STRINGS, GERBES, AND ALL THAT

2. SYMMETRIES AND GENERALISED GEOMETRY

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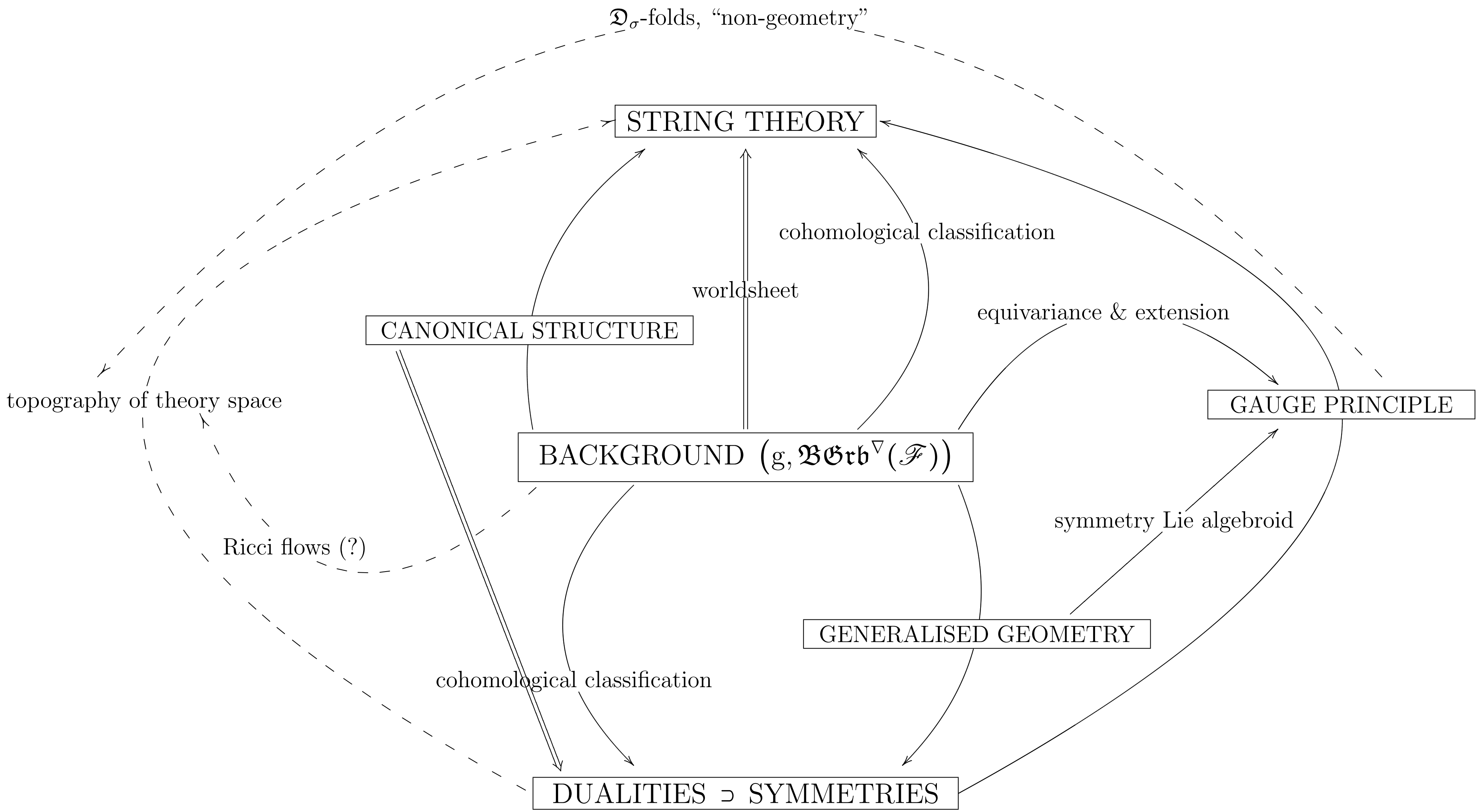
The topology of the subject:



I. The field theory of interest – the two-dimensional σ-model

The physics: propagation of loops in an ambient space(time) target.

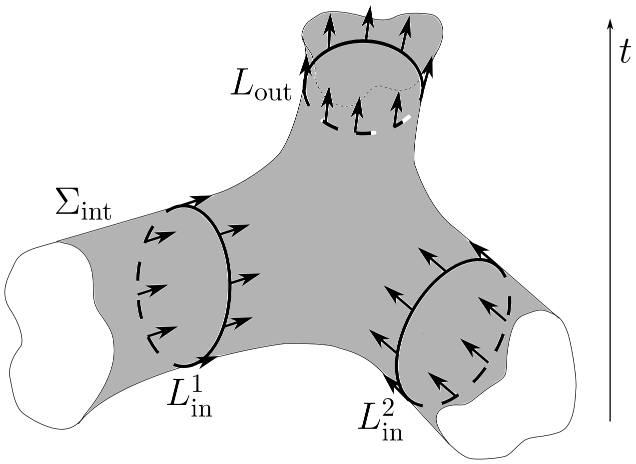
I.1. The mono-phase σ-model

The standard setting: loops sweep out metric **WORLD-SHEETS** (Σ,γ)

$$X\ :\ \Sigma\xrightarrow{C^1}M$$

embedded in a metric **TARGET SPACE** (M,g), and undergo

SPLITTING-JOINING INTERACTIONS



as determined by the **TWO-DIMENSIONAL NON-LINEAR σ-MODEL**

$$S_\sigma[X;\gamma]=-\tfrac{1}{2}\int_\Gamma \mathrm{g}_X(\mathsf{d}X\overset{\wedge}{,}\star_\gamma\,\mathsf{d}X)+S_{\rm top}[X]\,,\qquad \mathsf{d}X\xrightarrow{\text{loc.}}\partial_a X^\mu\,\mathsf{d}\sigma^a\otimes\partial_\mu\,,$$

with the topological term

$$S_{\rm top}[X]=-\mathsf{i}\,\log{\rm Hol}_{\mathcal{G}}(X)$$

written in terms of the **2-SURFACE HOLONOMY** ${\rm Hol}_{\mathcal{G}}$ along X of an **ABELIAN GERBE $\mathcal{G}$ WITH CONNECTION**, i.e. a Cheeger–Simons differential character that explicitly realises

$$H^2\big(\Sigma,{\rm U}(1)\big)\cong{\rm U}(1)\qquad\text{as per}\qquad {\rm Hol}_{\mathcal{G}}(X)=[X^*\mathcal{G}]\,.$$

The gerbe $\mathcal{G}$ is a differential-geometric structure

$$\begin{tikzcd} (L, \nabla_L, \mu_L) \arrow[d, "\pi_L"'] & \\ \mathsf{Y}^{[2]}M \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & (\mathsf{Y}M, \mathrm{B}) \arrow[d, "\pi_{\mathsf{Y}M}"] \\ & (M, \mathrm{H}) \end{tikzcd}$$

associated with a class $[\frac{1}{2\pi}\,\mathrm{H}] \in H^3(M,\mathbb{Z}) \subset H^3(M,\mathbb{R})$ such that

$$R_{\mu\nu}(\nabla^{\mathrm{g}}_{\mathrm{L-C}}) - \tfrac{1}{4}\left(\mathrm{g}^{-1}\right)^{\alpha\beta}\left(\mathrm{g}^{-1}\right)^{\gamma\delta}\mathrm{H}_{\mu\alpha\gamma}\,\mathrm{H}_{\nu\beta\delta} = O(\alpha')\,,$$

and admitting the following cohomological description:

Given an open cover $\mathcal{O}_M = \{\mathcal{O}_i^M\}_{i\in\mathscr{I}_M}$ of M, there exist local

CURVINGS $\quad B_i \in \Omega^2(\mathcal{O}_i)\,,$ $\qquad$ **CONNECTIONS** $\quad A_{ij} \in \Omega^1(\mathcal{O}_{ij})\,,$

TRANSITION FUNCTIONS $\quad g_{ijk} \in \mathrm{U}(1)_{\mathcal{O}_{ijk}}\,,$

subject to cohomological constraints

$$\begin{aligned} &\mathrm{H}|_{\mathcal{O}_i} =: \mathsf{d}B_i \\ &(B_j - B_i)|_{\mathcal{O}_{ij}} =: \mathsf{d}A_{ij} \\ &(A_{jk} - A_{ik} + A_{ij})|_{\mathcal{O}_{ijk}} =: \mathsf{i}\,\mathsf{d}\log g_{ijk} \\ &(g_{jkl}\cdot g_{ikl}^{-1}\cdot g_{ijl}\cdot g_{ijk}^{-1})|_{\mathcal{O}_{ijkl}} = 1 \end{aligned} \qquad \mathrm{mod} \qquad \begin{aligned} &B_i \mapsto B_i + \mathsf{d}\Pi_i \\ &A_{ij} \mapsto A_{ij} + (\Pi_j - \Pi_i)|_{\mathcal{O}_{ij}} - \mathsf{i}\,\mathsf{d}\log\chi_{ij} \\ &g_{ijk} \mapsto g_{ijk}\cdot(\chi_{jk}^{-1}\cdot\chi_{ik}\cdot\chi_{ij}^{-1})|_{\mathcal{O}_{ijk}} \end{aligned}$$

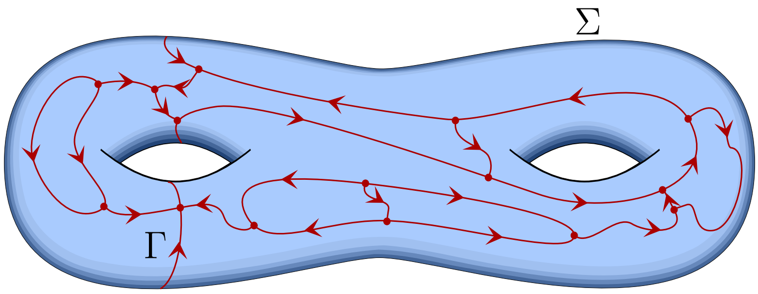
These define, for a triangulation $\triangle_\Sigma$ of Σ subordinate to $\mathcal{O}_M$ wrt. X,

$$S_{\mathrm{top}}[X] = \sum_{p\in\triangle_\Sigma}\left[\int_p X_p^* B_{i_p} + \sum_{e\subset p}\left(\int_e X_e^* A_{i_p i_e} - \mathsf{i}\sum_{v\in e}\log X^* g_{i_p i_e i_v}(v)^{\varepsilon_{pev}}\right)\right].$$

<u>**N.B.**</u> Local data (B_i, A_{ij}, g_{ijk}) define a class in $\mathbb{H}^2\big(M, \mathcal{D}(2)^\bullet\big)$.

I.1. The poly-phase σ-model

A generic setting: the mono-phase picture valid only *locally* on Σ, i.e. we have a poly-phase field theory over



with phases supported on patches $\wp \in \mathfrak{P}_\Sigma$,

$$X \; : \; \wp \to M \,, \qquad \text{with} \qquad (\mathrm{g}, \mathcal{G}) \,,$$

separated by **DEFECT LINES** $\ell \in \mathfrak{E}_\Gamma$ of a **DEFECT QUIVER** $\Gamma \subset \Sigma$,

$$X \; : \; \ell \to Q \,, \qquad \text{with} \qquad \iota_\alpha \; : \; Q \to M \,, \; \alpha \in \{1,2\} \qquad \text{and}$$

$$\text{1-ISOMORPHISM} \qquad \Phi \; : \; \iota_1^*\mathcal{G} \xrightarrow{\cong} \iota_2^*\mathcal{G} \otimes I_\omega \,, \qquad \omega \in \Omega^2(Q) \,,$$

joining at **DEFECT JUNCTIONS** $\jmath_n \in \mathfrak{V}_\Gamma^{(n)}$ (of valence $n \in \mathbb{N}_{\geq 3}$),

$$X \; : \; \{\jmath_n\} \to T_n \,, \qquad \text{with} \qquad \pi_n^{k,k+1} \; : \; T_n \to Q \,, \; k \in \overline{1,n} \qquad \text{and}$$

$$\text{2-ISOMORPHISM} \qquad \varphi_n \; : \; \circ_{k=1}^n \, \pi_n^{k,k+1\,*} \Phi^{\varepsilon_n^{k,k+1}} \xRightarrow{\cong} \mathrm{id} \,, \qquad \varepsilon_n^{k,k+1} \in \{-1,+1\} \,.$$

Thus, a σ-model for **NETWORK-FIELD CONFIGURATIONS** $(X \,|\, \Gamma)$ prerequires

$$\text{FIELD SPACE} \qquad \mathscr{F} = M \sqcup Q \sqcup \textstyle\bigsqcup_{n \in \mathbb{N}_{\geq 3}} T_n \,, \qquad \text{together with}$$

$$\text{BACKGROUND} \qquad \mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{I}) \qquad \text{with components}$$

$$\text{TARGET} \qquad \mathcal{M} = (M, \mathrm{g}, \mathcal{G}) \,, \qquad\qquad \mathcal{G}\text{-BI-BRANE} \qquad \mathcal{B} = (Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\}) \,,$$

$$(\mathcal{G},\mathcal{B})\text{-INTER-BI-BRANE} \qquad \mathcal{I} = \big(T_n, \big(\varepsilon_n^{k,k+1}, \pi_n^{k,k+1} \mid k \in \overline{1,n}\big), \varphi_n \mid n \in \mathbb{N}_{\geq 3}\big) \,.$$

Given a family of open covers:

$$\mathcal{O}_M = \{\mathcal{O}^M_i\}_{i\in\mathscr{I}_M}\,,\qquad \left\{\begin{array}{c}\mathcal{O}_Q = \{\mathcal{O}^Q_A\}_{A\in\mathscr{I}_Q}\\[6pt] \text{with } \phi_\alpha\ :\ \mathscr{I}_Q \to \mathscr{I}_M\\[6pt] \text{s.t. } \iota_\alpha(\mathcal{O}^Q_A) \subset \mathcal{O}^M_{\phi_\alpha(A)}\end{array}\right.\,,\qquad \left\{\begin{array}{c}\mathcal{O}_{T_n} = \{\mathcal{O}^{T_n}_\lambda\}_{\lambda\in\mathscr{I}_{T_n}}\\[6pt] \text{with } \psi_n^{k,k+1}\ :\ \mathscr{I}_{T_n} \to \mathscr{I}_Q\\[6pt] \text{s.t. } \pi_n^{k,k+1}(\mathcal{O}^{T_n}_\lambda) \subset \mathcal{O}^Q_{\psi_n^{k,k+1}(\lambda)}\end{array}\right.\,,$$

the background $\mathfrak{B}$ can be presented by its local data

$$\begin{aligned}&\mathcal{G} \xrightarrow{\text{loc.}} (B_i, A_{ij}, g_{ijk}) \in \underline{\Omega}^2(M) \times \underline{\Omega}^1(M) \times \underline{\mathrm{U}(1)}_M\,,\\ &\Phi \xrightarrow{\text{loc.}} (P_A, K_{AB}) \in \underline{\Omega}^1(Q) \times \underline{\mathrm{U}(1)}_Q\,,\\ &\varphi_n \xrightarrow{\text{loc.}} (f_{n,\lambda}) \in \underline{\mathrm{U}(1)}_{T_n}\,,\end{aligned}$$

subject to cohomological constraints

$$\big(\mathsf{d}B_i, \mathsf{d}A_{ij} - B_j + B_i, -\mathsf{i}\,\mathsf{d}\log g_{ijk} + A_{jk} - A_{ik} + A_{ij}, g^{-1}_{jkl}\cdot g_{ikl}\cdot g^{-1}_{ijl}\cdot g_{ijk}\big) = 0\,,$$

$$\begin{aligned}&\big(\mathsf{d}P_A, -\mathsf{i}\,\mathsf{d}\log K_{AB} + P_B - P_A, K^{-1}_{BC}\cdot K_{AC}\cdot K^{-1}_{AB}\big)\\ &= \sum_{\alpha\in\{1,2\}} (-1)^\alpha\, \iota^*_\alpha\big(B_{\phi_\alpha(A)}, A_{\phi_\alpha(A)\phi_\alpha(B)}, g_{\phi_\alpha(A)\phi_\alpha(B)\phi_\alpha(C)}\big) + (\omega, 0, 1)\,,\end{aligned}$$

$$\big(-\mathsf{i}\,\mathsf{d}\log f_{n,\lambda}, f_{n,\lambda}\cdot f^{-1}_{n,\mu}\big) = -\sum_{k=1}^{n} \varepsilon^{k,k+1}_n\, \pi^{k,k+1\,*}_n\big(P_{\psi^{k,k+1}_n(\lambda)}, K_{\psi^{k,k+1}_n(\lambda)\psi^{k,k+1}_n(\mu)}\big)\,,$$

and serving to define the action functional of the field theory:

$$S_\sigma[(X\,|\,\Gamma);\gamma] \;=\; -\tfrac{1}{2}\int_\Gamma \mathrm{g}_X(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) - \mathsf{i}\,\log \mathrm{Hol}_{\mathcal{G},\Phi,\varphi_n}(X)\,,$$

$$\begin{aligned}-\mathsf{i}\,\log \mathrm{Hol}_{\mathfrak{B}}(X\,|\,\Gamma) \;=\;& \sum_{p\in\triangle_\Sigma}\Big[\int_p X^*_p B_{i_p} + \sum_{e\subset p}\Big(\int_e X^*_e A_{i_p i_e} - \mathsf{i}\sum_{v\in e}\log X^* g^{\varepsilon_{pev}}_{i_p i_e i_v}(v)\Big)\Big]\\ &+ \sum_{e\in\triangle_\Gamma\smallsetminus\mathfrak{V}_\Gamma}\Big(\int_e X^*_e P_{A_e} - \mathsf{i}\sum_{v\in e}\log X^* K^{-\varepsilon_{ev}}_{A_e B_v}(v)\Big)\\ &- \mathsf{i}\sum_{\jmath\in\mathfrak{V}_\Gamma}\log X^* f_{n_\jmath,\lambda_\jmath}(\jmath)\,.\end{aligned}$$

II. Rigid symmetries of the σ-model

We want to understand the geometry of those field transformations $X \mapsto f \circ X,\ f \in \mathscr{D}iff(\mathscr{F})$ which preserve S_σ and hence descend the space of classical field configurations.

II.1. The infinitesimal picture

Consider a flow $\psi_\cdot : [-1,1] \times \mathscr{F} \to \mathscr{F}$ of a vector field $\mathscr{K}$ on $\mathscr{F}$, with restrictions $\mathscr{K}|_{\mathscr{M}} \equiv {}^{\mathscr{M}}\mathscr{K},\ \mathscr{M} \in \{M, Q, T_n\}$ aligned as

$$\pi_{n\,*}^{k,k+1}\big({}^{T_n}\mathscr{K}\big) = {}^{Q}\mathscr{K}\big|_{\pi_n^{k,k+1}(T_n)}\,, \qquad \iota_{\alpha\,*}\big({}^{Q}\mathscr{K}\big) = {}^{M}\mathscr{K}\big|_{\iota_\alpha(Q)}\,.$$

Since $\mathrm{Hol}_{\mathcal{G},\Phi,\varphi_n}$ is a generalised differential character, we find

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} S_\sigma[(\psi_t \circ X\,|\,\Gamma);\gamma] = -\tfrac{1}{2}\int_\Sigma \big(\mathscr{L}_{{}^M\mathscr{K}}\mathrm{g}\big)_X(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) + \int_\Sigma X^*\big({}^M\mathscr{K} \lrcorner \mathrm{H}\big) + \int_\Gamma X_\Gamma^*\big({}^Q\mathscr{K} \lrcorner \omega\big)\,,$$

and so symmetries correspond to those globally smooth sections

$$\mathfrak{K} = \big({}^M\mathscr{K} \oplus \kappa, {}^Q\mathscr{K} \oplus k, {}^{T_n}\mathscr{K} \oplus c\big) \in \Gamma_\sigma(\mathcal{E}\mathscr{F})$$

of the **GENERALISED TANGENT SHEAVES**

$$\mathcal{E}\mathscr{F} \equiv \mathcal{E}^{(1,1)}M \sqcup \mathcal{E}^{(1,0)}Q \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} \mathcal{E}^{(1,-1)}T_n \to M \sqcup Q \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} T_n\,, \qquad \mathcal{E}^{(1,q)}\mathscr{M} := \mathcal{T}\mathscr{M} \oplus \mathcal{T}_q^*\mathscr{M}\,,$$

written in terms of sheaf components $\mathcal{T}_{-1}^*\mathscr{M} := \underline{\mathbb{R}}$ and $\mathcal{T}_{p\geq 0}^*\mathscr{M} := \underline{\Omega}^p(\mathscr{M})$ of

$$\mathcal{T}_\bullet^*\mathscr{M}\ :\ 0 \to \mathcal{T}_{-1}^*\mathscr{M} \overset{\mathsf{d}^{(-1)}=\mathrm{id}}{\hookrightarrow} \mathcal{T}_0^*\mathscr{M} \xrightarrow{\mathsf{d}^{(0)}=\mathsf{d}} \mathcal{T}_1^*\mathscr{M} \xrightarrow{\mathsf{d}^{(1)}=\mathsf{d}} \cdots,$$

that are Killing for g and satisfy the section descent relations

$$\mathsf{d}_{\mathrm{H}}^{(1)}\big({}^M\mathscr{K} \oplus \kappa\big) = 0\,, \qquad \mathsf{d}_\omega^{(0)}\big({}^Q\mathscr{K} \oplus k\big) = -\Delta_Q\kappa\,, \qquad \mathsf{d}_0^{(-1)}\big({}^{T_n}\mathscr{K} \oplus c\big) = -\Delta_{T_n}k$$

wrt. $\Delta_Q := \iota_2^* - \iota_1^*,\ \Delta_{T_n} := \sum_{k=1}^n \varepsilon_n^{k,k+1}\,\pi_n^{k,k+1\,*}$ and $\mathsf{d}^{(q)}_{\mathrm{H}_{(q+2)}}(\mathscr{V} \oplus \upsilon) := \mathsf{d}^{(q)}\upsilon + \mathscr{V} \lrcorner \mathrm{H}_{(q+2)}$.

On $\Gamma(\mathscr{E}\mathscr{F})$, there exists a canonical **ANCHOR (MAP)**

$$\alpha_{\mathcal{T}\mathscr{F}} \;:\; \mathscr{E}\mathscr{F} \to \mathcal{T}\mathscr{F}$$

a **CANONICAL CONTRACTION** with restrictions

$$\begin{aligned}
(\cdot,\cdot)_{\lrcorner} \quad &: \quad \Gamma\big(\mathcal{E}^{(1,1)}\mathscr{M}\big)^{\times 2} \to \Gamma\big(\mathcal{T}^*_0\mathscr{M}\big) \;:\; (\mathscr{V}\oplus\upsilon, \mathscr{W}\oplus\varpi) \mapsto \tfrac{1}{2}\left(\mathscr{V}\lrcorner\varpi + \mathscr{W}\lrcorner\upsilon\right),\\
(\cdot,\cdot)_{\lrcorner} \quad &: \quad \Gamma\big(\mathcal{E}^{(1,m<1)}\mathscr{M}\big)^{\times 2} \to \Gamma\big(\mathcal{T}^*_{-1}\mathscr{M}\big) \;:\; (\mathscr{V}\oplus\upsilon, \mathscr{W}\oplus\varpi) \mapsto 0\,.
\end{aligned}$$

and an essentially unique $(\mathrm{H},\omega;\Delta_Q)$**-TWISTED BRACKET** such that

$$[\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)} \;:\; \Gamma_\sigma(\mathcal{E}\mathscr{F})^{\times 2} \to \Gamma_\sigma(\mathcal{E}\mathscr{F})\,, \qquad \alpha_{\mathcal{T}\mathscr{F}} \circ [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)} = [\cdot,\cdot] \circ \alpha_{\mathcal{T}\mathscr{F}}\,.$$

Given $\mathfrak{V}_i = \left({}^{M}\!\mathscr{V}_i \oplus \upsilon_i, {}^{Q}\!\mathscr{V}_i \oplus \xi_i, {}^{T_n}\!\mathscr{V} \oplus c_i\right)$, $i \in \{1,2\}$, it restricts as

$$\begin{aligned}
[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_M &= [{}^{M}\!\mathscr{V}_1, {}^{M}\!\mathscr{V}_2] \oplus \left(\mathscr{L}_{{}^{M}\!\mathscr{V}_1}\upsilon_2 - \mathscr{L}_{{}^{M}\!\mathscr{V}_2}\upsilon_1 - \tfrac{1}{2}\,\mathsf{d}\left({}^{M}\!\mathscr{V}_1 \lrcorner \upsilon_2 - {}^{M}\!\mathscr{V}_2 \lrcorner \upsilon_1\right) + {}^{M}\!\mathscr{V}_1 \lrcorner {}^{M}\!\mathscr{V}_2 \lrcorner \mathrm{H}\right),\\
[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_Q &= [{}^{Q}\!\mathscr{V}_1, {}^{Q}\!\mathscr{V}_2] \oplus \left({}^{Q}\!\mathscr{V}_1 \lrcorner \mathsf{d}\xi_2 - {}^{Q}\!\mathscr{V}_2 \lrcorner \mathsf{d}\xi_1 + {}^{Q}\!\mathscr{V}_1 \lrcorner {}^{Q}\!\mathscr{V}_2 \lrcorner \omega + \tfrac{1}{2}\left({}^{Q}\!\mathscr{V}_1 \lrcorner \Delta_Q\upsilon_2 - {}^{Q}\!\mathscr{V}_2 \lrcorner \Delta_Q\upsilon_1\right)\right),\\
[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_{T_n} &= [{}^{T_n}\!\mathscr{V}_1, {}^{T_n}\!\mathscr{V}_2] \oplus 0\,.
\end{aligned}$$

We thus obtain a $(\mathrm{H},\omega;\Delta_Q)$**-TWISTED BRACKET STRUXTURE**

$$\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F}) = \left(\mathcal{E}\mathscr{F}, [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}, (\cdot,\cdot)_{\lrcorner}, \alpha_{\mathcal{T}\mathscr{F}}\right).$$

<u>N.B.</u> The restriction $\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F})|_M$ yields the familiar Courant algebroid of $\mathcal{E}^{(1,1)}M$ with the Courant bracket twisted by H à la Ševera–Weinstein. The algebroid is central to the Gaultieri–Hitchin definition of **GENERALISED GEOMETRY**.

The $(\mathcal{G},\Phi,\varphi_n)$ enter the definition of the bracket structure through the tensorial twist fields $\mathrm{curv}(\mathcal{G})$ and $\mathrm{curv}(\mathcal{B})$. However, ...

II.2. The gerbe-theoretic interpretation

Prop$^{\underline{\mathrm{n}}}$: [rrS '10] Automorphisms of $\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F})$ are of the form

$$\begin{pmatrix} {}^M\!f_* & 0 \\ 0 & \left({}^M\!f^{-1}\right)^* \end{pmatrix} \circ \mathrm{e}^{{}^M\mathrm{B}} \sqcup \begin{pmatrix} {}^Q\!f_* & 0 \\ 0 & \left({}^Q\!f^{-1}\right)^* \end{pmatrix} \circ \mathrm{e}^{{}^Q\mathrm{B}} \sqcup \begin{pmatrix} {}^{T_n}\!f_* & 0 \\ 0 & \left({}^{T_n}\!f^{-1}\right)^* \end{pmatrix},$$

with ${}^M\!f \sqcup {}^Q\!f \sqcup {}^{T_n}\!f \in \mathscr{D}iff(\mathscr{F})$ such that

$${}^M\!f \circ \iota_\alpha = \iota_\alpha \circ {}^Q\!f\,, \qquad {}^Q\!f \circ \pi_n^{k,k+1} = \pi_n^{k,k+1} \circ {}^{T_n}\!f\,,$$

$${}^M\!f^*\mathrm{H} = \mathrm{H}\,, \qquad {}^Q\!f^*\omega = \omega\,,$$

and with ${}^M\mathrm{B} \in Z^2(M)$ and ${}^Q\mathrm{B} \in Z^1(Q)$ acting as per

$$\mathrm{e}^{{}^{\mathscr{M}}\mathrm{B}} \triangleright (\mathscr{V} \oplus \upsilon) := \mathscr{V} \oplus (\upsilon + \mathscr{V} \lrcorner\, {}^{\mathscr{M}}\mathrm{B})$$

and such that $\forall_{\mathscr{V} \in \Gamma(\mathcal{T}M)}\ :\ \Delta_Q(\mathscr{V} \lrcorner\, {}^M\mathrm{B}) = 0.$

Furthermore, using the proof of the above, we readily establish

Prop$^{\underline{\mathrm{n}}}$: [rrS '10] (Hitchin-type isomorphisms)

$$\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}_{(\iota_\alpha,\pi_n^{k,k+1})}(\mathscr{F}) \xrightarrow{\left(\mathrm{e}^{-B_i}\right)\sqcup\left(\mathrm{e}^{-P_A}\right)\sqcup\mathrm{id}} \Big(\mathcal{E}^{(1,1)}_{\{\mathrm{e}^{-\mathrm{d}A_{ij}}\}}M \sqcup \mathcal{E}^{(1,0)}_{\{\mathrm{e}^{P_A-P_B}\}}Q \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} \mathcal{E}^{(1,-1)}T_n, [\![\cdot,\cdot]\!]^{(0,0;\Delta_Q)}, (\cdot,\cdot)_\lrcorner, \alpha_{\mathcal{T}\mathscr{F}}\Big)_{(\iota_\alpha,\pi_n^{k,k+1})}$$

where the $\mathcal{E}^{(1,q)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M} \to \mathscr{M}$ are **TWISTED GENERALISED TANGENT SHEAVES**, with **TRANSITION OPERATORS**

$$\mathfrak{g}_{ij} \in \mathrm{End}\big(\mathcal{E}^{(1,q)}_{\{\mathfrak{g}_{ij}\}}\mathscr{M}(\mathcal{O}_i^{\mathscr{M}} \cap \mathcal{O}_j^{\mathscr{M}})\big)$$

subject to the cocycle relation

$$(\mathfrak{g}_{ij} \circ \mathfrak{g}_{jk})|_{\mathcal{O}_i^{\mathscr{M}} \cap \mathcal{O}_j^{\mathscr{M}} \cap \mathcal{O}_k^{\mathscr{M}}} = \mathfrak{g}_{ik}|_{\mathcal{O}_i^{\mathscr{M}} \cap \mathcal{O}_j^{\mathscr{M}} \cap \mathcal{O}_k^{\mathscr{M}}}$$

and defining the gluing of the local sections of the sheaf as per

$$\mathfrak{V}_j|_{\mathcal{O}_i^{\mathscr{M}} \cap \mathcal{O}_j^{\mathscr{M}}} = \mathfrak{g}_{ij} \triangleright \mathfrak{V}_i|_{\mathcal{O}_i^{\mathscr{M}} \cap \mathcal{O}_j^{\mathscr{M}}}\,.$$

II.3. The canonical interpretation

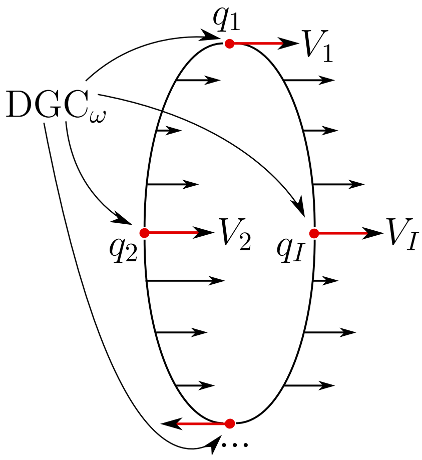
The phase space of the σ-model is neatly parameterised by Cauchy data localised on twisted (space-like) loops, i.e.

$$\mathsf{P}_{\sigma,\mathcal{B}|\{\varepsilon_k\}} = \Big\{ \big(X, \mathsf{p}, q_k, V_k \mid k \in \overline{1,I}\big) \in \mathsf{T}^*C^\infty\big(\mathbb{S}^1_{\{P_k\}}, M\big) \times \mathsf{T}Q^{\times I} \ \Big|$$

$$\wedge \quad \left\{ \begin{array}{l} \lim_{\epsilon\to 0^+} \mathsf{p}\big(P_k + (-1)^{\alpha+1}\,\epsilon\big) = \mathrm{g}_{\iota_\alpha^{\varepsilon_k}(P_k)}\big(\varepsilon_k\, \iota^{\varepsilon_k}_{\alpha\,*} V_k, \cdot\big) \\ \\ \mathrm{g}\big(\iota_1(P_k)\big)\big(\widehat{\tau}_1(P_k), \iota_{1\,*}(\cdot)\big) - \mathrm{g}\big(\iota_2(P_k)\big)\big(\widehat{\tau}_2(P_k), \iota_{2\,*}(\cdot)\big) = V_k \lrcorner\, \omega(q_k) \end{array} \right\} .$$

where $\mathbb{S}^1_{\{P_k\}} = \mathbb{S}^1 \smallsetminus \{P_k\}_{k\in\overline{1,I}}$ for arbitrary P_k, where $\widehat{\tau}_\alpha(P_k) := -\varepsilon_k \lim_{\epsilon\to 0^+} X_*\widehat{t}\big(P_k + (-1)^{\alpha+1}\,\varepsilon_k\,\epsilon\big)$, and where $(\iota_1^{+1}, \iota_2^{+1}) := (\iota_1, \iota_2)$ and $(\iota_1^{-1}, \iota_2^{-1}) := (\iota_2, \iota_1)$, i.e.

$\mathsf{P}_{\sigma,\mathcal{B}|\{\epsilon_k\}} = \Bigg\{$



$\Bigg\}$

Using the first-order formalism of Gawędzki–Kijowski-Szczyrba–Tulczyjew, we derive from S_σ a **SYMPLECTIC FORM** for $\mathsf{P}_{\sigma,\mathcal{B}|\{\varepsilon_k\}} \subset \mathsf{P}_\sigma$,

$$\Omega_{\sigma,\mathcal{B}|\{\varepsilon_k\}}[(X, \mathsf{p}, q_k, V_k)] = \mathrm{pr}^*_{\mathsf{T}^*C^\infty(\mathbb{S}^1_{\{P_k\}}, M)}\Big(\delta\theta + \pi^* \int_{\mathbb{S}^1_{\{P_k\}}} \mathrm{ev}_k^*\mathrm{H}\Big) + \sum_{k=1}^{I} \varepsilon_k\, \mathrm{pr}^*_{Q^{(k)}}\omega\,,$$

the latter being written in terms of the canonical maps:

$$\mathrm{pr}_X \ : \ \mathsf{P}_{\sigma,\mathcal{B}|\{\varepsilon_k\}} \to X\,, \qquad\qquad \pi \ : \ \mathsf{T}^*C^\infty\big(\mathbb{S}^1_{\{P_k\}}, M\big) \to C^\infty\big(\mathbb{S}^1_{\{P_k\}}, M\big)\,,$$

$$\mathrm{ev}_k \ : \ C^\infty\big(\mathbb{S}^1_{\{P_k\}}, M\big) \times \mathbb{S}^1_{\{P_k\}} \to M\,,$$

and the canonical 1-form $\theta[(X, \mathsf{p})] = \int_{\mathbb{S}^1_{\{P_k\}}} \mathrm{Vol}(\mathbb{S}^1_{\{P_k\}}) \wedge \mathsf{p}$.

II.3*. Reminder on the KGST formalism

To a theory given in terms of an action functional ($D = \dim \mathscr{M}$)

$$S[\phi^A] = \int_{\mathscr{M}} \mathsf{d}^D x\, \mathcal{L}(x^\mu, \phi^A, \xi^B_\nu)|_{\xi^B_\nu = \partial_\nu \phi^B}\,, \qquad \mathsf{d}^D x = \mathsf{d}x^1 \wedge \mathsf{d}x^2 \wedge \cdots \wedge \mathsf{d}x^D$$

on sections $(\phi^A)^{A \in \overline{1,N}}$ of the **CONFIGURATION BUNDLE** $\pi_{\mathscr{F}} : \mathscr{F} \to \mathscr{M}$, we associate the **CARTAN FORM** on the first-jet bundle $J^1\mathscr{F} \to \mathscr{M}$,

$$\Theta(x^\mu, \phi^A, \xi^B_\nu) = \big(\mathcal{L} - \xi^C_\lambda \tfrac{\delta\mathcal{L}}{\delta\xi^C_\lambda}\big)(x^\mu, \phi^A, \xi^B_\nu)\, \mathsf{d}^D x + \tfrac{\delta\mathcal{L}}{\delta\xi^C_\lambda}(x^\mu, \phi^A, \xi^B_\nu)\, \delta\phi^C \wedge \big(\partial_\lambda \lrcorner\, \mathsf{d}^D x\big)\,.$$

The latter has the all-important properties:

(i) the PLA for the functional

$$S_\Theta[\Psi] := \int_{\mathscr{M}} \Psi^* \Theta\,, \qquad \Psi \in \Gamma(J^1\mathscr{F})$$

yields the Euler–Lagrange equations of S;

(ii) upon defining a functional

$$S_{12}[\Psi_{\rm cl.}] := \int_{\mathscr{M}_{12}} \big(\Psi_{\rm cl.}|_{\mathscr{M}_{12}}\big)^* \Theta\,,$$

for a region $\mathscr{M}_{12} \subset \mathscr{M}$ cobounded by two homotopic Cauchy surfaces $\mathscr{C}_1$ and $\mathscr{C}_2$, we readily establish

$$\delta S_{12}[\Psi_{\rm cl.}] = \Xi_{\mathscr{C}_2}[\Psi_{\rm cl.}] - \Xi_{\mathscr{C}_1}[\Psi_{\rm cl.}]\,,$$

and so Θ canonically defines a closed 2-form

$$\Omega[\Psi_{\rm cl.}] := \delta\Xi_{\mathscr{C}}[\Psi_{\rm cl.}]\,, \qquad \mathscr{C} \in [\mathscr{C}_1]_{\rm hom.}$$

on the space $\mathsf{P}_{([\mathscr{C}_1]_{\rm hom.})}$ of extremal sections of $J^1\mathscr{F}$, i.e. also a symplectic structure on the phase space $\overline{\mathsf{P}}_{([\mathscr{C}_1]_{\rm hom.})}$ of the field theory.

We have a Noether mapping

$$\Gamma_\sigma(\mathcal{E}(M\sqcup Q))\to\Gamma\big(\mathcal{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}|\{\varepsilon_k\}}\big)\cap\ker\delta^{(1)}_{\Omega_{\sigma,\mathcal{B}|\{\varepsilon_k\}}}\ :\ \mathfrak{K}\mapsto\widetilde{\mathfrak{K}}\qquad \textbf{HAMILTONIAN SECTION.}$$

E.g.: Introduce the (1-)twisted loop space

$$\mathsf{L}_{Q|\varepsilon}=\{\ (X,q)\in C^\infty(\mathbb{S}^1_\pi,M)\times Q\quad|\quad\lim_{\epsilon\to 0^+}X\big(\pi+(-1)^{\alpha+1}\varepsilon\,\epsilon\big)=q\ \}\,,$$

and the canonical projection $\mathrm{pr}_{\mathsf{L}}:\mathsf{P}_{\sigma,\mathcal{B}|\varepsilon}\to\mathsf{L}_{Q|\varepsilon}$.

For any $({}^M\mathscr{V},\upsilon)\sqcup_{\iota_\alpha}({}^Q\mathscr{V},f)\in\Gamma\big(\mathcal{E}^{(1,1)}M\sqcup\mathcal{E}^{(1,0)}Q)\big)$, we have

CANONICAL L-LIFTS

$$\begin{aligned}\mathsf{L}_*\equiv({}^M\mathsf{L}_*,{}^Q\mathsf{L}_*)\quad&:\quad({}^M\mathscr{V},{}^Q\mathscr{V})\mapsto\Big(\int_{\mathbb{S}^1_\pi}{}^M\mathscr{V},{}^Q\mathscr{V}\circ\mathrm{pr}_Q\Big)\in\Gamma(\mathcal{T}\mathsf{L}_{Q|\varepsilon})\,,\\ ({}^M\mathsf{L}^*,{}^Q\mathsf{L}^*)\quad&:\quad(\upsilon,f)\mapsto\Big(\int_{\mathbb{S}^1_\pi}\mathrm{ev}_1^*\upsilon,\mathrm{pr}_Q^*f\Big)\in\Gamma(\mathcal{T}_0^*\mathsf{L}_{Q|\varepsilon})^{\times 2}\,.\end{aligned}$$

These can be used to induce

CANONICAL P-LIFTS

$$\begin{aligned}\widetilde{\mathsf{L}}_*\quad&:\quad\Gamma(\mathcal{T}M\sqcup_{\iota_\alpha}\mathcal{T}Q)\to\Gamma(\mathcal{T}\mathsf{P}_{\sigma,\mathcal{B}|\varepsilon})\,,\\ ({}^M\widetilde{\mathsf{L}}^*,{}^Q\widetilde{\mathsf{L}}^*):=\mathrm{pr}_{\mathsf{L}}^*\circ({}^M\mathsf{L}^*,{}^Q\mathsf{L}^*)\quad&:\quad\Gamma(\mathcal{T}_1^*M\sqcup\mathcal{T}_0^*Q)\to\Gamma(\mathcal{T}_0^*\mathsf{P}_{\sigma,\mathcal{B}|\varepsilon})^{\times 2}\,,\end{aligned}$$

with the former fixed by the standard conditions

$$\mathrm{pr}_{\mathsf{L}\,*}\circ\widetilde{\mathsf{L}}_*=\mathsf{L}_*\qquad\wedge\qquad\mathscr{L}_{\mathsf{L}_*({}^M\mathscr{V},{}^Q\mathscr{V})}\mathrm{pr}^*_{\mathsf{T}^*C^\infty(\mathbb{S}^1_\pi,M)}\theta=0\,.$$

This gives us a lift

$$\begin{aligned}\widetilde{\mathsf{L}}^*_*(\varepsilon)\quad&:\quad\Gamma\big(\mathcal{E}^{(1,1)}M\sqcup_{\iota_\alpha}\mathcal{E}^{(1,0)}Q)\big)\to\Gamma(\mathcal{E}^{(1,0)}\mathsf{P}_{\sigma,\mathcal{B}|\varepsilon})\\ &:\quad({}^M\mathscr{V},\upsilon)\sqcup({}^Q\mathscr{V},f)\mapsto\widetilde{\mathsf{L}}_*({}^M\mathscr{V},{}^Q\mathscr{V})\oplus\big({}^M\widetilde{\mathsf{L}}^*\upsilon+\varepsilon\,{}^Q\widetilde{\mathsf{L}}^*f\big)\,.\end{aligned}$$

Prop$^{\underline{\mathrm{n}}}$: [rrS '10]

$$\widetilde{\mathfrak{K}}=\mathrm{e}^{\mathrm{pr}^*_{\mathsf{T}^*C^\infty(\mathbb{S}^1_\pi,M)}\theta}\triangleright\widetilde{\mathsf{L}}^*_*(\varepsilon)\mathfrak{K}\,,\qquad\qquad\big[\widetilde{\mathfrak{K}}_1,\widetilde{\mathfrak{K}}_2\big]^{\Omega_{\sigma,\mathcal{B}|\varepsilon}}_{\mathrm{V}}=\widetilde{[\![\mathfrak{K}_1,\mathfrak{K}_2]\!]}^{(\mathrm{H},\omega;\Delta_Q)}\,.$$

III. The Gauge Principle

The next logical step consists in understanding the mechanism of gauging for rigid symmetries G_σ of the σ-model.

Motivation:

(i) **The topography of the theory space**: Working out systematic tools for constructing new σ-models, with field spaces given by G_σ-cosets of the original ones.

(ii) **Stringy dualities**: Obtaining ancillary tools for a rigorous study of *bona fide* dualities of the σ-model (e.g., the mirror symmetry for Calabi–Yau field spaces).

(iii) **"Non-geometry"**: Getting hints as to possible extensions of the smooth category $\mathfrak{Man}$ via stringy-duality quotients.

Challenges:

(i) G_σ**-equivariance**: Lifting the geometric symmetry from $\mathscr{F}$ to $\mathfrak{B}$.

(ii) **A principal extension**: In the case of continuous symmetries, the introduction of the world-sheet G-gauge field and coupling them to $X^*\mathfrak{B}$, in particular in the topologically non-trivial setting.

(iii) **The coset construction**: Understanding the descent $\mathfrak{B} \to \mathfrak{B}/\mathrm{G}_\sigma$ in purely geometric terms.

III.1. Insights from the study of the next-to-trivial case

Observation: $\mathfrak{g}_\sigma := \alpha_{\mathcal{T}\mathscr{F}}\big(\Gamma_\sigma(\mathcal{E}\mathscr{F})\big)$ is a Lie subalgebra of $\Gamma(\mathcal{T}\mathscr{F})$.

Let $\mathscr{K}_a,\ a \in \overline{1,D}$ be generators of $\mathfrak{g}_\sigma \equiv \mathrm{Lie}\mathrm{G}_\sigma$, satisfying

$$\textsc{structure relations} \qquad \big[\mathscr{K}_a, \mathscr{K}_b\big] = f_{abc}\,\mathscr{K}_c\,, \qquad f_{abc} \in \mathbb{R}\,.$$

Complete the $\mathscr{K}_a$ to the respective

$$\mathfrak{K}_a = \big({}^M\mathscr{K}_a \oplus \kappa_a\big) \sqcup \big({}^Q\mathscr{K}_a \oplus k_a\big) \sqcup \big({}^{T_n}\mathscr{K}_a \oplus 0\big) \in \Gamma_\sigma(\mathcal{E}\mathscr{F})\,.$$

Gauging G_σ calls for the introduction of

$$\textsc{principal } \mathrm{G}_\sigma\textsc{-bundle} \qquad \mathrm{G}_\sigma \hookrightarrow \mathsf{P} \xrightarrow{\pi_{\mathsf{P}}} \Sigma \qquad \text{with} \qquad r\ :\ \mathsf{P} \times \mathrm{G}_\sigma \to \mathsf{P}\ :\ (p,g) \mapsto p.g$$

$$\textsc{principal } \mathrm{G}_\sigma\textsc{-connection} \qquad \mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathrm{g}_\sigma \qquad \text{s.t.} \qquad \begin{cases} {}^{\mathsf{P}}\mathscr{K}_a \lrcorner\, \mathcal{A} = t_a \\ \mathcal{A}(p.g^{-1}) = \mathrm{Ad}_g \mathcal{A}(p) \end{cases}.$$

Consider, first, a G_σ-invariant top.-trivial background

$$\mathrm{H} = \mathsf{d}B\,, \qquad \Delta_Q B + \omega = \mathsf{d}P\,, \qquad \Delta_{T_n} P = -\mathsf{i}\,\mathsf{d}\log f_n\,,$$

$$\mathscr{L}_{{}^M\mathscr{K}_a} B = 0 = \mathscr{L}_{{}^Q\mathscr{K}_a} P = 0 = \mathscr{L}_{{}^{T_n}\mathscr{K}_a} f_n\,, \qquad \text{with} \qquad \mathfrak{K}_a = \big(\mathrm{e}^{\mathrm{B}} \sqcup \mathrm{e}^{P}\big)\big(\mathscr{K}_a \oplus 0\big)\,,$$

and a top.-trivial principal G_σ-bundle, $\mathsf{P} = \Sigma \times \mathrm{G}_\sigma$, with $\mathrm{A} \in \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$.

Particle-physics' intuition:

$$\textsc{minimal coupling} \qquad \mathsf{d}X^\mu(\sigma) \mapsto \mathrm{e}^{-\mathrm{A}^a(\sigma)\,\mathscr{K}_a(X(\sigma))\lrcorner}\,\mathsf{d}X^\mu(\sigma) \equiv D_{\mathrm{A}} X^\mu(\sigma)\,,$$

$$D_{\mathrm{A}}(g.X)^\mu = \tfrac{\partial (g.X)^\mu}{\partial X^\nu}\,D_{\mathrm{A}} X^\nu\,.$$

Upshot: Upon simple rearrangement, we obtain

$$\textsc{extended field space}\qquad \begin{aligned} & M \mapsto \Sigma \setminus \Gamma \times M\,, \\ & Q \mapsto \Gamma \setminus \mathfrak{V}_\Gamma \times Q\,, \\ & T_n \mapsto \mathfrak{V}_\Gamma^{(n)} \times T_n \end{aligned}$$

$$\textsc{extended background}\qquad \begin{aligned} & \mathrm{g} \mapsto \mathrm{pr}_2^*\mathrm{g}\,, \qquad\qquad \mathcal{G} \mapsto \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_{\mathrm{A}}} \\ & \Phi \mapsto \mathrm{pr}_2^*\Phi \otimes J_{\lambda_{\mathrm{A}}}\,, \\ & \varphi_n \mapsto \mathrm{pr}_2^*\varphi_n\,, \end{aligned}$$

where

$$\rho_{\mathrm{A}} = \mathrm{pr}_2^*\kappa_a \wedge \mathrm{pr}_1^*\mathrm{A}^a - \tfrac{1}{2}\,\mathrm{pr}_2^*({}^{\scriptscriptstyle M}\mathscr{K}_a \lrcorner\, \kappa_b)\,\mathrm{pr}_1^*(\mathrm{A}^a \wedge \mathrm{A}^b)\,, \qquad\qquad \lambda_{\mathrm{A}} = -\mathrm{pr}_2^* k_a\,\mathrm{pr}_1^*\mathrm{A}^a\,.$$

Ansatz: For $\mathsf{P} = \Sigma \times \mathrm{G}_\sigma$ with $\mathrm{A} \in \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$, we take

(i) S_{kin} – minimal coupling;

(ii) S_{top} – decorated-surface holonomy for an extended background

$$\big((\Sigma \setminus \Gamma, \mathrm{pr}_2^*\mathrm{g}, \mathrm{pr}_2^*\mathcal{G} \otimes I_{\varsigma_{\mathrm{A}}}), (\Gamma \setminus \mathfrak{V}_\Gamma, \mathrm{pr}_2^*\Phi \otimes J_{\mu_{\mathrm{A}}}), (\mathfrak{V}_\Gamma^{(n)} \times T_n, \mathrm{pr}_2^*\varphi_n)\big)\,.$$

Upshot: Infinitesimal-invariance analysis yields

$$\varsigma_{\mathrm{A}} \stackrel{!}{=} \rho_{\mathrm{A}}\,, \qquad\qquad \mu_{\mathrm{A}} \stackrel{!}{=} \lambda_{\mathrm{A}}\,, \qquad \text{with the } \mathfrak{K}_a \text{ subject to}$$

$$\textsc{gaugeability constraints}\qquad \begin{cases} \mathscr{L}_{{}^{\scriptscriptstyle M}\mathscr{K}_a}\kappa_b = f_{abc}\kappa_c \qquad \wedge \qquad \mathscr{L}_{{}^{\scriptscriptstyle Q}\mathscr{K}_a} k_b = f_{abc} k_c\,, \\ \qquad\qquad {}^{\scriptscriptstyle M}\mathscr{K}_a \lrcorner\, \kappa_b + {}^{\scriptscriptstyle M}\mathscr{K}_b \lrcorner\, \kappa_a = 0\,. \end{cases}$$

III.2. An algebroidal interpretation of the gaugeability constraints

The action $\ell \;:\; \mathrm{G}_\sigma \times \mathscr{F} \to \mathscr{F}$ gives rise to

$$\textbf{ACTION GROUPOID} \qquad \mathrm{G}\ltimes\mathscr{F} \quad : \; \mathrm{G} \times \mathscr{F} \overset{s=\mathrm{pr}_2}{\underset{t=\ell}{\rightrightarrows}} \mathscr{F} \;,$$

i.e. the small category

$$\mathrm{G}\ltimes\mathscr{F} = \left(\mathscr{F}, \mathrm{G}_\sigma \times \mathscr{F}, \mathrm{pr}_2, \ell, m \xrightarrow{\mathrm{Id}} (e,m), \circ\right)$$

with all morphisms invertible, as per

$$\mathrm{Inv} \;:\; \mathrm{G}_\sigma \times \mathscr{F} \to \mathrm{G}_\sigma \times \mathscr{F} \;:\; (g,m) \mapsto (g^{-1}, g.m)\,.$$

As for any Lie groupoid, we define its

$$\textbf{TANGENT (LIE) ALGEBROID} \qquad \mathfrak{g}_\sigma\ltimes\mathscr{F} = \left(\mathrm{Id}^*\mathrm{ker}(\mathsf{d}s), [\cdot,\cdot], \alpha_{\mathsf{T}(\mathrm{Ob}(\mathrm{G}_\sigma\ltimes\mathscr{F}))}\right),$$

with $\alpha_{\mathsf{T}(\mathrm{Ob}\,\mathrm{Gr})}$ inducing the map $\mathsf{d}t \circ i$ between spaces of sections, defined in terms of the canonical vector-space isomorphism

$$i \;:\; \Gamma(\mathrm{Id}^*\mathrm{ker}\,(\mathsf{d}s)) \xrightarrow{\cong} \mathfrak{X}^s_{\mathrm{R-inv}}(\mathrm{Mor}\,\mathrm{Gr})\,,$$

and with $[\cdot,\cdot]$ given by the unique bracket on $\Gamma(\mathrm{Id}^*\mathrm{ker}\,(\mathsf{d}s))$ for which i is an isomorphism of Lie algebras.

In the case in hand,

$$\mathfrak{g}_\sigma\ltimes\mathscr{F} \cong \left(\oplus_{a=1}^D C^\infty(\mathscr{F},\mathbb{R})\,\mathscr{R}_a, [\cdot,\cdot]_{\mathfrak{g}_\sigma\ltimes\mathscr{F}}, \alpha_{\mathsf{T}\mathscr{F}}\right), \qquad\qquad \mathscr{R}_a \equiv R_a \circ \mathrm{pr}_1|_{\mathrm{Id}(\mathscr{F})}$$

$$[\,\lambda^a\,\mathscr{R}_a\,,\,\mu^b\,\mathscr{R}_b\,]_{\mathfrak{g}_\sigma\ltimes M} = f_{abc}\,\lambda^a\,\mu^b\,\mathscr{R}_c + \left(\mathscr{L}_{\lambda^a\,\mathscr{K}_a}\mu^b - \mathscr{L}_{\mu^a\,\mathscr{K}_a}\lambda^b\right)\mathscr{R}_b\,.$$

<u>**Prop$^{\underline{\mathrm{n}}}$:**</u> [rrS '10]

$$\mathfrak{g}_\sigma\ltimes\mathscr{F} \cong \left(\oplus_{a=1}^D C^\infty(\mathscr{F},\mathbb{R})\mathfrak{K}_a, [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}, \alpha_{\mathcal{T}\mathscr{F}}\right).$$

VII.3. The global gauge anomaly

Invariance of the gauged σ-model under *large* gauge transformations calls – via a cohomological argument – for the existence of

$$\Upsilon\ :\ \ell^*\mathcal{G} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{G}\otimes I_{\rho_{\theta_L}} \qquad \text{over} \qquad \mathrm{Mor}(\mathrm{G}_\sigma{\ltimes}M)\,,$$

and a consistent 2-extension thereof to Φ and φ_n.

At this stage, we need to comply with the following requirements

(i) Incorporation of topologically non-trivial gauge bundles ($\sim$ G_σ-twisted sectors, or – less evidently – a solution to the field-identification problem).

(ii) Preservation of the original count of the physical degrees of freedom, given by $\dim\mathscr{F}$.

Problem: Goal (i) readily achieved via

PRINCIPAL EXTENSION $\qquad \mathscr{F}\mapsto (\mathsf{P}|_{\Sigma\setminus\Gamma}\times M)\sqcup(\mathsf{P}|_{\Gamma\setminus\mathfrak{V}_\Gamma}\times Q)\sqcup\bigsqcup_{n\in\mathbb{N}_{\geq 3}}(\mathsf{P}|_{\mathfrak{V}_\Gamma^{(n)}}\times T_n)\equiv\widetilde{\mathscr{F}}\,,$

with obvious Ansätze:

$$\widetilde{\mathcal{G}}_{\mathcal{A}}=\mathrm{pr}_2^*\mathcal{G}\otimes I_{\rho_{\mathcal{A}}}\,,\qquad \widetilde{\Phi}_{\mathcal{A}}=\mathrm{pr}_2^*\Phi\otimes J_{\lambda_{\mathcal{A}}}\,,\qquad \widetilde{\varphi}_{n,\mathcal{A}}=\mathrm{pr}_2^*\varphi_n\,.$$

However, the typical fibres here are

$$\mathrm{G}_\sigma\times M\hookrightarrow\widetilde{M}\to\Sigma\setminus\Gamma\,,\qquad \mathrm{G}_\sigma\times Q\hookrightarrow\widetilde{Q}\to\Gamma\setminus\mathfrak{V}_\Gamma\,,$$
$$\mathrm{G}_\sigma\times T_n\hookrightarrow\widetilde{T}_n\to\mathfrak{V}_\Gamma^{(n)}\,.$$

Idea: Lift the geometric action of G_σ from $\widetilde{\mathscr{F}}$ to the extended background.

VII.3*. Intermezzo: The Descent Principle, or (bundling,) gerbing and gauging

Given a pair $(\widetilde{M}, M)$ of manifolds and a *surjective submersion* $\varpi : \widetilde{\mathscr{M}} \to \mathscr{M}$, define over the simplicial manifold

$$\cdots \overset{\mathrm{pr}_{i,j,k}}{\Rrightarrow} \widetilde{M} \times_M \widetilde{M} \times_M \widetilde{M} \overset{\mathrm{pr}_{i,j}}{\Rrightarrow} \widetilde{M} \times_M \widetilde{M} \overset{\mathrm{pr}_i}{\rightrightarrows} \widetilde{M} \xrightarrow{\varpi} M$$

the **DESCENT 2-CATEGORY** $\mathfrak{Desc}(\varpi)$, with objects $(\mathcal{G}, \Psi, \chi)$, where

$$\mathrm{pr}_1^* \mathcal{G} \underset{\Psi}{\overset{\cong}{\to}} \mathrm{pr}_2^* \mathcal{G}\,, \qquad \mathrm{pr}_{2,3}^* \Psi \circ \mathrm{pr}_{1,2}^* \Psi \underset{\chi}{\overset{\cong}{\Longrightarrow}} \mathrm{pr}_{1,3}^* \Psi\,,$$

$$\mathrm{pr}_{1,3,4}^* \chi \bullet (\mathrm{id} \circ \mathrm{pr}_{1,2,3}^* \chi) = \mathrm{pr}_{1,2,4}^* \chi \bullet (\mathrm{pr}_{2,3,4}^* \chi \circ \mathrm{id})\,,$$

1-cells $(\Phi, \eta) : (\mathcal{G}_1, \Psi_1, \chi_1) \overset{\cong}{\to} (\mathcal{G}_2, \Psi_2, \chi_2)$, where

$$\mathcal{G}_1 \underset{\Phi}{\overset{\cong}{\to}} \mathcal{G}_2\,, \qquad \mathrm{pr}_2^* \Phi \circ \Psi_1 \underset{\eta}{\overset{\cong}{\Longrightarrow}} \Psi_2 \circ \mathrm{pr}_1^* \Phi\,,$$

$$(\chi_2 \circ \mathrm{id}) \bullet (\mathrm{id} \circ \mathrm{pr}_{1,2}^* \eta) \bullet (\mathrm{pr}_{2,3}^* \eta \circ \mathrm{id}) = \mathrm{pr}_{1,3}^* \eta \bullet (\mathrm{id} \circ \chi_1)\,,$$

and 2-cells $\varphi : (\Phi_1, \eta_1) \overset{\cong}{\Longrightarrow} (\Phi_2, \eta_2)$, where

$$\Phi_1 \underset{\varphi}{\overset{\cong}{\Longrightarrow}} \Phi_2\,, \qquad (\mathrm{id} \circ \mathrm{pr}_1^* \varphi) \bullet \eta_1 = \eta_2 \bullet (\mathrm{pr}_2^* \varphi \circ \mathrm{id})$$

<u>Th^m:</u> [Stevenson '00]

$$\varpi^* \; : \; \mathfrak{BGrb}^\nabla(M) \overset{\cong}{\to} \mathfrak{Desc}(\varpi) \; : \; \begin{cases} \mathcal{G} & \mapsto \quad (\varpi^* \mathcal{G}, \mathrm{id}, \mathrm{id}) \\ \Phi & \mapsto \quad (\varpi^* \Phi, \mathrm{id}) \\ \varphi & \mapsto \quad \varpi^* \varphi \end{cases} .$$

<u>The beautiful:</u>

(i) $\mathfrak{BGrb}^\nabla(\mathscr{M}) \equiv \big(\pi_{Y\mathscr{M}}^*\big)^{-1}\big(\mathfrak{Triv}\text{–}\mathfrak{BGrb}^\nabla(Y\mathscr{M})\big)$, the latter being defined in terms of smooth 2-forms and $\mathfrak{Bun}^\nabla(Y\mathscr{M})$, with $\mathfrak{Bun}^\nabla(Y\mathscr{M}) \equiv \big(\pi_{Y'Y\mathscr{M}}^*\big)^{-1}\big(\mathfrak{Triv}\text{–}\mathfrak{Bun}^\nabla(Y'Y\mathscr{M})\big)$.

(ii) Descent for the action groupoid over $\mathrm{G}_\sigma \hookrightarrow M \overset{\varpi}{\to} M/\mathrm{G}_\sigma$, where $\mathrm{G}_\sigma \subset \mathrm{Iso}(M, \mathrm{g})$ is a group of σ-model symmetries, determines the Gauge Principle (due to a remarkable interplay between Σ and $\mathscr{F}$).

A 2-birds-with-1-stone solution:

(i) Demand of $(\mathcal{G}, \Phi, \varphi_n)$ a full-blown G_σ**-EQUIVARIANT STRUCTURE** (i.e., morally speaking, pass from Čech–Deligne- to Čech–Deligne–G_σ-hypercohomology).

Prop$^{\underline{\mathrm{n}}}$: [Gawędzki, Waldorf & rrS '10] A G_σ-equivariant structure on $\mathfrak{B}$ relative to *arbitrary* $(\rho_\cdot, \lambda_\cdot)$ canonically induces a G_σ-equivariant structure on $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ relative to $(\rho_\cdot, \lambda_\cdot) = (0, 0)$.

(ii) Employ the Principle of Descent, in the guise

$$\mathfrak{BGrb}^\nabla_{(\rho_\cdot,\lambda_\cdot)=(0,0)}(\widetilde{\mathscr{F}}) \equiv \mathfrak{BGrb}^\nabla(\widetilde{\mathscr{F}}/\mathrm{G}_\sigma)$$

valid for the distinguished surjective submersions $\varpi_{\widetilde{\mathscr{F}}}\widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}}/\mathrm{G} \equiv \mathsf{P} \times_{\mathrm{G}_\sigma} \mathscr{F}$ (engendered by the *free* action $\widetilde{\ell} : \mathrm{G}_\sigma \times \widetilde{\mathscr{F}} \to \mathscr{F}$), to descend

$$(\widetilde{\mathcal{G}}_{\mathcal{A}}, \widetilde{\Phi}_{\mathcal{A}}, \widetilde{\varphi}_{n,\mathcal{A}}) \to \big(\underline{\mathcal{G}}(\mathcal{A}), \underline{\Phi}(\mathcal{A}), \underline{\varphi}_n(\mathcal{A})\big)$$

to the associated bundles.

Upshot: The **GAUGED σ-MODEL**

$$S_\sigma[(\underline{X} \,|\, \Gamma); \gamma, \mathcal{A}] = S^{\mathrm{MC}}_{\mathrm{kin}}[\underline{X}; \gamma, \mathcal{A}] - \mathsf{i}\, \log \mathrm{Hol}_{\underline{\mathcal{G}}(\mathcal{A}), \underline{\Phi}(\mathcal{A}), \underline{\varphi}_n(\mathcal{A})}(\underline{X}),$$

manifestly invariant under the action of the **GAUGE GROUP**

$$\Gamma(\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}_\sigma) \;:\; [(p, g_1)] \cdot [(p, g_2)] := [(p, g_1 \cdot g_2)].$$

The latter is induced by the action

$$\lambda \;:\; (\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}_\sigma) \times \mathsf{P} \to \mathsf{P} \;:\; \big([(p, g_1)], p.g_2\big) \mapsto p.(g_1 \cdot g_2)$$

and reads

$$(\chi, \underline{X}) \mapsto (\lambda_\chi, \mathrm{id}_M) \circ \underline{X}, \qquad\qquad (\chi, \mathcal{A}) \mapsto \lambda^*_{\chi^{-1}} \mathcal{A}.$$

VII.4. The coset model

For the topologically trivial gauge field (or locally), we may define the **COSET σ-MODEL** as

$$\mathrm{e}^{-W_{\rm eff}[(\underline{X}\,|\,\Gamma);\gamma]} := \int_{[\mathrm{A}]} \mathscr{D}\mathrm{A}\, \mathrm{e}^{-S_\sigma[(\underline{X}\,|\,\Gamma);\gamma,\mathrm{A}]}$$

N.B. The above path integral is gaussian, whence

$$W_{\sigma,\rm eff}[(\underline{X}\,|\,\Gamma);\gamma] \sim S_\sigma[(\underline{X}\,|\,\Gamma);\gamma,\mathrm{A}_{\rm cl.}]\,.$$

Under certain (mild) technical assumptions regarding $\mathfrak{B}$, the effective field theory is, indeed, a σ-model with a field space $\mathscr{F}$ and

EFFECTIVE BACKGROUND $\qquad \varpi^*_{\widetilde{\mathscr{F}}}\underline{\mathrm{G}}\,, \qquad \mathcal{G}\otimes I_\Delta\,, \qquad \Phi\otimes J_\delta\,, \qquad \varphi_n\,.$

The remarkable, again: The effective background is G_σ-equivariant relative to $(\rho_\cdot,\lambda_\cdot)=(0,0)$ iff the original one is G_σ-equivariant.

Conclusion: $\mathfrak{B}_{\rm eff}$ descends to a unique equivalence class $\underline{\mathfrak{B}}$ over the coset space $\mathscr{F}/\mathrm{G}_\sigma$ iff $\mathfrak{B}$ is endowed with a G_σ-equivariant structure.

Outlook: Towards “non-geometry” via gauged stringy dualities associated with groupoidal backgrounds…