sim}{K}_N \longrightarrow \tilde{\underset{\sim}{Y}}_N^{s-2} \\ & \qquad\qquad\qquad\qquad s \in R \\ \Lambda_D^s &: \underset{\sim}{X}_{D,\lambda}^s / \underset{\sim}{K}_D \longrightarrow \tilde{\underset{\sim}{Y}}_{D,\lambda}^s \end{aligned}$$ ◆

3.4.3. Remarks

(i) This method can be applied to non-selfadjoint operators and other boundary problems.

(ii) It is interesting to compare our results with that of Rojtberg ([23] - [26]). It appears that for self-adjoint problems his theorems are weaker (for $s < 2$) and do not have symplectic interpretation.

(iii) Similar results can be obtained for higher order theories. Corresponding results will be published elsewhere.

(iv) For non-symplectic interpretations of the isomorphisms theorems (especially for negative s) see [22] - [26] , [18] .

(v) Isomorphisms theorems are useful in the study of mixed problem for nonlinear hyperbolic equations (see e.g. [32]).

CHAPTER IV

DYNAMICS AS A LAGRANGIAN SUBSPACE

In this chapter we shall give a symplectic interpretation of the results of previous sections.

1. Lagrangian subspaces and self-adjoint operators.

We shall give an important example of a Lagrangian subspace

4.1.1. Lemma - Let V be a Banach space, V^* its dual. The graph of an operator $L: V \supset D \longrightarrow V^*$, where D is a dense subspace of V, is a Lagrangian subspace iff L is self-adjoint.

Proof. - Let L be self-adjoint, $G(L)$ - its graph. Let $v_o \oplus v_o^* \varepsilon G(L)^§$ (see 1.6.). It means that for each $v \varepsilon D$ $\omega(v_o \oplus v_o^*, v \oplus L(v)) = 0 = $ $= <v, v_o^*> - <v_o, L(v)>$. Hence v_o is in the domain of L^* and $v_o^* = L^*(v_o) = L(v_o)$. On the other hand, if $G(L)$ is Lagrangian subspace then L is maximal symmetric.

But $G(L)^§ \supset G(L^*)$ and $G(L) = G(L^*)$ since $G(L^*) \supset G(L) = G(L)^§$. ◆

2. Dynamics as Lagrangian subspaces in $\underset{\sim}{S}_N^s$ and $\underset{\sim}{S}_{D,}^s$.

First, we have to look at the dynamics as a subspace of a symplectic space. But for $s < 1$ $\underset{\sim}{X}_N^s$ and $\underset{\sim}{Y}_N^{s-2}$, $\underset{\sim}{X}_{D,\lambda}^s$ and $\underset{\sim}{Y}_{D,\lambda}^{s-2}$ cannot be treated as dual pairs.

Hence the graphs of Λ_D^s , Λ_N^s $s < 1$ do not have a symplectic interpretations, althought for some reson (see Chapter VI) are of interest in a symplectic approach too.

Let us look at dynamics in $\underset{\sim}{S}_N^1$, $\underset{\sim}{S}_{D,\lambda}^1$.

The following proposition is a simple consequence of 3.4.2. (see 1.6.2.)

4.2.1. *Proposition* – $\underset{\sim}{D}_N^1 := \mathrm{Gr}(\Lambda_N^1)$ and $\underset{\sim}{D}_D^1 = \mathrm{Gr}(\Lambda_D^1)$ are Lagrangian submanifolds of $\underset{\sim}{S}_N^s$ and $\underset{\sim}{S}_{D,\lambda}^s$ respectively.

There are canonical symplectic relations (for the concept of a symplectic relation see [29] or Chapter V) between $\underset{\sim}{S}_N^1$ and $\underset{\sim}{S}_N^s$, $\underset{\sim}{S}_{D,\lambda}^1$ and $\underset{\sim}{S}_{D,\lambda}^s$ defined by injections 3.1.5.

4.2.2. *Theorem* – The image of $\underset{\sim}{D}_N^1$ ($\underset{\sim}{D}_D^1$) in $\underset{\sim}{S}_N^s$ ($\underset{\sim}{S}_{D,\lambda}^s$) under the canonical symplectic relations is a Lagrangian subspace.

Proof. – We already know that the image of a Lagrangian subspace under a symplectic relation is an isotropic subspace.

But the mapping

$$\Lambda_N^1 : \underset{\sim}{X}_N^1 / \underset{\sim}{K}_N \longrightarrow \tilde{\underset{\sim}{Y}}_N^{-1}$$

is an isomorphism and there is a symplectic relation between $\underset{\sim}{X}_N^1 / \underset{\sim}{K}_N \oplus \tilde{\underset{\sim}{Y}}_N^{-1}$

and $X_N^s/K_N \oplus \tilde{Y}_N^{-s}$ induced by injections

$$X_N^s/K_N \longrightarrow X_N^1/K_N \ , \quad \tilde{Y}_N^{-s} \longleftarrow \tilde{Y}_N^{-1} \quad \text{for} \ s>1$$

and

$$X_N^s/K_N \longleftarrow X_N^1/K_N \ , \quad \tilde{Y}_N^{-s} \longrightarrow \tilde{Y}_N^{-1} \quad \text{for} \ s<1 \ .$$

With 1.6.2. it means that the image of $GR(\Lambda_N^1)$ in $X_N^s/K_N \oplus \tilde{Y}_N^{-s}$ is a Lagrangian subspace.

Now, since K_N is finite-dimensional, it is obvious that the inverse image of this subspace under the reduction from S_N^s to $X_N^s/K_N \oplus \tilde{Y}_N^{-s}$ is again Lagrangian. We shall denote it by D_N^s. In the same way we get a Lagrangian subspace D_D^s in $S_{D,\lambda}^s$.

Q.E.D.

We have the symplectic relation between S_N^s and $S_N^{s'}$ ($S_{D,\lambda}^s$ and $S_{D,\lambda}^{s'}$) for each pair (s,s'), but in general $D_N^{s'}$ is not the image of D_N^s (e.g. $s \neq 1$, $s' = 1$).

But one can easily check using isomorphism theorems that if $s>1$, $s' = s-2$ then D_N^s is an image of $D_N^{s'}$ and vice-versa. In fact, it is the symplectic content of 3.4.2. Theorem.

4.2.3. Theorem – Suppose that the dynamics in S_N^1 ($S_{D,\lambda}^1$) is an inverse image of the graph of the isomorphism $X_N^1/K_N \longrightarrow \tilde{Y}_N^1$, ($X_{D,\lambda}^1/K_D \longrightarrow \tilde{Y}_{D,\lambda}^1$) under the symplectic relation. Let K_N (K_D) be finite dimensional and $K_N \subset \bigcap_s X_N^s$ ($K_D \subset \bigcap_s X_{D,\lambda}^s$).

Then for $s>1$ D_N^s (D_D^s) is the image of D_N^{-s+2} (D_D^{-s+2}) and vice-versa under the symplectic relation between S_N^s and S_N^{-s+2} ($S_{D,\lambda}^s$ and $S_{D,\lambda}^{s-2}$) iff the isomorphism theorem 3.4.2. holds.

Proof. – It is obvious that relations to be proved follow from 3.4.2.

Now, suppose that $\underset{\sim}{D}_N^s$ is the image of $\underset{\sim}{D}_N^{-s+2}$. It means that $(\Lambda_N^{-s+2})^{-1}\ \underset{\sim}{Y}_N^{s-2} \subset \underset{\sim}{X}_N^s$. Since $\underset{\sim}{D}_N^{-s+2}$ is the image of $\underset{\sim}{D}_N^s$, we have that

$$\Lambda_N^s(\underset{\sim}{X}_N^s) \subset \underset{\sim}{Y}_N^{s-2} \ .$$

Since $\Lambda_N^1 : \underset{\sim}{X}_{N/\underset{\sim}{K}_N}^1 \longrightarrow \tilde{\underset{\sim}{Y}}_N^{-1}$ is the isomorphism and $\underset{\sim}{K}_N \subset \bigcap_s \underset{\sim}{X}_N^s$ we have $\Lambda_N^s(\underset{\sim}{X}_N^s) = \tilde{\underset{\sim}{Y}}_N^{s-2} : \tilde{\underset{\sim}{Y}}_N^{-1} \cap \tilde{\underset{\sim}{Y}}^{s-2}$ and $(\Lambda_N^{-s+2})^{-1}\ \underset{\sim}{Y}_N^{s-2} = \underset{\sim}{X}_N^s$.

Q.E.D.

3. Dynamics as Lagrangian subspaces in $S^{s,s'}$.

We saw in 3.1. that the spaces $\underset{\sim}{S}_N^s$ $(s \neq \frac{1}{2})$ and $\underset{\sim}{S}_{D,\lambda}^s$ $(s \neq \frac{3}{2})$ result from $\underset{\sim}{S}^{s,s-\frac{1}{2}}$ and $\underset{\sim}{S}^{s,-s+\frac{3}{2}}$ by reductions. According to 5.1.2. the inverse image of the dynamics is a Lagrangian subspace. We denote it by $\underset{\sim}{D}^{s,s'}$ $(s' = s - \frac{1}{2}$ or $s' = -s + \frac{3}{2})$. First, we define dynamics $\underset{\sim}{D}^{\frac{1}{2},0}$ and $\underset{\sim}{D}^{\frac{3}{2},0}$.

By 3.2.11. $\underset{\sim}{X}_N^{\frac{1}{2}}$ is a dense subspace of $\underset{\sim}{X}^0 \oplus \underset{\sim}{X}^{\frac{1}{2}}$. Hence $\underset{\sim}{P}^0 \oplus \underset{\sim}{F}^{-\frac{1}{2}}$ is a dense subspace of $\underset{\sim}{Y}_N^{-\frac{1}{2}}$. Inclusions give a symplectic relation between $\underset{\sim}{S}_N^{\frac{1}{2}}$ and $\underset{\sim}{S}^{\frac{1}{2},0}$. It is easy to check that the image of $\underset{\sim}{D}_N^{\frac{1}{2}}$ is again a Lagragian subspace. Let us denote it by $\underset{\sim}{D}^{\frac{1}{2},0}$. By the same arguments we get $\underset{\sim}{D}^{\frac{3}{2},0}$.

We see, that the projection of $\underset{\sim}{D}^{\frac{1}{2},0}$ onto $\underset{\sim}{P}^0 \oplus \underset{\sim}{F}^{-\frac{1}{2}}$ is a closed subspace.

4. The dynamics in $\underset{\sim}{S}^{1,\frac{1}{2}}$.

For $s=1$ we have $-s + \frac{3}{2} = s - \frac{1}{2}$ and both dynamics: obtained from

$\underset{\sim}{D}^1_N$ and $\underset{\sim}{D}^1_D$ are in the same space $\underset{\sim}{S}^{1,\frac{1}{2}}$. The question arises: Have we obtained one or two different Lagrangian subspaces?

We know that for smooth sections $\underset{\sim}{x}$ the elements $\underset{\sim}{x}|_{\partial\Omega} \oplus \underset{\sim}{x} \oplus \chi^t(\underset{\sim}{x}) \oplus \varepsilon(\underset{\sim}{x})$ belong to both dynamics and form a dense set in $\underset{\sim}{D}^1_D \subset \underset{\sim}{S}^{1,\frac{1}{2}}$. Since both subspaces are Lagrangian, they are equal.

Now. we can try to get a dynamics in $\underset{\sim}{S}^{s,s'}$ as an image of the dynamics in $\underset{\sim}{S}^{1,\frac{1}{2}}$ under the canonical symplectic relation. We have the version of 4.2.2.

4.4.1. Proposition – The image of $\underset{\sim}{D}^{1,\frac{1}{2}}$ in $\underset{\sim}{S}^{s,s-\frac{1}{2}}$ $(s < 1)$ and $\underset{\sim}{S}^{s,-s+\frac{3}{2}}$ $(s < \frac{3}{2})$ is $\underset{\sim}{D}^{s,s-\frac{1}{2}}$ and $\underset{\sim}{D}^{s,-s+\frac{3}{2}}$. ◆

Remark. We may define a dynamics in $\underset{\sim}{S}^{s,s'}$ for other values of (s,s'), but it is out of our interest.

The presented approach is based on the isomorphism theorems 3.4.2. This theorems have been obtained with the interpolation theorems.

Isomorphism theorems may be obtained in this way for non-self-adjoint problems too. But since we deal with Lagrangian subspaces all information is contained in generating functions.

For example the isomorphism theorem for Λ^1_N can be obtained directly by standard estimations of the Lagrangian ([33]). Direct estimations do not exist for Λ^1_D, but stil the information is in the Lagrangian. Here we show how to get the isomorphism theorem for Λ^1_D by analysis of the Lagrangian only. (For other aspects of this approach see [33]).

4.4.2. Theorem – Let an operator Λ^1_N and its reduction with respect to the subspace $\underset{\sim}{X}^1_o = \{\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x} \in \underset{\sim}{X}^1_N : \bar{\underset{\sim}{x}} = 0\}$ be continuous and elliptic (i.e. with a finite-dimensional kernel and a closed co-domain). Then Λ^1_D is continuous and elliptic too.

Remark. We do not assume any special property of a generating function (e.g. existence of the E – L operator).

Proof.

Suppose that Λ_N^1 is elliptic. From 2.5.7. the existence of Λ_D^1 follows.

(i) We show that the kernel of Λ_D^1 is finite-dimensional.

Let be $\underset{\sim}{x} \oplus \underset{\sim}{p} \in \underset{\sim}{X}^1 \oplus P^{-\frac{1}{2}}$ and $\underset{\sim}{x} \oplus \underset{\sim}{p} \in \underset{\sim}{K}_D$ where $\underset{\sim}{K}_D$ is the kernel of Λ_D^1. This is equivalent to saying that

$\underset{\sim}{S}^{1,\frac{1}{2}} \supset \underset{\sim}{D}^{1,\frac{1}{2}} = \underset{\sim}{D}_D^1 \ni \underset{\sim}{0} \oplus \underset{\sim}{x} \oplus \underset{\sim}{p} \oplus \underset{\sim}{0}$. But the dynamics is contained in the subspace defined by $\bar{\underset{\sim}{x}} = \underset{\sim}{x}\big|_{\partial\Omega}$, so $\underset{\sim}{x}\big|_{\partial\Omega} = 0$. (4.4.3)

On the other hand, $D^{1,\frac{1}{2}}$ is in the image of D_N^1. It means that

$\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x} \oplus \underset{\sim}{p} \oplus \underset{\sim}{f} \in D^{1,\frac{1}{2}}$ iff $\bar{\underset{\sim}{x}} \oplus \underset{\sim}{x} \oplus \underset{\sim}{0} \oplus (\underset{\sim}{f} - \tilde{\underset{\sim}{p}}) \in D^{1,\frac{1}{2}}$, (4.4.4)

where $\tilde{\underset{\sim}{p}}$ is an element of $\underset{\sim}{F}^{-1}$ equivalent to $\underset{\sim}{p}$ i.e.

$\underset{\sim}{p} \oplus \underset{\sim}{0} - \underset{\sim}{0} \oplus \tilde{\underset{\sim}{p}} \in K$, K defined by 3.1.3. .

We have then that $\underset{\sim}{0} \oplus \underset{\sim}{x} \oplus \underset{\sim}{0} \oplus (-\tilde{\underset{\sim}{p}}) \in \underset{\sim}{D}^{1,\frac{1}{2}}$.

Since $\underset{\sim}{x}\big|_{\partial\Omega} = 0$, $\underset{\sim}{x}$ is in the kernel of the dynamics reduced with respect to $\underset{\sim}{X}_0^1$, but this kernel is finite dimensional. Let us denote this kernel by K_0. If $\underset{\sim}{x} \in K_0$ then there exists $\underset{\sim}{p} \in \underset{\sim}{P}^{-\frac{1}{2}}$ such that $\underset{\sim}{0} \oplus \underset{\sim}{x} \oplus \underset{\sim}{0} \oplus \tilde{\underset{\sim}{p}} \in D^{1,\frac{1}{2}}$. But, since Λ_N^1 is a mapping, from 4.4.4. follows that $\tilde{p}$ is unique. Hence the kernel of Λ_D^1 is finite-dimensional and isomorphic to K_0.

(ii) Now, we prove that the image of Λ_D^1 is closed.

Let $\bar{\underset{\sim}{x}} \oplus \underset{\sim}{f} \in \bar{\underset{\sim}{X}}^{\frac{1}{2}} \oplus \underset{\sim}{F}^{-1}$ be in the annihilator of the kernel of Λ_D^1, i.e.

$$<\underset{\sim}{x},\underset{\sim}{p}> + <\underset{\sim}{x},\underset{\sim}{f}> = 0 \qquad \forall \underset{\sim}{x} \oplus \underset{\sim}{p} \in \operatorname{Ker} \Lambda_D^1 \tag{4.4.5}$$

We know, that for each $\bar{\underset{\sim}{x}} \in \underset{\sim}{X}^{\frac{1}{2}}$ there exists an element

$$\bar{\underset{\sim}{x}} \oplus \tilde{\underset{\sim}{x}} \oplus \underset{\sim}{p}_0 \oplus \underset{\sim}{f}_0 \in \underset{\sim}{D}^{1,\frac{1}{2}} \tag{4.4.6}$$

where $\tilde{\underset{\sim}{x}} \in \underset{\sim}{X}^1$ is such that $\tilde{\underset{\sim}{x}}\big|_{\partial\Omega} = \bar{\underset{\sim}{x}}$, i.e.

$$<\bar{\underset{\sim}{x}},\underset{\sim}{p}_0> + <\underset{\sim}{x},\underset{\sim}{f}_0> = 0 \qquad \forall \underset{\sim}{x} \oplus \underset{\sim}{p} \in \operatorname{Ker} \Lambda_D^1 .$$

With 4.4.5. we have then that

$$< \underset{\sim}{x}, \underset{\sim}{f} > = < \underset{\sim}{x}, \underset{\sim}{f}_o > \quad \forall \underset{\sim}{x} \in K_o \tag{4.4.7}$$

Since reduced Λ^1_N is elliptic 4.4.7. means that $\underset{\sim}{f} - \underset{\sim}{f}_o$ is in the image of the reduced Λ^1_N.

Hence there exist $\underset{\sim}{x}_1 \in X_o$ and $\underset{\sim}{p}_1 \in P^{-\frac{1}{2}}$ such that

that $\underset{\sim}{0} \oplus \underset{\sim}{x}_1 \oplus \underset{\sim}{p}_1 \oplus (\underset{\sim}{f} - \underset{\sim}{f}_o) \in \underset{\sim}{D}^{1,\frac{1}{2}}$.

Because of 4.4.6. we have

$$\underset{\sim}{x} \oplus (\underset{\sim}{x}_1 + \tilde{\underset{\sim}{x}}) \oplus (\underset{\sim}{p}_o \oplus \underset{\sim}{p}_1) \oplus \underset{\sim}{f} \in D^{\frac{1}{2}} .$$

Hence, the image of Λ^1_D is the annihilator of the kernel.

Q.E.D.

CHAPTER V

HOMOGENEOUS REDUCTIONS

In the variational approach to boundary problems we often deal with dinamics reduced with respect to homogeneous boundary data (or sources). In this Chapter we present a systematic analysis of reduced spaces and reduced dynamics in strong symplectic structures. The well known objects and methods will appear in a most natural way. Discussion of reduced dynamics in weark symplectic strutures will be presented in [33] .

1. Symplectic relations. Reductions.

Let (Σ,ω) and (Σ',ω') be symplectic spaces. There is a natural symplectic structure in $\Sigma \oplus \Sigma'$ denoted by $\omega \ominus \omega'$ and defined by $\omega - \omega' = - \omega' \circ pr' + \omega \circ pr$ where pr', pr are canonical projections onto Σ' , Σ.

A relation between Σ and Σ' is said to be symplectic iff its graph is a Lagrangian submanifold of $(\Sigma \oplus \Sigma', \omega \ominus \omega')$.

Example.

Let $W \subset \Sigma$ be a coisotropic subspace. The form ω on Σ induces a symplectic form ω' on $W/_{W}\S$. The symplectic space $(\Sigma' = W/_{W}\S, \omega')$ is called the symplectic space reduced with respect to a coisotropic subspace W.

One can easily check that the canonical relation in $\Sigma \oplus \Sigma'$ is a symplectic relation.

For example, if $\Sigma = V \oplus V^*$ and $V_o \subset V$ is an closed subspace, then $W = V_o \oplus V^*$ is a coisotropic subspace. The reduced symplectic space is isomorphic to $V_o \oplus V^*/_K$ where K is the annihilator of V_o in V^*.

The most important property of a symplectic relation in a finite dimensional case is that the image of a Lagrangian subspace is again a Lagrangian subspace. In general, it is not true in a infinite dimensional case. Now, we present some important cases when a reduced Lagrangian subspace is Lagrangian.

5.1.1. Lemma - Let $\Sigma = V \oplus V^*$ be a special symplectic space, $D \subset \Sigma$ - a Lagrangian subspace of Σ such that $pr(D)$ is closed of finite codimension, where $pr: \Sigma \to V$ is the canonical projection. Let $V_o \subset V$ be a closed subspace. Then the image of D under reduction with respect to the coisotropic subspace $V_o \oplus V^*$ is Lagrangian.

Proof. Let us denote the image of D by $\tilde{D}$. It is obvious that the constraint of $\tilde{D}$ in V_o is $V_o \cap pr\, D$. Hence $\tilde{D}$ is Lagrangian iff a kernel of the projection $\tilde{D} \to V_o \cap pr\, D$ is $0 \oplus (V_o \cap prD)^o$, where $(V_o \cap prD)^o$ is the annihilator of $V_o \cap prD$ in $V^*/_{V_o^\perp} =: V_o^*$ (V_o - the annihilator of V_o in V^*). This is equivalent to saying that $(prD \cap V_o)^\perp = V_o + (prD)$. Since prD is of finite codimension in V, $prD \cap V_o$ is of finite codimension in $(pr\, D \cap V_o)^\perp$. Hence the equality $(pr\, D \cap V_o)^\perp = V_o^\perp + (prD)$ follows immediately.◆ Q.E.D.

We need also

5.1.2. Lemma - Let Σ be a symplectic space, $W \subset \Sigma$- a coisotropic closed subspace. Suppose we have a Lagrangian subspace D in the reduced space $W/_W§$. There exists exactly one Lagrangian subspace $\tilde{D}$ in Σ such that $\tilde{D} \subset W$ and its reduced space is D.

Proof.

Let $\tilde{D}$ be an inverse image of D under the projection $W \to W/_W§$. D is isotropic and closed. But $\tilde{D} \subset W$, so $W \supset \tilde{D}$. If $\tilde{D}$ is not maximal, the D is not maximal too.◆

In the following we shall consider three kinds of reductions:

(i) *a homogeneous Neumann* (or stress) reduction is a reduction in $\underset{\sim}{S}^{s,s'}$ with respect to the coisotropic subspace $\{\underset{\sim}{p}=0\}$. The reduced space is again isomorphic to $\underset{\sim}{X}^s \oplus \underset{\sim}{F}^{-s}$,

(ii) *a homogeneous Dirichlet reduction* is a reduction in $\underset{\sim}{S}^{s,s'}$ with respect to the coisotropic subspace $\{\underset{\sim}{\bar{x}} = 0\}$. It is obvious that the reduced space is isomorphic to $\underset{\sim}{X}^s \oplus \underset{\sim}{F}^{-s}$ with its canonical duality,

(iii) *a homogeneous source reduction* is a reduction in $\underset{\sim}{S}^{s,s'}$ with respect to the subspace $\underset{\sim}{f} = 0$. The reduced space is isomorphic to $\underset{\sim}{X}^{s'} \oplus \underset{\sim}{P}^{-s'}$ i.e. to the boundary phase space.

2. Homogeneous Neumann reductions of dynamics

5.2.1. Proposition – Let be $s' = s - \frac{1}{2}$, then the image of $\underset{\sim}{D}^{s,s'}$ under the homogeneous Neumann reduction is a Lagrangian subspace.

Proof. For $s \leq 1$ the projection of $\underset{\sim}{D}^{s,s'}$ onto $\underset{\sim}{P}^{-s'} \oplus \underset{\sim}{F}^{-s}$ is closed. By (5.1.1) the reduced dynamics is a Lagrangian subspace.

For $s \leq \frac{1}{2}$ we have a 1-1 symplectic relation between reduced space $\underset{\sim}{X}^s \oplus \underset{\sim}{F}^{-s}$ and $\underset{\sim}{X}^s_N \oplus \underset{\sim}{X}^{-s}_N$ induced by the isomorphism $\underset{\sim}{X}^s_N \ni \bar{x} \oplus \underset{\sim}{x} \longrightarrow \underset{\sim}{x} \in \underset{\sim}{X}^s$. It is easy to see that the image of $\underset{\sim}{D}^s_N$ is the reduced dynamics. With 4.2.1. it follows that the reduced dynamics forms a Lagrangian subspace.

Q.E.D.

Since for $s > \frac{1}{2}$ the reduced dynamics is in a 1-1 correspondence with $\underset{\sim}{D}^s_N$, we say that the homogeneity is i n t r i n s i c o r h i d d e n . The sense of this concept is made clear by the following proposition.

5.2.2. Proposition – Suppose that $\underset{\sim}{x} \oplus \underset{\sim}{f}$ is in the reduced dynamics for some s. Let be $f \in F^{-s_1}$, $s_1 < \frac{1}{2}$, $\underset{\sim}{x} \in X^{s_2}$, $s_2 > \frac{3}{2}$. Then $\chi^t(x) = 0$.

Proof. – We have $\underset{\sim}{x}|_{\partial\Omega} \oplus \underset{\sim}{x} \oplus \underset{\sim}{0}_1 \oplus \underset{\sim}{f} \in \underset{\sim}{D}^{s,s-\frac{1}{2}}$. On the other hand $\underset{\sim}{x}|_{\partial\Omega} \oplus \underset{\sim}{x} \oplus \chi^t(x) \oplus \varepsilon(\underset{\sim}{x}) \in \underset{\sim}{D}^{s,s-\frac{1}{2}}$.

Hence (see the proof of 4.4.2.) $\underset{\sim}{f} - \varepsilon(x) = \widetilde{\chi^t}(x)$. But $\varepsilon(\underset{\sim}{x}) \in \underset{\sim}{F}^{s_2-2}$ and $\underset{\sim}{f} \in F^{-s_1}$ $s_1 < \frac{1}{2}$, $\chi^t(\underset{\sim}{x}) \in \underset{\sim}{F}^{-s_3}$ $s_3 < \frac{1}{2}$. It follows that $\chi^t(\underset{\sim}{x}) = 0$. ◆

5.2.3. Example

For $s = 0$ the reduced space is isomorphic to $\underset{\sim}{X}^o \oplus \underset{\sim}{F}^o$. Suppose that we have established an isomorphism between bundles X and F by choosing a volume element on M and a scalar product in fibers. Then $\underset{\sim}{F}^{o} = \underset{\sim}{X}^{o} = H^{o}(\Omega;X)$ and the dynamics is isomorphic to the Lagrangian subspace in $X^{o} \oplus X^{o}$. By 4.1.1. it is a graph of a self-adjoint operator. It is of course the self-adjoint realization of the Euler-Lagrange operator corresponding to the homogeneous Neumann boundary date.

We can reduce our dynamics in other spaces too. For example, we can take $s' = -s + \frac{3}{2}$.

5.2.4. Proposition - Let be $s' = -s + \frac{3}{2}$, $s \neq \frac{3}{2}$, then the reduced dyna mics is a Lagrangian subspace. For $s < \frac{3}{2}$ the reduced dynamics coincides with that of 5.2.1.

Proof. For $s \geq 1$, $s \neq \frac{3}{2}$ we make use of (5.1.1). We notice that for $s < 1$ the reduce dynamics is an image of the reduced dynamics for s=1 with respect to the symplectic relation induced by the inclusion $\underset{\sim}{F}^{-s} \subset \underset{\sim}{F}^{-1}$. As in 4.2. it follows that the reduced dynamics is Lagrangian. On the other hand, arguments used in 4.4. show that the reduced dynamics is for $s < \frac{3}{2}$ the same as in 5.2.1.

Q.E.D.

Remark. For $s > \frac{3}{2}$ we say that the reduced dynamics is s t a b l y homog eneous.

3. A homogeneous Dirichlet reduction

5.3.1. Proposition - Let be $s' = -s + \frac{3}{2}$, $s \neq \frac{3}{2}$, then the image of $\underset{\sim}{D}^{s,s'}$ under the homogeneous Dirichlet reduction is Lagrangian iff $s > \frac{3}{2}$ or $s \leq 1$.

Proof. Methods used in (5.2.1) prove that the reduced dynamics is Lagrangian. For $1 < s < \frac{3}{2}$, $\underset{\sim}{D}^{s,s'}$ consists of such elements $\underset{\sim}{\bar{x}} \oplus \underset{\sim}{x} \oplus \underset{\sim}{p} \oplus \underset{\sim}{f} \in \underset{\sim}{S}^{s,s'}$ that $\underset{\sim}{\bar{x}} \oplus \underset{\sim}{x} \oplus \underset{\sim}{p} \oplus \underset{\sim}{f} \in \underset{\sim}{D}^{1,\frac{1}{2}}$ (see 4.4.1.). The reduced dynamics is formed by elements $\underset{\sim}{x} \oplus (\underset{\sim}{f} - \tilde{\underset{\sim}{p}})$ where

$$\underset{\sim}{x}\big|_{\partial\Omega} = 0, \quad \underset{\sim}{p} \in \underset{\sim}{P}^{s-\frac{3}{2}}, \quad \underset{\sim}{0} \oplus \underset{\sim}{f} = \Lambda_{D}^{1}(\underset{\sim}{x} \oplus p).$$

But it is obvious, that the Lagrangian subspace beside $\underset{\sim}{x} \oplus \underset{\sim}{f}$ contains (it is maximal) elements of the form $\underset{\sim}{x} \oplus (\underset{\sim}{f} - \tilde{\underset{\sim}{p}})$ for

$p \in \underset{\sim}{P}^{-s+\frac{1}{2}}$. But for $1 < s < \frac{3}{2}$ $\underset{\sim}{P}^{-s+\frac{1}{2}} \supsetneq \underset{\sim}{P}^{s-\frac{3}{2}}$.

Q.E.D.

5.3.2. Remarks

(i) For $\frac{1}{2} < s \leq 1$ the reduced dynamics is determined by its further reduction with respect to the subspace $\{\underset{\sim}{x}|_{\partial\Omega} = 0\}$.

(ii) For $s > \frac{3}{2}$ we have, as in the case of the Neumann reduction the case of an intrinsic homogeneity.

(iii) Because of 4.4.1. $s \leq 1$ is the only case with an interesting interpretation.

5.3.3. Example

For $s=0$ we can get, as in 5.2.3. the self-adjoint realisation of the Euler-Lagrange operator corresponding to the homogeneous Dirichlet data.

The following proposition corresponds to 5.2.6.

5.3.4. Proposition - Let be $s' = s - \frac{1}{2}$. The reduced dynamics is a Lagrangian subspace. For $s \leq 1$ the reduced dynamics coincides with that of 5.3.1.

Proof. as in 5.2.6.◆

Remark. For $s > \frac{1}{2}$ the reduced dynamics is stably homogeneous.

4. A homogeneous source reduction

As in previous sections we see, that the reduction of the dynamics in $\underset{\sim}{S}^{s,s'}$ $(s \geq 1,\ s' = s - \frac{1}{2}$ or $s' = -s + \frac{3}{2})$ gives Lagrangian subspaces in $\underset{\sim}{X}^{s'} \oplus \underset{\sim}{P}^{-s}$. It is easy to check that the reduced dynamics in $\underset{\sim}{\bar{X}}^{s} \oplus \underset{\sim}{P}^{-s}$ is an image of the reduced dynamics in $\bar{X}^{\frac{1}{2}} + P^{-\frac{1}{2}}$ under the canonical symplectic relation. Moreover,

5.4.1. Proposition - The reduced dynamics in $\underset{\sim}{X}^{-\frac{1}{2}} \oplus \underset{\sim}{P}^{-\frac{1}{2}}$ is the graph of an bounded operator with index from $\underset{\sim}{X}^{\frac{1}{2}}$ to $\underset{\sim}{P}^{-\frac{1}{2}}$.◆

5. Generating functions of reduced dynamics

We shall consider the case of $\underset{\sim}{S}^{1,\frac{1}{2}}$ only. We make use of relations between generating functions of reduced and non-reduced dynamics ([29]).

5.5.1. Proposition

Generating functions of reduced dynamics in the field s.s.s. are

(i) for the Neumann reduction $\underset{\sim}{L}(\underset{\sim}{x}) = \int_\Omega L^N(\underset{\sim}{x})$

(ii) for the Dirichlet reduction $\underset{\sim}{L}(\underset{\sim}{x}) = \int_\Omega L^N(\underset{\sim}{x})$ on the constraint subspace $\underset{\sim}{X}^1_o$.

Proof.

(i) $\underset{\sim}{L}(\underset{\sim}{x})$ is a stationary value of the non-reduced Lagrangian with respect to $\bar{\underset{\sim}{x}}$. Because of the constraint $\bar{\underset{\sim}{x}} = \underset{\sim}{x}|_{\partial\Omega}$ it is exactly the value of the non-reduced Lagrangian in the point $\underset{\sim}{x}$.

(ii) As in (i) we know that $L(x)$ is a stationary value of $\int_\Omega L^N(\underset{\sim}{x}) - \int_{\partial\Omega}\underset{\sim}{p}\,\bar{\underset{\sim}{x}}$ with respect to p. Because of the condition $\bar{\underset{\sim}{x}} = 0$ the reduced dynamic has the constraint $\underset{\sim}{x}|_{\partial\Omega} = 0$. Hence

$$\underset{\underset{\sim}{p}}{\text{stat}}\left(\int_\Omega L^N(\underset{\sim}{x}) - \int_{\partial\Omega}\underset{\sim}{p}\,\bar{\underset{\sim}{x}}\right) = \int_\Omega L^N(\underset{\sim}{x}). \blacklozenge$$

5.5.2. Proposition – Generating functions of reduced dynamics in the source – s.s.s. are

(i) for the Neumann reduction

$$L(\underset{\sim}{f}) = \underset{\underset{\sim}{x}}{\text{stat}}\left(\int_\Omega L^N(\underset{\sim}{x}) + \int_\Omega \underset{\sim}{f}\,\underset{\sim}{x}\right)$$

(ii) for the Dirichlet reduction

$$L(f) = \underset{\bar{\underset{\sim}{x}},\underset{\sim}{p}}{\text{stat}}\left(\int_\Omega L^N(\underset{\sim}{x}) - \int_{\partial\Omega}\underset{\sim}{p}\,\bar{\underset{\sim}{x}} + \int_\Omega \underset{\sim}{f}\,\underset{\sim}{x}\right) =$$

$$= \underset{\underset{\sim}{x}\in\underset{\sim}{X}^1_o}{\text{stat}}\left(\int_\Omega L^N(\underset{\sim}{x}) + \int_\Omega \underset{\sim}{f}\,\underset{\sim}{x}\right).$$

Proof. Formulas for Legendre transformations ([29]) give (i) and the second part of (ii). But we can first make the Legendre transformation from the Neumann-field s.s.s. to the field-source s.s.s. and the reduce.

The transformed generating function is

$$L(\underset{\sim}{\bar{x}} \oplus \underset{\sim}{f}) = \underset{\underset{\sim}{x} \oplus \underset{\sim}{p}}{\text{stat}} \left(\int_\Omega L^N(\underset{\sim}{x}) - \int_{\partial\Omega} \underset{\sim}{p}\, \underset{\sim}{x} - \int_{\partial\Omega} \underset{\sim}{p}\underset{\sim}{\bar{x}} + \int_\Omega \underset{\sim}{f}\underset{\sim}{x}\right.$$

The reduction with respect to the subspace $\{\underset{\sim}{\bar{x}} = 0\}$ gives

$$L(\underset{\sim}{f}) = \underset{\underset{\sim}{x} \oplus \underset{\sim}{p}}{\text{stat}} \left(\int_\Omega L^N(\underset{\sim}{x}) - \int_{\partial\Omega} \underset{\sim}{p}\, \underset{\sim}{x} + \int_\Omega \underset{\sim}{f}\, \underset{\sim}{x}\right).$$

Q.E.D.

CHAPTER VI

GREEN'S FUNCTIONS. CONCLUDING REMARKS

In this chapter we give a symplectic interpretation of Green functions. For the sake of simplicity we assume that there is no constraint in $\underset{\sim}{Y}_N$ and $\underset{\sim}{Y}_D$.

1. Green's function for the Neumann problem.

We know that a Dirac delta δ_t, $t \in \Omega$ can be interpreted as an element of $\underset{\sim}{Y}_N^{-s}$, $s > \frac{m}{2}$. By the isomorphism theorem there exists $G_t^N \in \underset{\sim}{X}_N^{-s+2}$ such that $\Lambda_N^{-s+2}(G_t^N) = \delta_t$.

The function $\Omega \ni t \longrightarrow G_t^N \in X_N^{-s+2}$ is called the *Green's function for the Neumann problem.*

Since $\underset{\sim}{X}_N^{-s+2}$ is dual to $\underset{\sim}{Y}_N^{s-2}$, $< G_t^N , \underset{\sim}{y} >_T$ has sense for each $\underset{\sim}{y} \in \underset{\sim}{Y}_N^{s-2}$.

6.1.1. Theorem – For $\underset{\sim}{y} \in \underset{\sim}{Y}_N^{s-2}$, $s > \frac{m}{2}$ we define $\underset{\sim}{x}_{\underset{\sim}{y}}(t) := < G_t^N, \underset{\sim}{y} >_T$. Then $\underset{\sim}{x}_{\underset{\sim}{y}}(t) \in \underset{\sim}{X}_N^s$ and $\Lambda_N^s \underset{\sim}{x}_{\underset{\sim}{y}} = \underset{\sim}{y}$.

Proof.

$$< G_t^N, \underset{\sim}{y} >_T = < (\Lambda_N^{-s+2})^{-1}(\delta_t), \underset{\sim}{y} > = < \delta_t, (\Lambda_N^s)^{-1} \underset{\sim}{y} > = ((\Lambda_N^s)^{-1} \underset{\sim}{y})(t) \quad \text{i.e.}$$

$$\underset{\sim}{x}_{\underset{\sim}{y}} = (\Lambda_N^s)^{-1} \underset{\sim}{y}.$$

Q.E.D.

6.1.2. Corollary – The generating function of $\underset{\sim}{D}_N^{-s+2}$, $-s > \frac{m}{2}$ with respect to the source-stress s.s.s. is given by the formula:

$$W_N(\underset{\sim}{y}) = \frac{1}{2} \langle\langle G^N, \underset{\sim}{y} \rangle_T , \underset{\sim}{y} \rangle_T \blacklozenge$$

2. Green's function for the Dirichlet problem

Let be $s > \max(\frac{m}{2}, \frac{3}{2})$. We know that δ_t corresponds to an element of $\underset{\sim}{Y}_D^{-s}$. By the isomorphisms theorem there exists

$$\underset{\sim}{X}_D^{-s+2} \ni G_t^D := (\Lambda_D^{-s+2})^{-1} \delta_t .$$

As in 6.1.1. we get

6.2.1. Theorem – For $\underset{\sim}{y} \in \underset{\sim}{Y}_D^{s-2}$, $s > \max(\frac{m}{2}, \frac{3}{2})$ we put $\underset{\sim}{x}_{\underset{\sim}{y}}(t) = \langle G_t^D, \underset{\sim}{y} \rangle$ (duality between $\underset{\sim}{X}_D^{-s+2}$ and $\underset{\sim}{Y}_D^{s-2}$). Then $\underset{\sim}{x}_{\underset{\sim}{y}}(t) \in \underset{\sim}{X}_D^s$ and $\Lambda_D^s \underset{\sim}{x}_{\underset{\sim}{y}} = \underset{\sim}{y}$.

Remark. One can investigate further properties of Green's functions defined as above in this language.

3. Concluding remarks

The paper is the first step in a systematic analysis of the symplectic aspects of the theory of static fields. In our opinion the most important concepts and problems to be discussed are:

(i) physical and mathematical control modes (i.e. analysis of other then Dirichlet and Neumann boundary and non-boundary problems).

(ii) s-parameters and image control modes – existence and properties,

(iii) symplectic content of approximation methods, discrete control modes,

(iv) composition of dynamics,

(v) higher order theories.

APPENDIX

Proof of 3.3.1. Lemma

First, we need few informations concerning complex symplectic spaces. The presented version is not the most natural, but it is close to the standard approach to complex Hilbert spaces and sufficient for our goal.

Let be $\Sigma = V \oplus V^*$ where V, V^* are complex vector space with a duality $\langle , \rangle : V \times V^* \longrightarrow C$. Let the duality be linear with respect to the first and anti-linear with respect to the second argument. As in a real case we have a skew-symmetric R-linear form on Σ

$$\omega(v \oplus v^*, v_1 \oplus v_1^*) = \mathrm{Re}(\langle v_1, v^* \rangle - \langle v, v_1^* \rangle)$$

We can say on isotropic, co-isotropic and Lagrangian subspaces with respect to ω. Now, let be $\Lambda : V \longrightarrow V^*$ a C-linear, self-adjoint (i.e. $\langle v, \Lambda v' \rangle = \overline{\langle v', \Lambda v \rangle}$) mapping. It is uniquely determined by the generating function $L(v) = \frac{1}{2}\langle v, \Lambda v \rangle : \langle v', \Lambda v \rangle = \langle v', dLv \rangle - i \langle iv', dLv \rangle \; \forall v'$. ($dL$ defined as in 1.7.).

With these notions we can reformulate Chapters I,II for X being a complex vector bundle. For this we put $\langle \underset{\sim}{x}_1 \oplus \underset{\sim}{x}, \underset{\sim}{y} \rangle_T = \overline{\langle T, d(\underset{\sim}{p}\bar{\underset{\sim}{x}}_1) - \underset{\sim}{f}\, \bar{\underset{\sim}{x}} \rangle}$ where the bar denotes complex conjugate.

Now, let be $\lambda_{AB} = 0$, $\lambda^{\nu}_{AB} = 0$, $\lambda^{\nu\mu}_{AB}$ - elliptic, $L(\underset{\sim}{x}) = \lambda^{\nu\mu}_{AB}\, \underset{\sim}{x}^A_\nu\, \underset{\sim}{x}^B_\mu$.

We see, that for T defined by an oriented domain L_T is positively defined modulo constant sections.

Let us fix $\xi' \neq 0$ $(\xi' = (\xi^1, \ldots, \xi^{m-1}))$.

We put

$$\Omega = \{t:\ t^m \geq 0,\quad 0 \leq t^i \leq c^i \quad \text{where}$$

$$c^i = 2/_{\xi^i} \quad \text{for } \xi^i \neq 0, \quad c^i = 1 \quad \text{for } \xi^i = 0 \quad i=1,\ldots,m-1\}.$$

We can reduce the dynamic with respect to

$$\underset{\sim}{X}_o = \{\underset{\sim}{x} \in H^1(\Omega,X) \; : \; \underset{\sim}{x}(t^1,\dots,t^i=0) = \underset{\sim}{x}(t^1,\dots,t^i=c^i,\dots,t^m)$$
$$i=1,\dots,m-1\} \; .$$

The reduced dynamics can be described by the mapping

$$\underset{\sim}{X}_o \ni x \longrightarrow (\mathcal{K}(\underset{\sim}{x})\big|_{\partial_o\Omega} \, , \, \varepsilon(\underset{\sim}{x})) \quad \text{where}$$
$$\partial_o\Omega = \{t\varepsilon\Omega \; : \; t^m = 0\} \; .$$

Because of ellipticity the kernel of this mapping consists of constant sections only.

Hence the mapping

$$H^1(R_+,X) \ni \underset{\sim}{x}(t^m) \longrightarrow \underset{\sim}{x}(t) = e^{i<t',\xi'>} \underset{\sim}{x}(t^m) \; \varepsilon \; X_o \longrightarrow (\mathcal{K}(\underset{\sim}{x})\big|_{\partial_o\Omega}, \varepsilon(\underset{\sim}{x}))$$

is an injection. It is equivalent to ellipticity of the Neumann problem.

Q.E.D.

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