Z_3 GENERALIZATION OF PAULI'S PRINCIPLE APPLIED TO QUARK STATES AND THE LORENTZ INVARIANCE

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Summary

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- **We show how the discrete symmetries Z_2 and Z_3 combined with the superposition principle result in the $SL(2, \mathbf{C})$-symmetry.**

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- **Then Pauli's principle is generalized for the case of the Z_3 grading replacing the usual Z_2 grading, leading to ternary commutation relations**

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- Elementary properties and structures of such algebras are discussed, with special interest in low-dimensional ones, with two or three generators.
- Invariant cubic forms on such algebras are introduced, and it is shown how the $SL(2, C)$ group arises naturally as the symmetry group preserving these forms.

Summary

In the case of lowest dimension, with two generators only, it is shown how the cubic combinations of Z_3-graded elements behave like Lorentz spinors, and the binary product of elements of this algebra with an element of the conjugate algebra behave like Lorentz vectors.

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- **Finally, the wave equation generalizing the Dirac operator to the Z_3-graded case is introduced, whose diagonalization leads to a third-order equation. The solutions of this equation cannot propagate because their exponents always contain non-oscillating real damping factor.**

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- We show how certain cubic products can propagate nevertheless. The model suggests the origin of the color $SU(3)$ symmetry.

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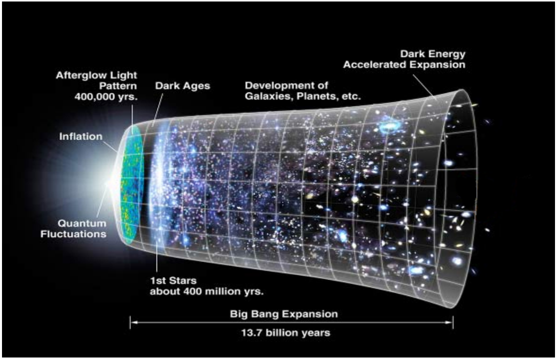
- **When extrapolated backwards in time, it leads to a compressed state of incredible density and temperature, containing nevertheless all baryonic matter and all photons, or their equivalent in some unknown form.**

Of matter and space-time



Artist's view of the Big Bang and Inflation

Of matter and space-time

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- **But it seems that matter was there even if it remains questionable whether space and time could be defined.**

Of matter and space-time

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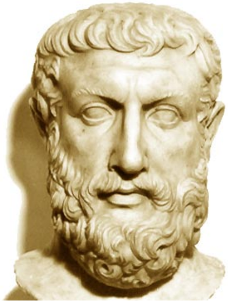
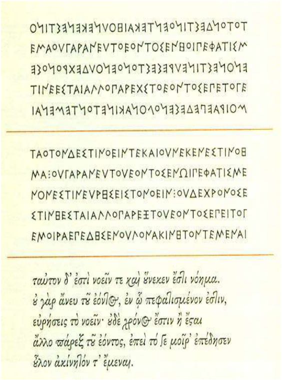
Of matter and space-time

- **Although hardly conceivable with our intuition, the hypothesis that matter can exist beyond space and time seems more acceptable than the contrary, the space-time totally deprived of matter.**
- **When one thinks of it a bit more, space-time without matter does not make a lot of sense.**
- **In any case, there would be nobody and nothing to check whether it exists, as pointed out by Parmenides more than twenty-four centuries ago.**





The Ancient Greek philosopher Parmenides of Elea

Of matter and space-time

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- In Parmenides' view, the totality of all things that exist can be defined as belonging to what he called *BEING*, and the only thing one can say about the *NON-BEING* is that it does not exist.
- Therefore a totally empty space devoid of any being (matter) cannot exist.
- As a corollary, one infers that space (and time) cannot exist by themselves, but only as by-products of matter. In today's fashionable language, they are *EMERGING ENTITIES*.

Of matter and space-time

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- As a follower of Maxwell and Faraday, he believed in the primary role of fields and tried to derive the equations of motion as characteristic behavior of singularities of the fields, or the singularities of the space-time curvature.

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- **But one can imagine an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time.**
- **Seen under this angle, the idea to derive the geometric properties of space-time, and perhaps its very existence, from fundamental symmetries and interactions proper to matter's most fundamental building blocks seems quite natural.**

Of matter and space-time

- **Many of those fundamental properties, especially on the quantum mechanical level, do not need any mention of space and time, as was demonstrated by Born and Heisenberg in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras.**

Of matter and space-time

- Many of those fundamental properties, especially on the quantum mechanical level, do not need any mention of space and time, as was demonstrated by Born and Heisenberg in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras.
- The non-commutative geometry is another example of formulation of space-time relationships in purely algebraic terms.

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- More precisely, they were conceived in order to take into account the relations between electric and magnetic fields as seen by different Galilean observers.
- Only later on Einstein extended the Lorentz transformations to space and time coordinates, giving them a universal meaning. As a result, the Lorentz symmetry became perceived as group of invariance of Minkowskian space-time metric.

Of matter and space-time

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- Extending the Lorentz transformations to space and time coordinates modified also Newtonian mechanics so that it could become invariant under the Lorentz instead of the Galilei group.
- In the textbooks introducing the Lorentz and Poincaré groups the accent is put on the transformation properties of space and time coordinates, and the invariance of the Minkowskian metric tensor $g_{\mu\nu} = diag(+, -, -, -)$.
- But neither the components of $g_{\mu\nu}$, nor the space-time coordinates of an observed event can be given an intrinsic physical meaning; they are not related to any conserved or directly observable quantities.

The Lorentz covariance

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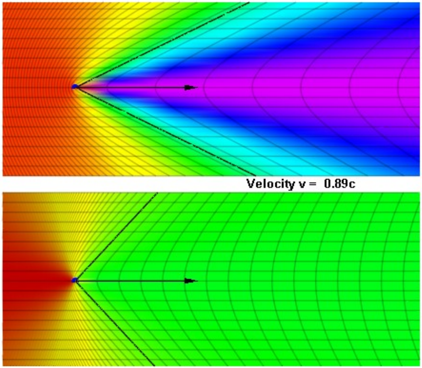
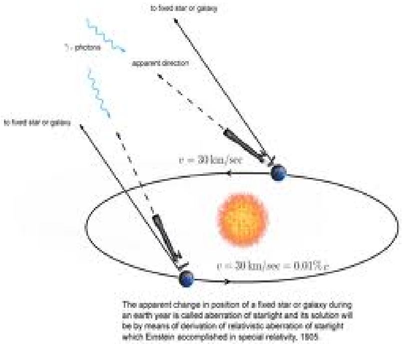
- Under a closer scrutiny, it turns out that only **TIME** - the proper time of the observer - can be measured directly. The notion of space variables results from the convenient description of experiments and observations concerning the propagation of photons, and the existence of the universal constant c.
- Consequently, with high enough precision one can infer that the Doppler effect is relativistic, i.e. the frequency ω and the wave vector $\mathbf{k}$ form an entity that is seen differently by different inertial observers, and passing from $\frac{\omega}{c}, \mathbf{k}$ to $\frac{\omega'}{c}, \mathbf{k}'$ is the Lorentz transformation.

The Lorentz covariance



Relativistic versus Galilean Doppler effect.

The Lorentz covariance



Aberration of light from stars. (Bradley, 1729)

The Lorentz covariance

Both effects, proving the relativistic formulae

$$\omega' = \frac{\omega - Vk}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad k' = \frac{k - \frac{V}{c^2}\omega}{\sqrt{1 - \frac{V^2}{c^2}}},$$

have been checked experimentally by Ives and Stilwell in 1937.

The Lorentz covariance

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- Reliable experimental confirmations of the validity of Lorentz transformations concern measurable quantities such as charges, currents, energies (frequencies) and momenta (wave vectors) much more than the less intrinsic quantities which are the *differentials* of the space-time variables.
- In principle, the Lorentz transformations could have been established by very precise observations of the Doppler effect alone.

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- **It should be stressed that had we only the light at our disposal, i.e. massless photons propagating with the same velocity c, we would infer that the general symmetry of physical phenomena is the** *Conformal Group*, **and not the Poincaré group.**

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- **It should be stressed that had we only the light at our disposal, i.e. massless photons propagating with the same velocity c, we would infer that the general symmetry of physical phenomena is the** *Conformal Group*, **and not the Poincaré group.**
- **To the observations of light must be added the** *the principle of inertia*, **i.e. the existence of massive bodies moving with speeds lower than c, and constant if not sollicited by external influence.**

First principles

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- **The distinctive feature of such particles is their** *inertial mass*, **equivalent with their energy at rest, which can be measured classically via Newton's law:**
- $$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{1}$$

First principles

The fundamental equation

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{2}$$

relates the only **observable** (using clocks and light rays as measuring rods) quantity, the acceleration **a**, with a combination of less evidently defined quantities, *mass* and *force*, which is interpreted as a **causality relation**, the force being the cause, and acceleration the effect.

First principles

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- **The more realistic sources of acceleration - or rather of the variation of energy and momenta - are the intensities of electric, magnetic or gravitational fields.**

First principles

The differential form of the Lorentz force,

$$\frac{d\mathbf{p}}{dt} = q\,\mathbf{E} + q\,\frac{\mathbf{v}}{c} \wedge \mathbf{B} \tag{3}$$

combined with the energy conservation of a charged particle under the influence of electromagnetic field

$$\frac{d\mathcal{E}}{dt} = q\,\mathbf{E} \cdot \mathbf{v} \tag{4}$$

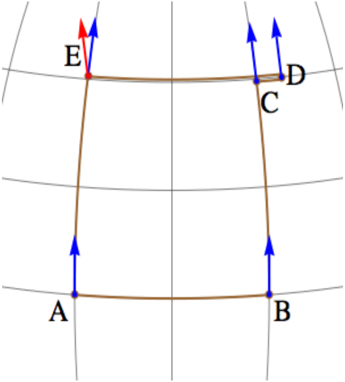
is also Lorentz-invariant:

$$dp^{\mu} = \frac{q}{mc}\,F^{\mu}{}_{\nu}\,p^{\nu}, \tag{5}$$

where $p^{\mu} = [p^0, \mathbf{p}]$ is the four-momentum and $F^{\mu}{}_{\nu}$ is the Maxwell-Faraday tensor.

First principles

Another example of an equation that relies the cause and its effect is the geodesic deviation equation:



First principles

- $$\frac{d^2 \delta x^\mu}{ds^2} = R^\mu{}_{\nu\lambda\rho} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds} \delta x^\rho, \tag{6}$$

  where δx^λ is the infinitesimal deviation between two free-falling objects, and $R^\mu{}_{\nu\lambda\rho}$ is the Riemann tensor.

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  where δx^λ is the infinitesimal deviation between two free-falling objects, and $R^\mu{}_{\nu\lambda\rho}$ is the Riemann tensor.
- **Here again, the measurable quantity is on the left, while the cause provoking the deviation phenomenon (i.e. the acceleration) is encoded in the components of the energy-momentum tensor on the right.**

First principles

- $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^3} T_{\mu\nu}. \tag{7}$$

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- **The hypothesis concerning the presence of dark matter in Galaxies results from the analysis of star motions, which combined with the deviation equation (7) lead to the conclusion that the energy-momentum tensor on the right-hand side must represent more masses than what can be estimated from the visible distribution of matter.**

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- Our questioning about the cause of measurable effects should not stop at the stage of *forces*, which are but expressions of effects of countless fundamental interactions, just like the thermodynamical pressure is in fact an averaged result of countless atomic collisions.
- On a classical level, when theory permits, the symbolical force can be replaced by a more explicit expression in which fields responsible for the forces do appear, like in the case of the Lorentz force (5).

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$$A_\mu(\mathbf{r}, t) = \frac{1}{4\pi c} \int\int\int \frac{j_\mu(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{\mid \mathbf{r} - \mathbf{r}' \mid} d^3\mathbf{r}'. \qquad (8)$$

then we get the field tensor given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

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- **with** $\psi^\dagger = \bar{\psi}^T \gamma^5, \quad \text{where}$

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- **In fact, the four-component complex function ψ is composed of two two-component spinors,** ξ_α and $\chi_{\dot\beta}$,

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- **which are supposed to transform under two non-equivalent representations of the $SL(2, \mathbf{C})$ group:**

$$\xi_{\alpha'} = S^{\alpha}_{\alpha'} \xi_\alpha, \quad \chi_{\dot\beta'} = S^{\dot\beta}_{\dot\beta'} \chi_{\dot\beta'}, \tag{10}$$

- **The electric charge conservation is equivalent to the annulation of the four-divergence of j^{μ}:**

$$\partial_\mu j^\mu = \left(\partial_\mu \psi^\dagger \gamma^\mu\right) \psi + \psi^\dagger \left(\gamma^\mu \partial_\mu \psi\right) = 0, \tag{11}$$

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- from which we infer that this condition will be satisfied if we have

$$\partial_\mu \psi^\dagger \gamma^\mu = -m\psi^\dagger \ \text{ and } \ \gamma^\mu \partial_\mu \psi = m\psi, \tag{12}$$

which is the Dirac equation.

- **In terms of the spinorial components ξ and χ the Dirac equation can be seen as a pair of two coupled equations which can be written in terms of Pauli's σ-matrices:**

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- The relativistic invariance imposed on this equation is usually presented as follows: under a Lorentz transformation Λ the 4-current j^μ undergoes the following change:

$$j^\mu \to j^{\mu'} = \Lambda^{\mu'}_{\mu} j^\mu. \tag{14}$$

- This means that the matrices γ^μ must transform as components of a 4-vector, too. Parallelly, the components of the bi-spinor ψ must be transformed in a way such as to leave the form of the equations (12) unchanged: writing symbolically the transformation of $\mid \psi >$ as $\mid \psi' >= S \mid \psi >$, and $< \psi' \mid = < \psi \mid S^{-1}$, we should have

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- $$j^{\mu'} =< \psi' \mid \gamma^{\mu'} \mid \psi' >=< \psi \mid S^{-1}\gamma^{\mu'} S \mid \psi >= \Lambda^{\mu'}_{\mu} j^\mu = \Lambda^{\mu'}_{\mu} < \psi \mid \gamma^\mu \mid \psi > \tag{15}$$

from which we infer the transformation rules for gamma-matrices:

$$S^{-1}\gamma^{\mu'} S = \Lambda^{\mu'}_{\mu} \gamma^\mu . \tag{16}$$

Quantum covariance

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- **Quantum Mechanics started as a non-relativistic theory, but very soon its relativistic generalization was created.**
- **As a result, the wave functions in the Schroedinger picture were required to belong to one of the linear representations of the Lorentz group, which means that they must satisfy the following covariance principle:**

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

Quantum covariance

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- The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial...
- As in many other fundamental relations, the seemingly simple equation

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creates a bridge between two totally different realms: the space-time accessible via classical macroscopic observations, and the Hilbert space of quantum states. It can be interpreted in two opposite ways, depending on which side we consider as the cause, and which one as the consequence.

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$$\Lambda^{\mu'}_{\mu}(S) j^{\mu} = j^{\mu'}(\psi') = j^{\mu'}(S(\psi)),$$

In the above formula

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

is the Dirac current, ψ is the electron wave function.

- **In view of the analysis of the causal chain, it seems more appropriate to write the same transformations with Λ depending on S:**

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- **This form of the same relation suggests that the transition from one quantum state to another, represented by the unitary transformation S is the primary cause that implies the transformation of observed quantities such as the electric 4-current, and as a final consequence, the apparent transformations of time and space intervals measured with classical physical devices.**

Quantum covariance

- **Although mathematically the two formulations are equivalent, it seems more plausible that the Lorentz group resulting from the averaging of the action of the $SL(2, \mathbf{C})$ in the Hilbert space of states contains less information than the original double-valued representation which is a consequence of the particle-anti-particle symmetry, than the other way round.**

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- **In what follows, we shall draw physical consequences from this approach, concerning the strong interactions in the first place.**

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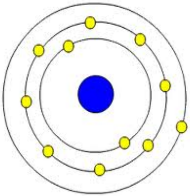
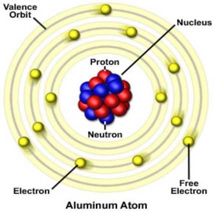
- **Not only does it explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse. as suggested by Ehrenfest, and proved later by Dyson.**
- **The link between the exclusion principle and particle's spin, known as the "spin-and-statistic theorem", is one of the deepest results in quantum field theory.**

Pauli's exclusion principle

Because fermionic operators must satisfy anti-commutation relations $\Psi^a \Psi^b = -\Psi^b \Psi^a$, two electrons (or other fermions) cannot coexist in the same state.





For the principal quantum number n there are only $2 \times n^2$ electrons in different states.

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- **In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.**
- **The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the "bra" and "ket" vectors.**

Pauli's exclusion principle

Consider the wave function of a particle in the state $\mid x >$,

$$\Phi(x) = < \psi \mid x > . \qquad (18)$$

A two-particle state of $(\mid x >, \mid y)$ is a tensor product

$$\mid \psi > = \sum \Phi(x, y) \, (\mid x > \otimes \mid y >). \qquad (19)$$

If the wave function $\Phi(x, y)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(x, y) = -\Phi(y, x), \qquad (20)$$

then $\Phi(x, x) = 0$ and such states have vanishing probability.

Pauli's exclusion principle

Conversely, suppose that $\Phi(x,x)$ does vanish. This remains valid in any basis provided the new basis $| x' >, | y' >$ was obtained from the former one via unitary transformation. Let us form an arbitrary state being a linear combination of $| x >$ and $| y >$,

$$| z >= \alpha | x > + \beta | y >, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z,z) = < \psi | (\alpha | x > + \beta | y >) \otimes (\alpha | x > + \beta | y >), \quad (21)$$

Pauli's exclusion principle

- **which develops as follows:**

$$\alpha^2 \; < \psi \mid x, x > + \alpha\beta < \psi \mid x, y >$$

$$+ \beta\alpha \; < \psi \mid y, x > + \beta^2 \; < \psi \mid y, y > =$$

$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (22)$$

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- Now, as $\Phi(x, x) = 0$ and $\Phi(y, y) = 0$, the sum of remaining two terms will vanish if and only if (20) is satisfied, i.e. if $\Phi(x, y)$ is anti-symmetric in its two arguments.

Pauli's exclusion principle

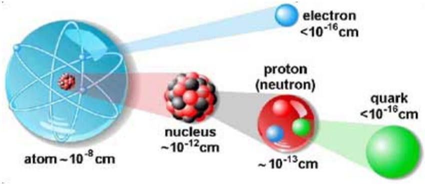
After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0 \tag{23}$$

where $\psi(x) \mid 0 >=\mid x >$.

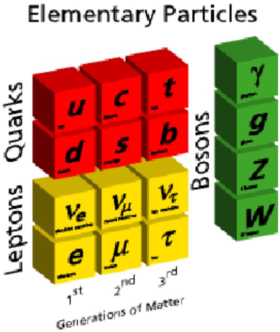
Quarks and Leptons

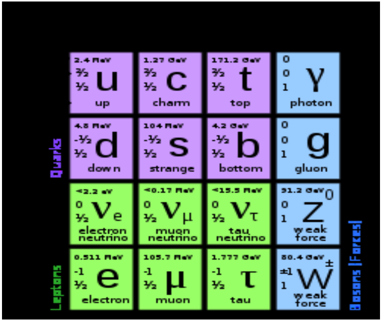
According to present knowledge, the ultimate undivisible and undestructible constituents of matter, called atoms by ancient Greeks, are in fact the QUARKS, carrying fractional electric charges and baryonic numbers, two features that appear to be undestructible and conserved under any circumstances.



Quarks and Leptons

The carriers of elementary charges also go by packs of three: three families of quarks, and three types of leptons.





Quarks and Leptons

- **In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.**

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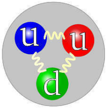
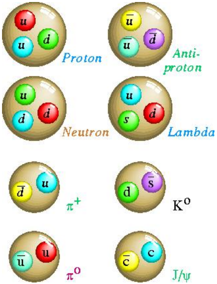
Quarks and Leptons

- **In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.**
- **Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.**
- **Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d ("up" and "down"), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.**

Quarks and Leptons





Baryons (hadrons) are composed of quarks,
which cannot be observed in a free (unbound) state.

Ternary exclusion principle

This suggests that a convenient generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can be present in a nucleon.
Let us require then the vanishing of wave functions representing the tensor product of *three* (but not necessarily two) identical states. That is, we require that $\Phi(x, x, x) = 0$ for any state $| \, x >$. **As in the former case, consider an arbitrary superposition of three different states,** $| \, x >$, $| \, y >$ and $| \, z >$,

$$| \, w >= \alpha \, | \, x > + \beta \, | \, y > + \gamma \, | \, z >$$

and apply the same criterion, $\Phi(w, w, w) = 0$.

Ternary exclusion principle

We get then, after developing the tensor products,

$$\Phi(w,w,w) = \alpha^3\Phi(x,x,x) + \beta^3\Phi(y,y,y) + \gamma^3\Phi(z,z,z)$$

$$+\alpha^2\beta[\Phi(x,x,y)+\Phi(x,y,x)+\Phi(y,x,x)]+\gamma\alpha^2[\Phi(x,x,z)+\Phi(x,z,x)+\Phi(z,x,x)]$$

$$+\alpha\beta^2[\Phi(y,y,x)+\Phi(y,x,y)+\Phi(x,y,y)]+\beta^2\gamma[\Phi(y,y,z)+\Phi(y,z,y)+\Phi(z,y,y)]$$

$$+\beta\gamma^2[\Phi(y,z,z)+\Phi(z,z,y)+\Phi(z,y,z)]+\gamma^2\alpha[\Phi(z,z,x)+\Phi(z,x,z)+\Phi(x,z,z)]$$

$$+\alpha\beta\gamma[\Phi(x,y,z)+\Phi(y,z,x)+\Phi(z,x,y)+\Phi(z,y,x)+\Phi(y,x,z)+\Phi(x,z,y)] = 0.$$

- The terms $\Phi(x,x,x)$, $\Phi(y,y,y)$ and $\Phi(z,z,z)$ do vanish by virtue of the original assumption; in what remains, combinations preceded by various powers of independent numerical coefficients α, β and γ, must vanish separately.

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- **This is achieved if the following Z_3 symmetry is imposed on our wave functions:**

$$\Phi(x,y,z) = j\,\Phi(y,z,x) = j^2\,\Phi(z,x,y)$$

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with $\quad j = e^{\frac{2\pi i}{3}}, \;\; j^3 = 1, \;\; j + j^2 + 1 = 0.$

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- Note that the complex conjugates of functions $\Phi(x,y,z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x,y,z) = j^2\,\Psi(y,z,x) = j\,\Psi(z,x,y).$$

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  After second quantization, when the fields become operator-valued, an alternative CUBIC commutation relations seems to be more appropriate:
- Instead of

$$\Psi^a \Psi^b = (-1)\, \Psi^b \Psi^a$$

we can introduce

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B,$$

with $j = e^{\frac{2\pi i}{3}}$

Algebraic properties of quark states

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Algebraic properties of quark states

- Our aim now is to derive the space-time symmetries from minimal assumptions concerning the properties of the most elementary constituents of matter, and the best candidates for these are quarks.
- To do so, we should explore algebraic structures that would privilege cubic or ternary relations, in other words, find appropriate cubic or ternary algebras reflecting the most important properties of quark states.

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- i) **The mathematical entities representing the quarks form a linear space over complex numbers, so that we could form their linear combinations with complex coefficients.**

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- *iii*) **There should exist two isomorphic algebras of this type corresponding to quarks and anti-quarks, and the conjugation that maps one of these algebras onto another, $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.**

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- *iv*) The three quark (or three anti-quark) and the quark-anti-quark combinations should be distinguished in a certain way, for example, they should form a subalgebra in the enveloping algebra spanned by the generators.

The principle of covariance

- **Any meaningful quantity described by a set of functions $\psi^A(x^\mu)$, $A, B, ... = 1, 2, ..., N, \quad \mu, \nu, ... = 0, 1, 2, 3$ defined on the Minkowskian space-time must be a representation of the Lorentz group, i.e. it should transform following one of its representations:**

$$\psi^{A'}(x^{\mu'}) = \psi^{A'}(\Lambda^{\mu'}_{\rho} x^\rho) = S^{A'}_B(\Lambda^{\mu'}_{\rho})\,\psi^B(x^\rho). \tag{24}$$

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which can be written even more concisely,

- $$\psi(x') = S(\Lambda)(\psi(x)). \tag{25}$$

The important assumption here being the representation property of the linear transformations $S(\Lambda)$:

$$S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1\Lambda_2). \tag{26}$$

The principle of covariance

- **In view of what has been said, we should write the same equation diffferently as:**

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and consequently,

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$$\Lambda(S_1)\Lambda(S_2) = \Lambda(S_1 S_2). \tag{28}$$

Covariance principle: the discrete case

- **A similar principle can be formulated in the discrete case of permutation groups, in particular for the Z_2 group. Instead of a set of functions defined on the space-time, we consider the mapping of two indices into the complex numbers, i.e. a matrix or a two-valenced complex-valued tensor.**

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- **Under the non-trivial permutation π of indices its value should change according to one of the possible representations of Z_2 in the complex plane. This leads to the following four possibilities:**

Covariance principle: the discrete case

i) The trivial representation defines the symmetric tensors:

$$S_{\pi(AB)} = S_{BA} = S_{AB},$$

ii) The sign inversion defines the anti-symmetric tensors:

$$A_{\pi(CD)} = A_{DC} = -A_{CD},$$

iii) The complex conjugation defines the hermitian tensors:

$$H_{\pi(AB)} = H_{BA} = \bar{H}_{AB},$$

iv) $(-1)\times$ complex conjugation defines the anti-hermitian tensors.

$$T_{\pi(AB)} = T_{BA} = -\bar{T}_{AB},$$

The symmetric S_3 group

- **The symmetric group S_3 containing all permutations of three different elements is a special case among all symmetry groups S_N. It is the first in the row to be non-abelian, and the last one that possesses a faithful representation in the complex plane $\mathbf{C}^1$.**

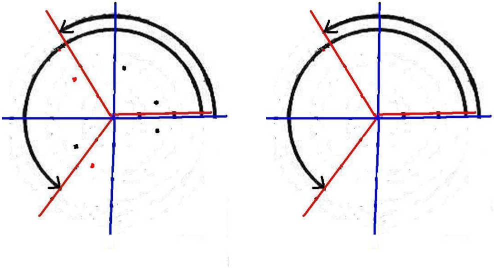
The symmetric S_3 group

- **The symmetric group S_3 containing all permutations of three different elements is a special case among all symmetry groups S_N. It is the first in the row to be non-abelian, and the last one that possesses a faithful representation in the complex plane $\mathbf{C}^1$.**
- **It contains six elements, and can be generated with only two elements, corresponding to one cyclic and one odd permutation, e.g. $(abc) \rightarrow (bca)$, and $(abc) \rightarrow (cba)$. All permutations can be represented as different operations on complex numbers as follows.**

The cyclic group Z_3

Let us denote the primitive third root of unity by $j = e^{2\pi i/3}$. The cyclic abelian subgroup Z_3 contains three elements corresponding to the three cyclic permutations, which can be represented via multiplication by j, j^2 and $j^3 = 1$ (the identity).

$$\begin{pmatrix} ABC \\ ABC \end{pmatrix} \to \mathbf{1}, \quad \begin{pmatrix} ABC \\ BCA \end{pmatrix} \to \mathbf{j}, \quad \begin{pmatrix} ABC \\ CAB \end{pmatrix} \to \mathbf{j}^2, \qquad (29)$$



The six S_3 symmetry transformations contain the identity, two rotations, one by 120^O, another one by 240^0, and three reflections, in the x-axis, in the j-axis and in the j^2-axis. The Z_3 subgroup contains only the three rotations.

Representation od S_3 in the complex plane

Odd permutations must be represented by idempotents, i.e. by operations whose square is the identity operation. We can make the following choice:

$$\begin{pmatrix} ABC \\ CBA \end{pmatrix} \rightarrow (\mathbf{z} \rightarrow \bar{\mathbf{z}}), \quad \begin{pmatrix} ABC \\ BAC \end{pmatrix} \rightarrow (\mathbf{z} \rightarrow \hat{\mathbf{z}}), \quad \begin{pmatrix} ABC \\ CBA \end{pmatrix} \rightarrow (\mathbf{z} \rightarrow \mathbf{z}^*), \tag{30}$$

Here the bar $(\mathbf{z} \rightarrow \bar{\mathbf{z}})$ denotes the complex conjugation, i.e. the reflection in the real line, the hat $\mathbf{z} \rightarrow \hat{\mathbf{z}}$ denotes the reflection in the root j^2, and the star $\mathbf{z} \rightarrow \mathbf{z}^*$ the reflection in the root j. The six operations close in a non-abelian group with six elements, and the corresponding multiplication table is shown in the following table:

The group S_3 - the multiplication table

	1	j	j^2	$-$	$\wedge$	$*$
1	1	j	j^2	$-$	$\wedge$	$*$
j	j	j^2	1	$*$	$-$	$\wedge$
j^2	j^2	1	j	$\wedge$	$*$	$-$
$-$	$-$	$\wedge$	$*$	1	j	j^2
$\wedge$	$\wedge$	$*$	$-$	j^2	1	j
$*$	$*$	$-$	$\wedge$	j	j^2	1

Table I: The multiplication table for the S_3 symmetric group

Basic definitions and properties

- Let us introduce N generators spanning a linear space over complex numbers, satisfying the following cubic relations:

$$\theta^A \theta^B \theta^C = j \, \theta^B \theta^C \theta^A = j^2 \, \theta^C \theta^A \theta^B, \tag{31}$$

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- with $j = e^{2i\pi/3}$, the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$.

Basic definitions and properties

We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, ... = 1, 2, ..., N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} = j^2\,\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}} = j\,\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}, \tag{32}$$

Let us denote this algebra by $\mathcal{A}$.

The Z_3 graded algebra $\mathcal{A}$

- **Let us denote the algebra spanned by the θ^A generators by $\mathcal{A}$. We shall endow it with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.**

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- **The grades add up modulo 3, so that the products $\theta^A\theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A\theta^B\theta^C$ being of grade 0.**

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- **The grades add up modulo 3, so that the products $\theta^A\theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A\theta^B\theta^C$ being of grade 0.**
- **Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}$ are of grade $2+2=4_{mod\,3}=1$, whereas their cubic products are again of grade 0, like the cubic products od θ^A's.**

The Z_3 graded algebra $\mathcal{A}$

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the Z_3 graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A\theta^B\theta^C\theta^D = j\,\theta^B\theta^C\theta^A\theta^D = j^2\,\theta^B\theta^A\theta^D\theta^C =$$

$$= j^3\,\theta^A\theta^D\theta^B\theta^C = j^4\,\theta^A\theta^B\theta^C\theta^D, \tag{33}$$

and because $j^4 = j \neq 1$, the only solution is

$$\theta^A\theta^B\theta^C\theta^D = 0. \tag{34}$$

The Z_3 graded algebra $\mathcal{A}$

- The total dimension of the algebra defined via the cubic relations (31) is equal to $N + N^2 + (N^3 - N)/3$: the N generators of grade 1, the N^2 independent products of two generators, and $(N^3 - N)/3$ independent cubic expressions, because the cube of any generator must be zero by virtue of (31), and the remaining $N^3 - N$ ternary products are divided by 3, also by virtue of the constitutive relations (31).

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- The conjugate generators $\bar{\theta}^{\dot{B}}$ span an algebra $\bar{\mathcal{A}}$ isomorphic with $\mathcal{A}$.

The Z_3 graded algebra $\mathcal{A}$

- **Both algebras split quite naturally into sums of linear subspaces with definite grades:**

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2, \qquad \bar{\mathcal{A}} = \bar{\mathcal{A}}_0 \oplus \bar{\mathcal{A}}_1 \oplus \bar{\mathcal{A}}_2,$$

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- **The subspaces $\mathcal{A}_0$ and $\bar{\mathcal{A}}_0$ form zero-graded subalgebras. These algebras can be made *unital* if we add to each of them the unit element 1 acting as identity and considered as being of grade 0.**

The Z_3 graded algebra $\mathcal{A}$

- **If we want the products between the generators θ^A and the conjugate ones $\bar{\theta}^{\dot{B}}$ to be included into the greater algebra spanned by both types of generators, we should consider all possible products, which will be included in the linear subspaces with a definite grade. of the resulting algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$.**

The Z_3 graded algebra $\mathcal{A}$

- **If we want the products between the generators θ^A and the conjugate ones $\bar{\theta}^{\dot{B}}$ to be included into the greater algebra spanned by both types of generators, we should consider all possible products, which will be included in the linear subspaces with a definite grade. of the resulting algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$.**
- **The grade 1 component will contain now, besides the generators of the algebra $\mathcal{A}$, also the products like**

$$\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{D}}, \quad \theta^A\theta^B\theta^C\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{G}}, \quad \text{and} \quad \theta^A\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{G}},$$

and of course all possible monomials resulting from the permutations of factors in the above expressions.

The Z_3 graded algebra $\mathcal{A}$

The grade two component will contain, along with the conjugate generators $\bar{\theta}^{\dot{B}}$ and the products of two grade 1 generators $\theta^A\theta^B$, the products of the type

$$\theta^A\theta^B\theta^C\bar{\theta}^{\dot{D}} \quad \text{and} \quad \theta^A\theta^B\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}\bar{\theta}^{\dot{G}}$$

and all similar monomials obtained via permutations of factors in the above.

The Z_3 graded algebra $\mathcal{A}$

Finally, the grade 0 component will contain now the binary products

$$\theta^A \bar{\theta}^{\dot{B}}, \qquad \bar{\theta}^{\dot{B}} \theta^A,$$

the cubic monomials

$$\theta^A \theta^B \theta^C, \;\; \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}},$$

and the products of four and six generators, with the equal number of θ^A and $\bar{\theta}^{\dot{B}}$ generators:

$$\theta^A \theta^B \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{D}}, \quad \theta^A \bar{\theta}^{\dot{C}} \theta^B \bar{\theta}^{\dot{D}}, \quad \theta^A \theta^B \theta^C \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}}, \quad \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}} \theta^A \theta^B \theta^C,$$

and other expressions of this type that can be obtained by permutations of factors.

The Z_3 graded algebra $\mathcal{A}$

- The fact that the conjugate generators are endowed with grade 2 could suggest that they behave just like the products of two ordinary generators $\theta^A \theta^B$. However, such a choice does not enable one to make a clear distinction between the conjugate generators and the products of two ordinary ones, and it would be much better, to be able to make the difference.

The Z_3 graded algebra $\mathcal{A}$

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- Due to the binary nature of the products, another choice is possible, namely, to require the following commutation relations:

$$\theta^A \bar{\theta}^{\dot{B}} = -j\, \bar{\theta}^{\dot{B}} \theta^A, \qquad \bar{\theta}^{\dot{B}} \theta^A = -j^2\, \theta^A \bar{\theta}^{\dot{B}}, \tag{35}$$

Symmetries and tensors on Z_3-graded algebras

- **As all bilinear maps of vector spaces into numbers can be divided into irreducible symmetry classes according to the representations of the Z_2 group, so can the tri-linear mappins be distinguished by their symmetry properties with respect to the permutations belonging to the S_3 symmetry group.**

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- **As all bilinear maps of vector spaces into numbers can be divided into irreducible symmetry classes according to the representations of the Z_2 group, so can the tri-linear mappins be distinguished by their symmetry properties with respect to the permutations belonging to the S_3 symmetry group.**
- **There are several different representations of the action of the S_3 permutation group on tensors with three indices. Consequently, such tensors can be divided into irreducible subspaces which are conserved under the action of S_3.**

- There are three possibilities of an action of Z_3 being represented by multiplication by a complex number: the trivial one (multiplication by 1), and the two other representations, the multiplication by $j = e^{2\pi i/3}$ or by its complex conjugate, $j^2 = \bar{j} = e^{4\pi i/3}$.

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-

$$T \in \mathcal{T}: \quad T_{ABC} = T_{BCA} = T_{CAB}, \tag{36}$$

(totally symmetric)

$$S \in \mathcal{S}: \quad S_{ABC} = j\, S_{BCA} = j^2\, S_{CAB}, \tag{37}$$

(j-skew-symmetric)

$$\bar{S} \in \bar{\mathcal{S}}; \quad \bar{S}_{ABC} = j^2\, \bar{S}_{BCA} = j\, \bar{S}_{CAB}, \tag{38}$$

(j^2-skew-symmetric).

Tri-linear forms

- The space of all tri-linear forms is the sum of three irreducible subspaces,

$$\Theta_3 = \mathcal{T} \oplus \mathcal{S} \oplus \bar{\mathcal{S}}$$

the corresponding dimensions being, respectively, $(N^3 + 2N)/3$ for $\mathcal{T}$ and $(N^3 - N)/3$ for colorred $\mathcal{S}$ and for $\bar{\mathcal{S}}$.

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- Any three-form W^{α}_{ABC} mapping $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ into a vector space $\mathcal{X}$ of dimension k, $\alpha, \beta = 1, 2, ...k$, so that $X^{\alpha} = W^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C$ can be represented as a linear combination of forms with specific symmetry properties,

$$W^{\alpha}_{ABC} = T^{\alpha}_{ABC} + S^{\alpha}_{ABC} + \bar{S}^{\alpha}_{ABC},$$

Irreducible three-linear forms

$$T^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + W^{\alpha}_{BCA} + W^{\alpha}_{CAB}), \tag{39}$$

$$S^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + j\,W^{\alpha}_{BCA} + j^2\,W^{\alpha}_{CAB}), \tag{40}$$

$$\bar{S}^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + j^2\,W^{\alpha}_{BCA} + j\,W^{\alpha}_{CAB}), \tag{41}$$

As in the Z_2 case, the three symmetries above define irreducible and mutually orthogonal 3-forms

The simplest case: two generators

Let us consider the simplest case of cubic algebra with two generators, $A, B, ... = 1, 2$. Its grade 1 component contains just these two elements, θ^1 and θ^2; its grade 2 component contains four independent products,

$$\theta^1\theta^1,\ \theta^1\theta^2,\ \theta^2\theta^1,\quad \text{and}\quad \theta^2\theta^2.$$

Finally, its grade 0 component (which is a subalgebra) contains the unit element 1 and the two linearly independent cubic products,

$$\theta^1\theta^2\theta^1 = j\,\theta^2\theta^1\theta^1 = j^2\,\theta^1\theta^1\theta^2,$$

and

$$\theta^2\theta^1\theta^2 = j\,\theta^1\theta^2\theta^2 = j^2\,\theta^2\theta^2\theta^1.$$

General definition of invariant forms

Let us consider multilinear forms defined on the algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. Because only cubic relations are imposed on products in $\mathcal{A}$ and in $\bar{\mathcal{A}}$, and the binary relations on the products of ordinary and conjugate elements, we shall fix our attention on tri-linear and bi-linear forms.

Consider a tri-linear form ρ^{α}_{ABC}. We shall call this form Z_3-invariant if we can write, by virtue of (31).:

$$\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\,\theta^B\theta^C\theta^A + \rho^{\alpha}_{CAB}\,\theta^C\theta^A\theta^B\right] =$$

$$= \frac{1}{3}\left[\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\,(j^2\,\theta^A\theta^B\theta^C) + \rho^{\alpha}_{CAB}\,j\,(\theta^A\theta^B\theta^C)\right],$$

General definition of invariant forms

From this it follows that we should have

$$\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC} + j^2\, \rho^{\alpha}_{BCA} + j\, \rho^{\alpha}_{CAB}\right] \theta^A \theta^B \theta^C, \tag{42}$$

from which we get the following properties of the ρ-cubic matrices:

$$\rho^{\alpha}_{ABC} = j^2\, \rho^{\alpha}_{BCA} = j\, \rho^{\alpha}_{CAB}. \tag{43}$$

General definition of invariant forms

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2, so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{B}\dot{A}}$ satisfying the relations similar to (43) with j replaced be its conjugate, j^2:

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j\,\bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j^2\,\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}} \tag{44}$$

Invariant forms: the two-generator case

- **In the simplest case of two generators, the j-skew-invariant forms have only two independent components:**

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$$\rho^2_{212} = j\,\rho^2_{122} = j^2\,\rho^2_{221},$$

Invariant forms: the two-generator case

- **In the simplest case of two generators, the j-skew-invariant forms have only two independent components:**
- $$\rho^1_{121} = j\,\rho^1_{211} = j^2\,\rho^1_{112},$$
  $$\rho^2_{212} = j\,\rho^2_{122} = j^2\,\rho^2_{221},$$
- and we can set

$$\rho^1_{121} = 1,\ \rho^1_{211} = j^2,\ \rho^1_{112} = j,$$
$$\rho^2_{212} = 1,\ \rho^2_{122} = j^2,\ \rho^2_{221} = j.$$

Cubic matrices

A tensor with three covariant indices can be interpreted as a "cubic matrix". One can introduce a ternary multiplication law for cubic matrices defined below:

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \tag{45}$$

in which any *cyclic* **permutation of the matrices in the product is equivalent to the same permutation on the indices:**

$$(a * b * c)_{ikl} = (b * c * a)_{kli} = (c * a * b*)_{lik} \tag{46}$$

Cubic matrices

- The ternary mutliplication law involving summation on certain indices should be in fact interpreted as contraction of covariant and contravariant indices, which requires the introduction of an analog of a metric tensor.

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$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \tag{47}$$

  the "metric" might be just the Kronecker delta:

- $$(a * b * c)_{ikl} := a_{nij} b_{pkr} c_{slm} \, \delta^{jp} \delta^{rs} \delta^{mn}. \tag{48}$$

Ternary multiplication law

Under the ternary multiplication with Kronecker delta's playing the role of the metric the matrices ρ^{α}_{ABC} do not form a closed ternary algebra.
However, they may form such an algebra if the symplectic two-form is used instead of the Kronecker delta:

$$\{\rho^{\alpha}, \rho\beta, \rho^{\gamma}\}_{ABC} = \rho^{\alpha}_{DAE}, \rho\beta_{FBG}, \rho^{\gamma}_{HCJ}\, \epsilon^{EF}\epsilon^{GH}\epsilon^{JD}, \tag{49}$$

with $\epsilon^{12} = -\epsilon^{21} = 1, \; \epsilon^{11} = \epsilon^{22} = 0.$

Cubic matrices

If we want to keep a particular symmetry under such ternary composition, we we should introduce a new composition law that follows the particular symmetry of the given type of cubic matrices. For example, let us define:

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\} := \rho^{(\alpha)} * \rho^{(\beta)} * \rho^{(\gamma)} + j\, \rho^{(\beta)} * \rho^{(\gamma)} * \rho^{(\alpha)} + j^2\, \rho^{(\gamma)} * \rho^{(\alpha)} * \rho^{(\beta)}$$

Ternary algebra of cubic matrices

Because of the symmetry of the ternary *j-bracket* one has

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{ABC} = j\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{BCA},$$

so that it becomes obvious that with respect to the *j-bracket* composition law the matrices $\rho^{(\alpha)}$ form a ternary subalgebra. Indeed, we have

$$\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\} = -\rho^{(2)}\,; \qquad \{\rho^{(2)}, \rho^{(1)}, \rho^{(2)}\} = -\rho^{(1)}\,; \tag{50}$$

all other combinations being proportional to the above ones with a factor j or j^2, whereas the j-brackets of three identical matrices obviously vanish.

The invariance group of cubic matrices

- The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.

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- Let $U_A^{A'}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U_B^{B'}\, \theta^B$.

The invariance group of cubic matrices

- The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
- Let $U^{A'}_{A}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_{B}\, \theta^B$.
- We are looking for the solution of the covariance condition for the ρ-matrices:

$$\Lambda^{\alpha'}_{\beta}\, \rho^{\beta}_{ABC} = U^{A'}_{A}\, U^{B'}_{B}\, U^{C'}_{C}\, \rho^{\alpha'}_{A'B'C'}. \tag{51}$$

The invariance group of cubic matrices

- Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

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$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

- which leads to the following equation:

$$\Lambda^{1'}_{1} = U^{1'}_{1}\, U^{2'}_{2}\, U^{1'}_{1} + j^2\, U^{2'}_{1}\, U^{1'}_{2}\, U^{1'}_{1} + j\, U^{1'}_{1}\, U^{1'}_{2}\, U^{2'}_{1} = U^{1'}_{1}\, (U^{2'}_{2}\, U^{1'}_{1} - U^{2'}_{1}\, U^{1'}_{2}),$$

because $j^2 + j = -1$.

The invariance group of cubic matrices

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$\Lambda^{2'}_1 = U^{2'}_1 \, U^{1'}_2 \, U^{2'}_1 + j^2 \, U^{1'}_1 \, U^{2'}_2 \, U^{2'}_1 + j \, U^{2'}_1 \, U^{2'}_2 \, U^{1'}_1 = U^{2'}_1 \, (U^{1'}_2 \, U^{2'}_1 - U^{1'}_1 \, U^{2'}_2),$$

The invariance group of cubic matrices

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ-matrix on the left hand side must be ρ^2 whose component ρ^2_{212} is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$\Lambda_2^{1'} = U_2^{1'}\,U_1^{2'}\,U_2^{1'} + j^2\,U_2^{2'}\,U_1^{1'}\,U_2^{1'} + j\,U_2^{1'}\,U_1^{1'}\,U_2^{2'} = U_2^{1'}\,(U_2^{1'}\,U_1^{2'} - U_1^{1'}\,U_2^{2'}),$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$\Lambda_2^{2'} = U_2^{2'}\,U_1^{1'}\,U_2^{2'} + j^2\,U_2^{1'}\,U_1^{2'}\,U_2^{2'} + j\,U_2^{2'}\,U_1^{2'}\,U_2^{1'} = U_2^{2'}\,(U_1^{1'}\,U_2^{2'} - U_1^{2'}\,U_2^{1'}).$$

The invariance group of cubic matrices

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side.

$$\Lambda^{2'}_{1} = -U^{2'}_{1}\,[det(U)], \tag{52}$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$\Lambda^{1'}_{2} = -U^{1'}_{2}\,[det(U)], \qquad \Lambda^{2'}_{2} = U^{2'}_{2}\,[det(U)]. \tag{53}$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $\Lambda^{\alpha'}_{\beta}$ one gets immediately

$$det\,(\Lambda) = [det\,(U)]^3. \tag{54}$$

The invariance group of cubic matrices

However, the U-matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations and they can take on three different values: 1, j or j^2,
i.e. the matrices $j\, U^{A'}_{B}$ or $j^2\, U^{A'}_{B}$ satisfy the same relations as the matrices $U^{A'}_{B}$ defined above.
The determinant of U can take on the values 1, j or j^2 if $det(\Lambda) = 1$
But for the time being, we have no reason yet to impose the unitarity condition. It can be derived from the conditions imposed on the invariance and duality.

Duality and covariance

In the Hilbert space of spinors the $SL(2, \mathbf{C})$ action conserved naturally two anti-symmetric tensors,

$$\varepsilon_{\alpha\beta} \quad \text{and} \quad \varepsilon_{\dot{\alpha}\dot{\beta}}.$$

and their duals,

$$\varepsilon^{\alpha\beta} \quad \text{and} \quad \varepsilon^{\dot{\alpha}\dot{\beta}}.$$

Spinorial indeces thus can be raised or lowered using these fundamental $SL(2, \mathbf{C})$ tensors:

$$\psi_\beta = \epsilon_{\alpha\beta}\, \psi^\alpha, \quad \psi^{\dot{\delta}} = \varepsilon^{\dot{\delta}\dot{\beta}}\, \psi_{\dot{\beta}}.$$

- In the space of quark states similar invariant form can be introduced, too. Theere is only one alternative: either the Kronecker delta, or the anti-symmetric 2-form ε.

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- Supposing that our cubic combinations of quark states behave like fermions, there is no choice left: if we want to define the duals of cubic forms ρ^{α}_{ABC} displaying the same symmetry properties, we must impose the covariance principle as follows:

$$\epsilon_{\alpha\beta}\, \rho^{\alpha}_{ABC} = \varepsilon_{AD}\varepsilon_{BE}\varepsilon_{CG}\, \rho^{DEG}_{\beta} \, .$$

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$$\epsilon_{\alpha\beta}\, \rho^{\alpha}_{ABC} = \varepsilon_{AD}\varepsilon_{BE}\varepsilon_{CG}\, \rho^{DEG}_{\beta}.$$

- The requirement of the invariance of tensor ε_{AB}, $A, B = 1, 2$ with respect to the change of basis of quark states leads to the condition $\det U = 1$, i.e. again to the $SL(2, \mathbf{C})$ group.

The vector representation

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic quark-anti-quark combinations into a four-dimensional linear real space. As we already saw, the symmetry (35) imposed on these expressions reduces their number to four. Let us define two quadratic forms, $\pi^{\mu}_{A\dot{B}}$ and its conjugate $\bar{\pi}^{\mu}_{\dot{B}A}$

$$\pi^{\mu}_{A\dot{B}}\,\theta^{A}\bar{\theta}^{\dot{B}} \quad \text{and} \quad \bar{\pi}^{\mu}_{\dot{B}A}\,\bar{\theta}^{\dot{B}}\theta^{A}. \tag{55}$$

The Greek indices $\mu, \nu \ldots$ take on four values, and we shall label them $0, 1, 2, 3$.

The vector representation

The four tensors $\pi^{\mu}_{A\dot{B}}$ and their hermitina conjugates $\bar{\pi}^{\mu}_{\dot{B}A}$ define a bi-linear mapping from the product of quark and anti-quark cubic algebras into a linear four-dimensional vector space, whose structure is not yet defined.

Let us impose the following invariance condition:

$$\pi^{\mu}_{A\dot{B}}\,\theta^{A}\bar{\theta}^{\dot{B}} = \bar{\pi}^{\mu}_{\dot{B}A}\bar{\theta}^{\dot{B}}\theta^{A}. \tag{56}$$

The vector representation

It follows immediately from (35) that

$$\pi^{\mu}_{A\dot{B}} = -j^2 \, \bar{\pi}^{\mu}_{\dot{B}A}. \tag{57}$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi^{\mu}_{A\dot{B}} = j^2 \, i \, \sigma^{\mu}_{A\dot{B}}, \quad \bar{\pi}^{\mu}_{\dot{B}A} = -j \, i \, \sigma^{\mu}_{\dot{B}A} \tag{58}$$

where $\sigma^{\mu}_{A\dot{B}}$ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1, 2, 3$.

The vector representation

Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and $\dot{B}$ undergo the transformation with matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$, we demand that there exist some 4×4 matrices $\Lambda^{\mu'}_{\nu}$ representing the transformation of lower indices by the matrices U and $\bar{U}$:

$$\Lambda^{\mu'}_{\nu} \, \pi^{\nu}_{A\dot{B}} = U^{A'}_{A} \, \bar{U}^{\dot{B}'}_{\dot{B}} \, \pi^{\mu'}_{A'\dot{B}'}, \tag{59}$$

this defines the vector (4×4) representation of the Lorentz group.

The vector representation

The first four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{0'}_{0} + \Lambda^{0'}_{3} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{1}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{1}}$$

$$\Lambda^{0'}_{0} - \Lambda^{0'}_{3} = U^{1'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{0'}_{0} - i\Lambda^{0'}_{2} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{0'}_{0} + i\Lambda^{0'}_{2} = U^{1'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{1}} + U^{2'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{1}}$$

The vector representation

The next four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{1'}_{0} + \Lambda^{1'}_{3} = U^{1'}_{1} \, \bar{U}^{\dot{2}'}_{\dot{1}} + U^{2'}_{1} \, \bar{U}^{\dot{1}'}_{\dot{1}}$$

$$\Lambda^{1'}_{0} - \Lambda^{1'}_{3} = U^{1'}_{2} \, \bar{U}^{\dot{2}'}_{\dot{2}} + U^{2'}_{2} \, \bar{U}^{\dot{1}'}_{\dot{2}}$$

$$\Lambda^{1'}_{0} - i\Lambda^{1'}_{2} = U^{1'}_{1} \, \bar{U}^{\dot{2}'}_{\dot{2}} + U^{2'}_{1} \, \bar{U}^{\dot{1}'}_{\dot{2}}$$

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The vector representation

The next four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{2'}_{0} + \Lambda^{2'}_{3} = -i\ U^{1'}_{1}\ \bar{U}^{\dot{2}'}_{\dot{1}} + i\ U^{2'}_{1}\ \bar{U}^{\dot{1}'}_{\dot{1}}$$

$$\Lambda^{2'}_{0} - \Lambda^{2'}_{3} = -i\ U^{1'}_{2}\ \bar{U}^{\dot{2}'}_{\dot{2}} + i\ U^{2'}_{2}\ \bar{U}^{\dot{1}'}_{\dot{2}}$$

$$\Lambda^{2'}_{0} - i\Lambda^{2'}_{2} = -i\ U^{1'}_{1}\ \bar{U}^{\dot{2}'}_{\dot{2}} + i\ U^{2'}_{1}\ \bar{U}^{\dot{1}'}_{\dot{2}}$$

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The vector representation

The last four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{3'}_{0} + \Lambda^{3'}_{3} = U^{1'}_{1}\, \bar{U}^{\dot{1}'}_{\dot{1}} - U^{2'}_{1}\, \bar{U}^{\dot{2}'}_{\dot{1}}$$

$$\Lambda^{3'}_{0} - \Lambda^{3'}_{3} = U^{1'}_{2}\, \bar{U}^{\dot{1}'}_{\dot{2}} - U^{2'}_{2}\, \bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{3'}_{0} - i\Lambda^{3'}_{2} = U^{1'}_{1}\, \bar{U}^{\dot{1}'}_{\dot{2}} - U^{2'}_{1}\, \bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{3'}_{0} + i\Lambda^{3'}_{2} = U^{1'}_{2}\, \bar{U}^{\dot{1}'}_{\dot{1}} - U^{2'}_{2}\, \bar{U}^{\dot{2}'}_{\dot{1}}$$

The metric tensor $g_{\mu\nu}$

With the invariant “spinorial metric” in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^{\nu\ A\dot{B}}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2}\left[\pi^{\mu}_{A\dot{B}}\,\pi^{\nu\ A\dot{B}}\right] = diag(+,-,-,-) \tag{60}$$

Together with the anti-commuting spinors ψ^{α} the four real coefficients defining a Lorentz vector, $x_{\mu}\,\pi^{\mu}_{A\dot{B}}$, can generate now the supersymmetry via standard definitions of super-derivations.

The invariance group of cubic matrices

Let us then choose the matrices $\Lambda^{\alpha'}_{\beta}$ to be the usual spinor representation of the $SL(2, \mathbf{C})$ group, while the matrices $U^{A'}_{B}$ will be defined as follows:

$$U^{1'}_{1} = j\Lambda^{1'}_{1}, U^{1'}_{2} = -j\Lambda^{1'}_{2}, U^{2'}_{1} = -j\Lambda^{2'}_{1}, U^{2'}_{2} = j\Lambda^{2'}_{2}, \tag{61}$$

the determinant of U being equal to j^2.

The invariance group of cubic matrices

Obviously, the same reasoning leads to the conjugate cubic representation of the same symmetry group $SL(2, \mathbf{C})$ if we require the covariance of the conjugate tensor

$$\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}} = j\,\bar{\rho}^{\dot{\beta}}_{\dot{E}\dot{F}\dot{D}} = j^2\,\bar{\rho}^{\dot{\beta}}_{\dot{F}\dot{D}\dot{E}},$$

by imposing the equation similar to (51)

$$\Lambda^{\dot{\alpha}'}_{\dot{\beta}}\;\bar{\rho}^{\dot{\beta}}_{\dot{A}\dot{B}\dot{C}} = \bar{\rho}^{\dot{\alpha}'}_{\dot{A}'\dot{B}'\dot{C}'}\,\bar{U}^{\dot{A}'}_{\dot{A}}\;\bar{U}^{\dot{B}'}_{\dot{B}}\;\bar{U}^{\dot{C}'}_{\dot{C}}. \tag{62}$$

The matrix $\bar{U}$ is the complex conjugate of the matrix U, and its determinant is equal to j.

The vector representation

Moreover, the two-component entities obtained as images of cubic combinations of quarks, $\psi^\alpha = \rho^\alpha_{ABC}\theta^A\theta^B\theta^C$ and $\bar{\psi}^{\dot\beta} = \bar{\rho}^{\dot\beta}_{\dot D\dot E\dot F}\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F}$ should anti-commute, because their arguments do so, by virtue of (35):

$$(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F}) = -(\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F})(\theta^A\theta^B\theta^C)$$

$SL(2, \mathbf{C})$ group conserves the ternary algebra

- **We have found the way to derive the covering group of the Lorentz group acting on spinors via the usual spinorial representation. The spinors are obtained as the homomorphic image of tri-linear combination of three quarks (or anti-quarks).**

$SL(2, \mathbf{C})$ group conserves the ternary algebra

- **We have found the way to derive the covering group of the Lorentz group acting on spinors via the usual spinorial representation. The spinors are obtained as the homomorphic image of tri-linear combination of three quarks (or anti-quarks).**
- **The quarks transform with matrices U (or $\bar{U}$ for the anti-quarks), but these matrices are not unitary: their determinants are equal to j^2 or j, respectively. So, quarks cannot be put on the same footing as classical spinors; they transform under a Z_3-covering of the Lorentz group.**

Fractional electric charge

- In the spirit of the Kaluza-Klein theory, the electric charge of a particle is the eigenvalue of the fifth component of the generalized momentum operator:

$$\hat{p}_5 = -i\hbar \frac{\partial}{\partial x^5},$$

where x^5 stays for the fifth coordinate.

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where x^5 stays for the fifth coordinate.

- Let the observed electric charge of the proton be e and that of the electron $-e$. If we put now the following factors multiplying the generators θ^1 and θ^2:

$$\Theta^1 = \theta^1 \, e^{-\frac{iqx^5}{3\hbar}}, \quad \Theta^2 = \theta^2 \, e^{\frac{2iqx^5}{3\hbar}},$$

- **The eigenvalues of the fifth component of the momentum operator are, respectively:**

$$\hat{p}\Theta^1 = -i\hbar\partial_5(\theta^1 \, e^{\frac{2iqx^5}{3\hbar}}) = -\frac{q}{3}\Theta^1,$$

$$\hat{p}\Theta^2 = -i\hbar\partial_5(\theta^2 \, e^{-\frac{iqx^5}{3\hbar}}) = \frac{2q}{3}\Theta^2$$

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- The only non-vanishing products of our generators being $\theta^1\theta^1\theta^2$ and $\theta^1\theta^2\theta^2$, for the admissible products of functions representing the ternary combinations we readily get:

$$\hat{p}\theta^1\theta^1\theta^2 = q\,\theta^1\theta^1\theta^2, \quad \hat{p}\theta^1\theta^2\theta^2 = 0,$$

which correspond to the usual combinations of (**udd**) and (**uud**) quarks, representing two baryons: the proton and the neutron.

Three generators of the Z_3-graded algebra

Consider now three generators, $Q^a, \; a = 1, 2, 3$, and their conjugates $\bar{Q}^{\dot{b}}$ satisfying similar cubic commutation relations as in the two-dimensional case:

$$Q^a Q^b Q^c = j\, Q^b Q^c Q^a = j^2\, Q^c Q^a Q^b,$$

$$\bar{Q}^{\dot{a}} \bar{Q}^{\dot{b}} \bar{Q}^{\dot{c}} = j^2\, \bar{Q}^{\dot{b}} \bar{Q}^{\dot{c}} \bar{Q}^{\dot{a}} = j\, \bar{Q}^{\dot{c}} \bar{Q}^{\dot{a}} \bar{Q}^{\dot{b}},$$

$$Q^a\, \bar{Q}^{\dot{b}} = -j\, \bar{Q}^{\dot{b}}\, Q^a.$$

With indices $a, b, c...$ **ranging from** 1 **to** 3 **we get** *eight* **linearly independent combinations of three undotted indices, and the same number of combinations of dotted ones.**

Three generators

They can be arranged as follows:

$$Q^3 Q^2 Q^3, \quad Q^2 Q^3 Q^2, \quad Q^1 Q^2 Q^1,$$
$$Q^3 Q^1 Q^3, \quad Q^1 Q^2 Q^1, \quad Q^2 Q^1 Q^2,$$
$$Q^1 Q^2 Q^3, \quad Q^3 Q^2 Q^1,$$

while the quadratic expressions of grade 0, $Q^a \, \bar{Q}^{\dot{b}}$ span a 9-dimensional subspace in the finite algebra generaterd by Q^a's.

Cubic matrices in three dimensions

The invariant 3-form mapping these combinations onto some eight-dimensional space must have also eight independent components (over real numbers). One can easily define these three-dimensional "cubic matrices" as follows:

$$K^{3+}_{121} = j \quad K^{3+}_{112} = j^2 \quad K^{3+}_{211} = 1; \quad K^{3-}_{212} = j \quad K^{3-}_{221} = j^2 \quad K^{3-}_{122} = 1;$$

$$K^{2+}_{313} = j \quad K^{2+}_{331} = j^2 \quad K^{2+}_{133} = 1; \quad K^{2-}_{131} = j \quad K^{2-}_{113} = j^2 \quad K^{2-}_{311} = 1;$$

$$K^{1+}_{232} = j \quad K^{1+}_{223} = j^2 \quad K^{1+}_{322} = 1; \quad K^{1-}_{323} = j \quad K^{1-}_{332} = j^2 \quad K^{1-}_{233} = 1;$$

$$K^{7}_{123} = j \quad K^{7}_{231} = j^2 \quad K^{7}_{312} = 1, \qquad K^{8}_{132} = j \quad K^{8}_{321} = j^2 \quad K^{8}_{213} = 1,$$

all other components being identically zero.

Cubic matrices in three dimensions

The structure of the set of cubic K-matrices is similar to the structure of the root diagram of the Lie algebra of the $SU(3)$ group.

We have three pairs of generators behaving like the three $SU(2)$ subgroups, K^{3+}, K^{3-}, K^{1+}, K^{1-}, K^{2+}, K^{2-}
and two extra generators behaving like the Cartan subalgebra of the $SU(3)$ Lie algebra, K^7 and K^8.

Cubic matrices in three dimensions

A similar covariance requirement can be applied to the K cubic matrices; We may ask for the following relation to be held:

$$\Lambda^{\alpha'}_{\beta}\, K^{\beta}_{abc} = U^{a'}_{a}\, U^{b'}_{b}\, U^{c'}_{c}\;\; K^{\alpha'}_{a'b'c'}$$

where the indices $\alpha, \beta ...$ run from 1 to 8, and the indices a, b, c run from 1 to 3.

We have checked that with these relations,

$$det(\Lambda) = [det(U)]^8$$

The matrices U are the fundamental representation of $SU(3)$, while the matrices Λ are the adjoint representation of $SU(3)$.

- Let us first underline the Z_2 symmetry of Maxwell and Dirac equations, which implies their hyperbolic character, which makes the propagation possible. Maxwell's equations *in vacuo* can be written as follows:

$$\begin{aligned} \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \nabla \wedge \mathbf{B}, \\ -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= \nabla \wedge \mathbf{E}. \end{aligned} \tag{63}$$

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$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \wedge \mathbf{B},$$
$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}. \qquad (63)$$

- These equations can be decoupled by applying the time derivation twice, which in vacuum, where $div\mathbf{E} = 0$ and $div\mathbf{B} = 0$ leads to the d'Alembert equation for both components separately:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0.$$

Nevertheless, neither of the components of the Maxwell tensor, be it E or B, can propagate separately alone. It is also remarkable that although each of the fields E and B satisfies a second-order propagation equation, due to the coupled system (63) there exists a quadratic combination satisfying the forst-order equation, the Poynting four-vector:

$$P^\mu = \left[P^0, \mathbf{P} \right], \quad P^0 = \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right), \quad \mathbf{P} = \mathbf{E} \wedge \mathbf{B},$$

$$\partial_\mu P^\mu = 0. \tag{64}$$

The Dirac equation for the electron displays a similar Z_2 symmetry, with two coupled equations which can be put in the following form:

$$i\hbar \frac{\partial}{\partial t}\psi_+ - mc^2 \psi_+ = i\hbar \sigma \cdot \nabla \psi_-,$$

$$-i\hbar \frac{\partial}{\partial t}\psi_- - mc^2 \psi_- = -i\hbar \sigma \cdot \nabla \psi_+, \tag{65}$$

where ψ_+ and ψ_- are the positive and negative energy components of the Dirac equation; this is visible even better in the momentum representation:

$$\left[E - mc^2\right]\psi_+ = c\sigma \cdot \mathbf{p}\psi_-,$$

$$\left[-E - mc^2\right]\psi_- = -c\sigma \cdot \mathbf{p}\psi_+. \tag{66}$$

The Dirac equation

The same effect (negative energy states) can be obtained by changing the direction of time, and putting the minus sign in front of the time derivative, as suggested by Feynman.

Each of the components satisfies the Klein-Gordon equation, obtained by successive application of the two operators and diagonalization:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - m^2\right]\psi_{\pm} = 0$$

As in the electromagnetic case, neither of the components of this complex entity can propagate by itself; only all the components can.

Generalized Dirac equation

Apparently, the two types of quarks, u **and** d**, cannot propagate freely, but can form a freely propagating particle perceived as a fermion, only under an extra condition: they must belong to three** *different* **species called** *colors***; short of this they will not form a propagating entity.**

- **Therefore, quarks should be described by *three fields* satisfying a set of coupled linear equations, with the Z_3-symmetry playing a similar role of the Z_2-symmetry in the case of Maxwell's and Dirac's equations. Instead of the "-" sign multiplying the time derivative, we should use the cubic root of unity j and its complex conjugate j^2 according to the following scheme:**

- **Therefore, quarks should be described by *three fields* satisfying a set of coupled linear equations, with the Z_3-symmetry playing a similar role of the Z_2-symmetry in the case of Maxwell's and Dirac's equations. Instead of the "-" sign multiplying the time derivative, we should use the cubic root of unity j and its complex conjugate j^2 according to the following scheme:**
-

$$\frac{\partial}{\partial t} \mid \psi >= \hat{H}_{12} \mid \phi >,$$

$$j \frac{\partial}{\partial t} \mid \phi >= \hat{H}_{23} \mid \chi >,$$

$$j^2 \frac{\partial}{\partial t} \mid \chi >= \hat{H}_{31} \mid \psi >, \tag{67}$$

- **We do not specify yet the number of components in each state vector, nor the character of the hamiltonian operators on the right-hand side; the three fields $\mid \psi >$, $\mid \phi >$ and $\mid \chi >$ should represent the three colors, none of which can propagate by itself.**

- We do not specify yet the number of components in each state vector, nor the character of the hamiltonian operators on the right-hand side; the three fields $\mid \psi >$, $\mid \phi >$ and $\mid \chi >$ should represent the three colors, none of which can propagate by itself.
- The quarks being endowed with mass, we can suppose that one of the main terms in the hamiltonians is the mass operator $\hat{m}$; and let us suppose that the remaining parts are the same in all three hamiltonians.

- **This will lead to the following three equations:**

$$\frac{\partial}{\partial t} \mid \psi > - \hat{m} \mid \psi >= \hat{H} \mid \phi >,$$
$$j \frac{\partial}{\partial t} \mid \phi > - \hat{m} \mid \phi >= \hat{H} \mid \chi >,$$
$$j^2 \frac{\partial}{\partial t} \mid \chi > - \hat{m} \mid \chi >= \hat{H} \mid \psi >, \tag{68}$$

- **This will lead to the following three equations:**

$$\frac{\partial}{\partial t} \mid \psi > - \hat{m} \mid \psi >= \hat{H} \mid \phi >,$$
$$j \frac{\partial}{\partial t} \mid \phi > - \hat{m} \mid \phi >= \hat{H} \mid \chi >,$$
$$j^2 \frac{\partial}{\partial t} \mid \chi > - \hat{m} \mid \chi >= \hat{H} \mid \psi >, \tag{68}$$

- **Supposing that the mass operator commutes with time derivation, by applying three times the left-hand side operators, each of the components satisfies the same common *third order* equation:**

$$\left[\frac{\partial^3}{\partial t^3} - \hat{m}^3 \right] \mid \psi >= \hat{H}^3 \mid \psi > . \tag{69}$$

The anti-quarks should satisfy a similar equation with the negative sign for the Hamiltonian operator. The fact that there exist two types of quarks in each nucleon suggests that the state vectors $\mid \psi >, \mid \phi >$ and $\mid \chi >$ should have two components each. When combined together, the two postulates lead to the conclusion that we must have three two-component functions and their three conjugates:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \bar{\psi}_{\dot{1}} \\ \bar{\psi}_{\dot{2}} \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \bar{\varphi}_{\dot{1}} \\ \bar{\varphi}_{\dot{2}} \end{pmatrix}, \quad \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \begin{pmatrix} \bar{\chi}_{\dot{1}} \\ \bar{\chi}_{\dot{2}} \end{pmatrix},$$

which may represent three colors, two quark states (e.g. “up” and “down”), and two anti-quark states (with anti-colors, respectively).

Finally, in order to be able to implement the action of the $SL(2, \mathbf{C})$ group via its 2×2 matrix representation defined in the previous section, we choose the Hamiltonian $\hat{H}$ equal to the operator $\sigma \cdot \nabla$, the same as in the usual Dirac equation. The action of the Z_3 symmetry is represented by factors j and j^2, while the Z_2 symmetry between particles and anti-particles is represented by the "-" sign in front of the time derivative.

The differential system that satisfies all these assumptions is as follows:

$$
\begin{aligned}
-i\hbar\frac{\partial}{\partial t}\psi &= mc^2\psi - i\hbar c\sigma\cdot\nabla\bar{\varphi},\\
i\hbar\frac{\partial}{\partial t}\bar{\varphi} &= jmc^2\bar{\varphi} - i\hbar c\sigma\cdot\nabla\chi,\\
-i\hbar\frac{\partial}{\partial t}\chi &= j^2mc^2\chi - i\hbar c\sigma\cdot\nabla\bar{\psi},\\
i\hbar\frac{\partial}{\partial t}\bar{\psi} &= mc^2\bar{\psi} = -i\hbar c\sigma\cdot\nabla\varphi,\\
-i\hbar\frac{\partial}{\partial t}\varphi &= j^2mc^2\varphi - i\hbar c\sigma\cdot\nabla\bar{\chi},\\
i\hbar\frac{\partial}{\partial t}\bar{\chi} &= jmc^2\bar{\chi} - i\hbar c\sigma\cdot\nabla\psi,
\end{aligned}
\tag{70}
$$

Here we made a simplifying assumption that the mass operator is just proportional to the identity matrix, and therefore commutes with the operator $\sigma \cdot \nabla$.
The functions ψ, φ and χ are related to their conjugates via the following third-order equations:

$$-i \frac{\partial^3}{\partial t^3} \psi = \left[\frac{m^3 c^6}{\hbar^3} - i(\sigma \cdot \nabla)^3 \right] \bar{\psi} = \left[\frac{m^3 c^6}{\hbar^3} - i\sigma \cdot \nabla \right] (\Delta \bar{\psi}),$$

$$i \frac{\partial^3}{\partial t^3} \bar{\psi} = \left[\frac{m^3 c^6}{\hbar^3} - i(\sigma \cdot \nabla)^3 \right] \psi = \left[\frac{m^3 c^6}{\hbar^3} - i\sigma \cdot \nabla \right] (\Delta \psi), \quad (71)$$

and the same, of course, for the remaining wave functions φ and χ.

The overall $Z_2 \times Z_3$ symmetry can be grasped much better if we use the matrix notation, encoding the system of linear equations (70) as an operator acting on a single vector composed of all the components. Then the system (70) can be written with the help of the following 6×6 matrices composed of blocks of 3×3 matrices as follows:

$$\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}, \qquad (72)$$

with I the 3×3 identity matrix, and the 3×3 matrices B_1, B_2 and Q defined as follows:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrices B_1 and Q generate the algebra of traceless 3×3 matrices with determinant 1, introduced by Sylvester and Cayley under the name of *nonionalgebra*. With this notation, our set of equations (70) can be written in a very compact way:

$$-i\hbar\Gamma^0 \frac{\partial}{\partial t}\Psi = \left[Bm - i\hbar Q\sigma \cdot \nabla\right]\Psi, \tag{73}$$

Here Ψ is a column vector containing the six fields,

$$[\psi, \varphi, \chi, \ \bar{\psi}, \bar{\varphi}, \bar{\chi}],$$

in this order.

But the same set of equations can be obtained if we dispose the six fields in a 6×6 **matrix,** on which the operators in (73) act in a natural way:

$$\Psi = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}, \quad \text{with} \quad X_1 = \begin{pmatrix} 0 & \psi & 0 \\ 0 & 0 & \phi \\ \chi & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & \bar{\chi} \\ \bar{\psi} & 0 & 0 \\ 0 & \bar{\varphi} & 0 \end{pmatrix} \tag{74}$$

By consecutive application of these operators we can separate the variables and find the common equation of sixth order that is satisfied by each of the components:

$$-\hbar^6 \frac{\partial^6}{\partial t^6}\psi - m^6 c^{12}\psi = -\hbar^6 \Delta^3 \psi. \tag{75}$$

- Identifying quantum operators of energy nand the momentum,

$$-i\hbar\frac{\partial}{\partial t} \rightarrow E, \quad -i\hbar\nabla \rightarrow \mathbf{p},$$

we can write (75) simply as follows:

$$E^6 - m^6 c^{12} = \mid \mathbf{p} \mid^6 c^6. \tag{76}$$

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- **This equation can be factorized showing how it was obtained by subsequent action of the operators of the system (70):**

$$E^6 - m^6c^{12} = (E^3 - m^3c^6)(E^3 + m^3c^6) =$$

$$(E-mc^2)(jE-mc^2)(j^2E-mc^2)(E+mc^2)(jE+mc^2)(j^2E+mc^2) = \mid \mathbf{p} \mid^6 c^6.$$

The equation (75) can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1\, e^{\omega\, t} + A_2\, e^{j\, \omega\, t} + A_3 e^{j^2\, \omega\, t}, \qquad B_1\, e^{\mathbf{k}.\mathbf{r}} + B_2\, e^{j\, \mathbf{k}.\mathbf{r}} + B_3\, e^{j^2\, \mathbf{k}.\mathbf{r}}$$

with ω and $\mathbf{k}$ satisfying the following dispersion relation:

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6, \tag{77}$$

where we have identified $E = \hbar\omega$ **and** $\mathbf{p} = \hbar\mathbf{k}$.

The relation

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6,$$

is invariant under the action of $Z_2 \times Z_3$ symmetry, because to any solution with given real ω and $\mathbf{k}$ one can add solutions with ω replaced by $j\omega$ or $j^2\omega$, $j\mathbf{k}$ or $j^2\mathbf{k}$, as well as $-\omega$; there is no need to introduce also $-\mathbf{k}$ instead of $\mathbf{k}$ because the vector $\mathbf{k}$ can take on all possible directions covering the unit sphere.

The nine complex solutions can be displayed in two 3×3 matrices as follows:

$$\begin{pmatrix} e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{\omega\, t - j\mathbf{k}\cdot\mathbf{r}} & e^{\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{j\omega\, t - j\mathbf{k}\cdot\mathbf{r}} & e^{j\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j^2\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix},$$

$$\begin{pmatrix} e^{-\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{-\omega\, t - j\mathbf{k}\cdot\mathbf{r}} & e^{-\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{-j\omega\, t - j\mathbf{k}\cdot\mathbf{r}} & e^{-j\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j^2\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega\, t - \mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega\, t - j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix}$$

and their nine independent products can be represented in a basis of real functions as

$$\begin{pmatrix} A_{11}\, e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} & A_{12}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\mathbf{k}\cdot\xi) & A_{13}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\mathbf{k}\cdot\xi) \\ A_{21}\, e^{-\frac{\omega\, t}{2} - \mathbf{k}\cdot\mathbf{r}} \cos\omega\tau & A_{22}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\omega\tau - \mathbf{k}\cdot\xi) & A_{23}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\omega\tau + \mathbf{k} \\ A_{31}\, e^{-\frac{\omega\, t}{2} - \mathbf{k}\cdot\mathbf{r}} \sin\omega\tau & A_{32}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\omega\tau + \mathbf{k}\cdot\xi) & A_{33}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\omega\tau - \mathbf{k} \end{pmatrix}$$

where $\tau = \frac{\sqrt{3}}{2}\, t$ and $\xi = \frac{\sqrt{3}}{2}\mathbf{kr}$; the same can be done with the conjugate solutions (with $-\omega$ instead of ω).

Cubic generalization of Dirac's equation

The functions displayed in the matrix do not represent a wave; however, one can produce a propagating solution by forming certain cubic combinations, e.g.

$$e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\omega\tau - \mathbf{k}\cdot\xi)\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\omega\tau - \mathbf{k}\cdot\xi) = \frac{1}{2}\, \sin(2\omega\tau - 2\mathbf{k}\cdot\xi).$$

Cubic generalization of Dirac's equation

- **What we need now is a multiplication scheme that would define triple products of non-propagating solutions yielding propagating ones, like in the example given above, but under the condition that the factors belong to three distinct subsets b(which can be later on identified as "colors").**

Cubic generalization of Dirac's equation

- **What we need now is a multiplication scheme that would define triple products of non-propagating solutions yielding propagating ones, like in the example given above, but under the condition that the factors belong to three distinct subsets b(which can be later on identified as "colors").**
- **This can be achieved with the 3×3 matrices of three types, containing the solutions displayed in the matrix, distributed in a particular way, each of the three matrices containing the elements of one particular line of the matrix:**

$$[A] = \begin{pmatrix} 0 & A_{12}\, e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} & 0 \\ 0 & 0 & A_{23}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos \mathbf{k}\cdot\xi \\ A_{31}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin \mathbf{k}\cdot\xi & 0 & 0 \end{pmatrix} \quad (78)$$

$$[B] = \begin{pmatrix} 0 & B_{12}\, e^{-\frac{\omega}{2}\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\tau + \mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & B_{23}\, e^{-\frac{\omega}{2}\, t - \mathbf{k}\cdot\mathbf{r}}\, \sin\tau \\ B_{31}\, e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}}\, \cos\tau & 0 & 0 \end{pmatrix} \quad (79)$$

$$[C] =$$

$$\begin{pmatrix} 0 & C_{12}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\tau+\mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & C_{23}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\tau \\ C_{31}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\tau+\mathbf{k}\cdot\xi) & 0 & 0 \end{pmatrix} \tag{80}$$

Now it is easy to check that in the product of the above three matrices, ABC all real exponentials cancel, leaving the periodic functions of the argument $\tau + \mathbf{k} \cdot \mathbf{r}$. The trace of this triple product is equal to $Tr(ABC) =$

$$[\sin\tau \; \cos(\mathbf{k}\cdot\mathbf{r}) + \cos\tau \; \sin(\mathbf{k}\cdot\mathbf{r})] \; \cos(\tau + \mathbf{k}\cdot\mathbf{r}) + \cos(\tau + \mathbf{k}\cdot\mathbf{r}) \; \sin(\tau + \mathbf{k}\cdot\mathbf{r}),$$

representing a plane wave propagating towards $-\mathbf{k}$. Similar solution can be obtained with the opposite direction. From four such solutions one can produce a propagating Dirac spinor.

This model makes free propagation of a single quark impossible, (except for a very short distances due to the damping factor), while three quarks can form a freely propagating state.