Lie systems: theory, generalizations, and applications

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- A **Lie system** is a non-autonomous system of first-order ordinary differential equations admitting its general solution to be written in terms of each finite 'generic' family of particular solutions and a set of constants by means of a function, the **superposition rule**.
- *During the last thirty years, the theory of Lie systems have frequently been analyzed*, many new results have been derived, and multiple generalizations of the Lie system notion have been proposed.
- As a result, *Lie systems and their generalizations can nowadays be used to analyze diverse problems in Classical Mechanics, Quantum Mechanics, Financial Mathematics, Control Theory, etc*. Moreover, applications of these systems have been found in the theory of integrability of differential equations, geometric phases, etc.

Linear superposition rules (I)

Each homogeneous system of first-order ordinary differential equations

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} a^i_j(t)x^j, \qquad i = 1, \ldots, n,$$

admits its general solution to be cast into the form

$$x(t) = \sum_{j=1}^{n} k_j x_{(j)}(t),$$

with $x_{(1)}(t), \ldots, x_{(n)}(t)$ being a set of *fundamental* (linearly independent) solutions and $k_1, \ldots, k_n$ a family of real constants.

Linear superposition rules (II)

Therefore, there exists a (linear) function $F : (\mathbb{R}^n)^n \times \mathbb{R}^n \to \mathbb{R}^n$, such that the general solution of the homogeneous system can be written

$$x(t) = F(x_{(1)}(t), \ldots, x_{(n)}(t); k_1, \ldots, k_n),$$

in terms of the family $x_{(1)}(t), \ldots, x_{(n)}(t)$ of linearly independent particular solutions and the set $k_1, \ldots, k_n$ of constants.

Note that:

- Not all the particular solutions are valid. Only those families made of n linearly independent ones.
- The number of constants and the dimension of the space coincide. Indeed, this is caused by the relation of constants and initial conditions.

Non-linear superposition rules

Now, every change of variables $y = \varphi(x)$ transforms the previous homogeneous system into the (generally nonlinear) one

$$\frac{dy^i}{dt} = X^i(t, y), \qquad i = 1, \ldots, n, \tag{1}$$

whose general solution can be expressed as

$$y(t) = \varphi(x(t)) = \varphi\left(\sum_{j=1}^{n} k_j \varphi^{-1}(y_{(j)}(t))\right), \tag{2}$$

with $y_{(1)}(t), \ldots, y_{(n)}(t)$ being any 'generic' family of particular solutions for equation (1) and $k_1, \ldots, k_n$ a set of constants.

Non-Linear superposition rules (II)

Therefore, there exists a non-linear function $F : (\mathbb{R}^n)^n \times \mathbb{R}^n \to \mathbb{R}^n$, such that the general solution of the previous non-linear system can be written

$$y(t) = F(y_{(1)}(t), \ldots, y_{(n)}(t); k_1, \ldots, k_n),$$

in terms of certain families $y_{(1)}(t), \ldots, y_{(n)}(t)$ of particular solutions of the non-linear system and the set $k_1, \ldots, k_n$ of constants.

Note that:

- Not all the particular solutions generate the general solution. It depends on φ.
- The number of constants and solutions coincides again.

Non-linear superposition rules (III)

Similarly, each linear systems of first-order ordinary differential equations

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} a^i_j(t)x^j + b^i(t), \qquad i = 1, \ldots, n,$$

admits its general solution to be cast into the form

$$x(t) = \sum_{j=1}^{n} k_j(x_{(j+1)}(t) - x_{(j)}(t)) + x_{(1)}(t),$$

with $x_{(1)}(t), \ldots, x_{(n+1)}(t)$ being any generic family of particular solutions and $k_1, \ldots, k_n$ a set of real constants.

Non-linear superposition rules (IV)

In this case, a nonlinear change of variables $y = \varphi(x)$ transforms the previous homogeneous system into the nonlinear one whose general solution can be expressed as

$$y(t) = \varphi\left(\sum_{j=1}^{n} k_j \left[\varphi^{-1}(y_{(j+1)}(t)) - \varphi^{-1}(y_{(j)}(t))\right] + \varphi^{-1}(y_{(1)}(t))\right),$$

with $y_{(1)}(t), \ldots, y_{(n+1)}(t)$ being a generic family of particular solutions for equation (1) and a set of constants $k_1, \ldots, k_n$. In other words, there exists a time-independent function $F : (\mathbb{R}^n)^{n+1} \to \mathbb{R}^n$ such that

$$y(t) = F(y_{(1)}(t), \ldots, y_{(n+1)}(t); k_1, \ldots, k_n). \tag{3}$$

Definition

A non-autonomous system of first-order differential equations is said to admit a **superposition rule**, if there exists a time independent function $F : \mathbb{R}^{nm} \times \mathbb{R}^n \to \mathbb{R}^n$ such that given each generic family of particular solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ and a set of constants $k_1, \ldots, k_n$, its general solution, $x(t)$, can be written as

$$x(t) = F(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n).$$

Question (Vessiot & Guldberg):

Which non-autonomous systems of first-order differential equations admit a superposition rule?

Answer (Lie 1893): Lie Theorem

The non-autonomous system

$$\frac{dx^i}{dt} = X^i(t,x), \qquad i = 1, \ldots, n,$$

admits a superposition rule if and only if the t-dependent vector field $X(t,x)$ which describes its integral curves, namely, $X(t,x) = \sum_{i=1}^n X^i(t,x)\partial/\partial x^i$, can be written as

$$X(t,x) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x), \tag{4}$$

in terms of a family of vector fields, X_α, closing on a finite-dimensional Lie algebra of vector fields V. The Lie algebra V is called a *Vessiot-Guldberg Lie algebra* of the *Lie system* (4).

Superposition Rule for Riccati equations

Riccati equations are differential equations of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2,$$

whose general solution can be cast into the form

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) - k\, x_2(t)(x_3(t) - x_1(t))}{(x_3(t) - x_2(t)) - k\,(x_3(t) - x_1(t))},$$

in terms of any family of generic (three different) particular solutions $x_1(t), x_2(t), x_3(t)$ and a constant $k \in \mathbb{R}$. In other words, the Riccati equation admits the superposition rule $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ of the form

$$x = F(x_1, x_2, x_3; k) = \frac{x_1(x_3 - x_2) - k\, x_2(x_3 - x_1)}{(x_3 - x_2) - k\,(x_3 - x_1)}.$$

Superposition Rule for Riccati equations

Riccati equations determine the integral curves for the t-dependent vector field

$$X(t,x) = (b_1(t) + b_2(t)x + b_3(t)x^2)\frac{\partial}{\partial x},$$

admitting the decomposition

$$X(t,x) = b_1(t)X_1(x) + b_2(t)X_2(x) + b_3(t)X_3(x),$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = x^2\frac{\partial}{\partial x},$$

closing the commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

In other words, the vector fields X_1, X_2, X_3 span a finite-dimensional Lie algebra of vector fields.

Some examples

- Homogeneous systems of linear differential equations (linear).
- Linear systems of linear differential equations (affine).
- Riccati equations and matrix Riccati equations (non-linear).
- Certain equations on Lie groups (non-linear).
- The Hamilton equations for second-order Riccati equations (non-linear).

Disadvantage

In general, respect to the general case of non-autonomous systems of first-order differential equations, there are a few systems (but very interesting) admitting a superposition rule.

Objectives

First aim: description of two geometric approaches to Lie systems

1. Group approach.
2. Distributional approach.

Second aim: detailing some generalizations of the Lie system notion

1. Quantum Lie systems.
2. Lie families.
3. Superposition rules and second-order differential equations.
4. Quasi-Lie schemes and quasi-Lie systems.

Third aim: application of our results

1. Quantum Lie systems $\Longrightarrow$ Spin Hamiltonian, Quantum time-dependent harmonic oscillators with dissipation, etc.
2. Lie families $\Longrightarrow$ Dissipative Milne–Pinney equations, Abel equations, etc.
3. Second-order Lie systems $\Longrightarrow$ Painleve–Ince equations, Kummer–Schwartz equations, time-dependent harmonic oscillators, Milne–Pinney equations.
4. Quasi-Lie systems $\Longrightarrow$ Non-linear oscillators, Emden equations, Abel equations, dissipative Milne–Pinney equations, second-order Riccati equations, soliton solutions, etc.

Lie systems and equations on Lie groups

Consider again the Lie system

$$X(t,x)=\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha(x),$$

with X_α spanning a Vessiot–Guldberg Lie algebra V. There always exists an effective left action $\Phi : G\times\mathbb{R}^n\to\mathbb{R}^n$, where $T_eG\simeq V$, whose fundamental vector fields are those in V. Take a basis $\{\mathrm{a}_\alpha\,|\,\alpha=1,\ldots,r\}$ of T_eG such that

$$X_\alpha(x)=\frac{d}{dt}\bigg|_{t=0}\Phi(\exp(-t\mathrm{a}_\alpha),x).$$

Note that the linear morphism $\rho : T_eG\to\mathfrak{X}(\mathbb{R}^n)$, satisfying $\rho(\mathrm{a}_\alpha)=X_\alpha$, is a Lie algebra morphism.

Consider the restriction map

$$\begin{array}{rccc}\Phi_x: & G & \longrightarrow & \mathbb{R}^n,\\ & g & \mapsto & \Phi_x(g)=\Phi(g,x).\end{array}$$

In terms of this map, we have that $X_\alpha(x)=-\mathrm{T}_e\Phi_x(\mathrm{a}_\alpha)$. Hence,

$$X(t,x)=\sum_{\alpha=1}^r b_\alpha(t)X_\alpha(x)\Longrightarrow X(t,x)=\mathrm{T}_e\Phi_x\left(-\sum_{\alpha=1}^r b_\alpha(t)\mathrm{a}_\alpha\right).$$

Consequently,

$$\text{if }\mathrm{a}(t)\equiv-\sum_{\alpha=1}^r b_\alpha(t)\mathrm{a}_\alpha,\text{ then }X(t,x)=\mathrm{T}_e\Phi_x(\mathrm{a}(t)).$$

Let $X_\alpha^R(g) = \mathrm{T}_e R_g \mathrm{a}_\alpha$ be a basis of the set of right-invariant vector fields on G, consider the solution, $g(t)$, of the Lie system

$$\frac{dg}{dt} = \mathrm{T}_e R_g(\mathrm{a}(t)) = -\sum_{\alpha=1}^{n} b_\alpha(t) X_\alpha^R(g), \qquad g(0) = e.$$

Now, given $x(t) = \Phi(g(t), x_0)$, we get that

$$\frac{dx}{dt}(t) = \mathrm{T}_{g(t)}\Phi_{x_0} \circ \mathrm{T}_e R_{g(t)}(\mathrm{a}(t)) = \mathrm{T}_e \Phi_{x(t)}(\mathrm{a}(t)) = X(t, x(t)).$$

The solution of every Lie system can be reduced to solve a Lie system on a Lie group!!!

Note also that given a Lie system on G of the form

$$\frac{dg}{dt}(t) = -\sum_{\alpha=1}^{n} b_\alpha(t) X_\alpha^R(g), \qquad g(0) = e,$$

then, if we take $\mathrm{a}_\alpha \equiv X_\alpha^R(e)$, we can define

$$\mathrm{a}(t) \equiv -\sum_{\alpha=1}^{n} b_\alpha(t)\mathrm{a}_\alpha \Longrightarrow \frac{dg}{dt} = \mathrm{T}_e R_g(\mathrm{a}(t)).$$

Now, for any effective action $\Phi' : G \times \mathbb{R}^n \to \mathbb{R}^n$, the curves $x'(t) = \Phi'(g(t), x_0)$ are the solutions of the equation

$$\frac{dx'}{dt}(t) = \mathrm{T}_{g(t)}\Phi'_{x_0} \circ \mathrm{T}_e R_{g(t)}(\mathrm{a}(t)) = \mathrm{T}_e \Phi'_{x'(t)}(\mathrm{a}(t)) \equiv X(t, x'(t)).$$

Scheme I

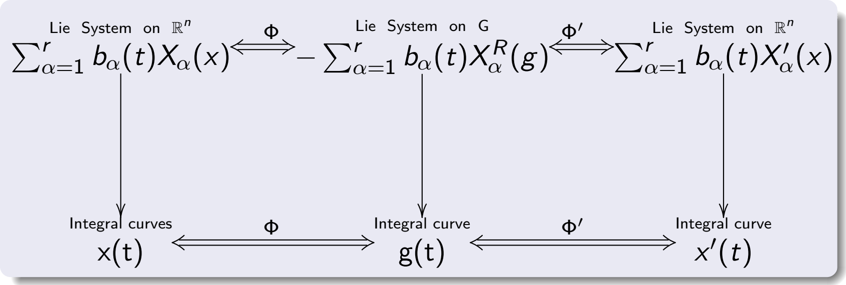
Lie System on $\mathbb{R}^n$ $\quad\overset{\Phi}{\Longleftrightarrow}\quad$ Lie System on G

$$X = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \qquad\qquad X^G = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g)$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

Integral curves x(t) $\quad\overset{\Phi}{\Longleftrightarrow}\quad$ Integral curve g(t)

Scheme II


Lie System on $\mathbb{R}^n$
$\sum_{\alpha=1}^r b_\alpha(t)X_\alpha(x)$
Φ
Lie System on G
$-\sum_{\alpha=1}^r b_\alpha(t)X_\alpha^R(g)$
Φ′
Lie System on $\mathbb{R}^n$
$\sum_{\alpha=1}^r b_\alpha(t)X'_\alpha(x)$
Integral curves
x(t)
Φ
Integral curve
g(t)
Φ′
Integral curve
$x'(t)$

Distributional approach

Consider again the Lie system $X(t,x)=\sum_{\alpha=1}^r b_\alpha(t)X_\alpha(x)$ admitting a superposition rule $F:\mathbb{R}^{nm}\times\mathbb{R}^n\to\mathbb{R}^n$. Its general solution, $x(t)$, can be written as

$$x(t)=F(x_{(1)}(t),\ldots,x_{(m)}(t);k_1,\ldots,k_n),$$

in terms of any generic family of particular solutions and a set of constants to be related to initial conditions. Indeed, note that at $t=0$, the above relation implies that, fixed $x_{(1)}(0),\ldots,x_{(m)}(0)$, for every $x(0)$, there must be a unique set of constants $k_1,\ldots,k_n$ such that

$$x(0)=F(x_{(1)}(0),\ldots,x_{(m)}(0);k_1,\ldots,k_n).$$

The Implicit Function Theorem yields that there exists a map $\Psi:\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that

$$k=\Psi(x_{(0)},x_{(1)},\ldots,x_{(m)}),$$

with $k=(k_1,\ldots,k_n)$ the only point in $\mathbb{R}^n$ such that

$$x_{(0)}=F(x_{(1)},\ldots,x_{(m)};k).$$

Consequently, given a $(m+1)$-tuple of particular solutions, we have that

$$x_{(0)}(t)=F(x_{(1)}(t),\ldots,x_{(m)}(t);k),$$

implies

$$k^i = \Psi^i(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)), \qquad i = 1, \ldots, n.$$

Superposition rules are related with the existence of constant of the motion.

Differentiating the above relation in terms of the time, we obtain

$$\frac{d}{dt}\Psi^i(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) = 0, \qquad i = 1, \ldots, n,$$

and

$$\sum_{j=1}^{n}\sum_{a=0}^{m}\frac{dx^j_{(a)}}{dt}(t)\frac{\partial\Psi^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n.$$

Consequently,

$$\sum_{j=1}^{n}\sum_{a=0}^{m} X^j(t, x_{(a)}(t))\frac{\partial \Psi^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n, \tag{5}$$

for every set of particular solutions $x_{(0)}(t), \ldots, x_{(m)}(t)$. Therefore,

$$\sum_{j=1}^{n}\sum_{a=0}^{m} X^j(t, x_{(a)})\frac{\partial \Psi^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n. \tag{6}$$

Diagonal prolongations

Given the time-dependent vector field $X = \sum_{j=1}^n X^j(t, x)\partial/\partial x^j$, we call its **diagonal prolongation**, hereby $\widehat{X}$, to $\mathbb{R}^{n(m+1)}$ the time-dependent vector field

$$\widehat{X} = \sum_{j=1}^{n}\sum_{a=0}^{m} X^j(t, x_{(a)})\frac{\partial}{\partial x^j_{(a)}}.$$

Taking into account the above definition and (6), we have that

$$\widehat{X}\Psi^i = 0, \qquad i = 1, \ldots, n.$$

The vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ span a distribution

$$D_p = \{\widehat{X}(t,p)\,|\,t\in\mathbb{R}\} \subset \mathrm{T}_p\mathbb{R}^{n(m+1)},$$

on $\mathbb{R}^{n(m+1)}$. Its vector fields are tangent to the level sets of the function Ψ. Consequently, the Lie brackets among vector fields of the above distribution are also tangent to the level sets of Ψ. Consider the distribution $\widehat{D}$ defined on $\mathbb{R}^{n(m+1)}$ spanned by the vector fields of D and their succesive Lie brackets, i.e.,

$$\widehat{D}_p = \langle (X_t)_p, [X_t, X_{t'}]_p, [X_{t''}, [X_t, X_{t'}]]_p, \ldots, \rangle, \quad t, t', t'', \ldots \in \mathbb{R}.$$

Obviously, the distribution $\widehat{D}$ is involutive and finite-dimensional. Hence, around a generic point where it becomes regular, it gives rise to a p-codimensional foliation $\mathcal{F}_0$, with $p \geq n$.

This gives rise to the existence of a n-codimensional foliation $\mathcal{F}$ whose leaves contain several leaves of $\mathcal{F}_0$. Note that if $p > n$, there are several ways to obtain a n-codimensional foliations $\mathcal{F}$ out of $\mathcal{F}_0$. This leads to the existence of different superposition rules for the same Lie system.

Given a point $(x_{(1)}, \ldots, x_{(m)})$ and a leaf $\mathcal{F}_k$ of $\mathcal{F}$, there exists just a $x_{(0)}$ such that $(x_{(0)}, x_{(1)}, \ldots, x_{(m)}) \in \mathcal{F}_k$. Indeed,

$$\Psi(x_{(0)}, x_{(1)}, \ldots, x_{(m)}) = k \Longrightarrow x_{(0)} = F(x_{(1)}, \ldots, x_{(m)}; k).$$

Hence, the projection $\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{nm}$ of the form

$$\mathrm{pr}(x_{(0)}, x_{(1)}, \ldots, x_{(m)}) = (x_{(1)}, \ldots, x_{(m)}),$$

induces diffeomorphisms among the leaves $\mathcal{F}_k$ and $\mathbb{R}^{nm}$ (and, therefore, among different leaves of $\mathcal{F}$ also).

Each superposition rule in terms of m solutions defines a local n-codimensional horizontal foliation $\mathcal{F}$ on $\mathbb{R}^{n(m+1)}$ with respect to pr and such that the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ are tangent to its leaves.

Conversely, a foliation $\mathcal{F}$ provides a superposition rule F. Actually, if we take a set of particular solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$ and a leaf $\mathcal{F}_k$ of the foliation $\mathcal{F}$, there exists just a unique curve $x_{(0)}(t)$ such that $(x_{(0)}(t),x_{(1)}(t),\ldots,x_{(m)}(t))\in\mathcal{F}_k$. Therefore, we can define

$$x_{(0)}(t)=F(x_{(1)}(t),\ldots,x_{(m)}(t);k),$$

where $x_{(0)}(t)$ is a new solution.

Each local n-codimensional foliation $\mathcal{F}$ on $\mathbb{R}^{n(m+1)}$ horizontal with respect to pr and with $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ tangent to the leaves, defines a superposition rule in terms of m particular solutions.

Proposition

Giving a superposition rule for a non-autonomous system of differential equations is equivalent to giving a n-codimensinal foliation on $\mathbb{R}^{n(m+1)}$ horizontal with respect to the projection $\mathrm{pr}:\mathbb{R}^{(m+1)n}\to\mathbb{R}^{nm}$, for which the diagonal prolongations $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ of the vector fields $\{X_t\}_{t\in\mathbb{R}}$ defining the system are tangent to its leaves.

Geometric Lie Theorem

Proposition

A system on $\mathbb{R}^n$ admits a superposition rule if and only if its associated t-dependent vector field $X(t,x)$ can be locally written in the form

$$X(t,x) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x)$$

where the vector fields X_α, $\alpha = 1,\ldots,r$, span a finite-dimensional Lie algebra.

Suppose that the non-autonomous system admits a superposition rule depending on m particular solutions and let $\mathcal{F}$ be its related foliation on $\mathbb{R}^{n(m+1)}$. We know that the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ are tangent to the leaves of $\mathcal{F}$. Consequently, we can choose, among these vector fields, a finite family $\widehat{X}_1,\ldots,\widehat{X}_s$ spanning the distribution $D_p=\{(\widehat{X}_t)_p\,|\,t\in\mathbb{R}\}$. By means of their successive Lie brackets, we can get a family of vector fields of the form $\{\widehat{X}_\alpha\mid\alpha=1,\ldots,r\}$ spanning an involutive distribution $\widehat{D}$ containing D. Therefore

$$[\widehat{X}_\alpha,\widehat{X}_\beta]=\sum_{\gamma=1}^{r}c_{\alpha\beta}{}^{\gamma}\widehat{X}_\gamma, \tag{7}$$

where the coefficients $c_{\alpha\beta}{}^{\gamma}$ are constant, so also

$$[X_\alpha,X_\beta]=\sum_{\gamma=1}^{r}c_{\alpha\beta}{}^{\gamma}\,X_\gamma\,.$$

Since every $\widehat{X}(t)$ is in D, there are functions $b_t^\alpha(x_{(0)},\ldots,x_{(m)})$ such that

$$\widehat{X}(t)=\sum_{\alpha=1}^{r} b_t^\alpha\,\widehat{X}_\alpha\,.$$

But $\widehat{X}(t)$ is a diagonal prolongation, so, using the fundamental lemma once more, we get that the $b_t^\alpha=b^\alpha(t)$ are independent on $x_{(0)},\ldots,x_{(m)}$. Hence

$$\widehat{X}(t)=\sum_{\alpha=1}^{r} b^\alpha(t)\,\widehat{X}_\alpha \Longrightarrow X(t)=\sum_{\alpha=1}^{r} b^\alpha(t)\,X_\alpha\,. \tag{8}$$

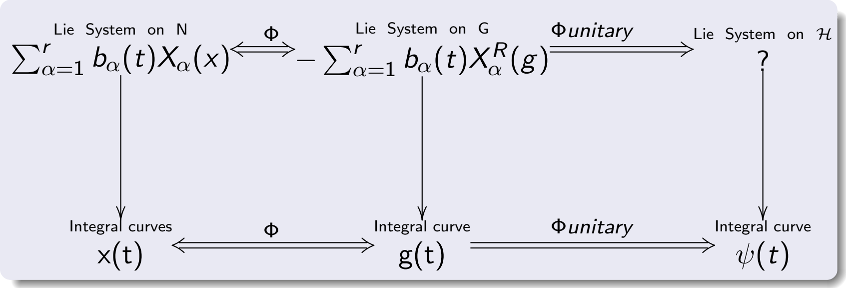
(Converse property) Assume that $X(t,x)$ is a Lie system, i.e. $X(t,x)=\sum_{\alpha=1}^r b^\alpha(t)X_\alpha \Longrightarrow \widehat{X}(t)=\sum_{\alpha=1}^r b^\alpha(t)\,\widehat{X}_\alpha$.

Thus they define an r-dimensional Lie algebra with structure constants $c_{\alpha\beta\gamma}$. There is a number $m \leq rn$ such that their diagonal prolongations to $\mathbb{R}^{nm}$ are generically linearly independent at each point. The distribution spanned by the diagonal prolongations $\widehat{X}_1, \ldots, \widehat{X}_r$ to $\mathbb{R}^{n(m+1)}$ is clearly involutive, so it defines an r-dimensional foliation $\mathcal{F}_0$ on $\mathbb{R}^{nm}$.

Moreover, the leaves of this foliation project onto the product of the last m factors diffeomorphically and they are at least n-codimensional. Now, we can extend this foliation to an n-codimensional foliation $\mathcal{F}$ with the latter property, and this foliation defines a superposition rule.

Scheme of generalization

Our aim is to "copy" the structure of the previous diagram in order to construct Lie systems on a Hilbert space $\mathcal{H}$, i.e.



Quantum Lie systems
Geometrization

- Hilbert space $\mathcal{H} \Longrightarrow$ real manifold $\mathcal{H}$.
- The $\mathbb{C}$-operator $A : \mathcal{H} \to \mathcal{H} \Longrightarrow$ vector field $X^A : \psi \in \mathcal{H} \mapsto (\psi, A\psi) \in \mathrm{T}\mathcal{H}$.
- The t-dependent operator $A(t) : \mathbb{R} \times \mathcal{H} \to \mathcal{H} \Longrightarrow$ $X^{A(t)} : (t, \psi) \in \mathbb{R} \times \mathcal{H} \mapsto (\psi, A(t)\psi) \in \mathrm{T}\mathcal{H}$.

The integral curves, $t \mapsto \psi_t$, of a vector field X^A hold the equation

$$\frac{d\psi_t}{dt} = A\psi_t \equiv X^A_{\psi_t} \quad \text{where } X^A(\psi) = (\psi, X^A_\psi).$$

Vector fields and Quantum Mechanics

Note that each vector field X^A related to a skew-self adjoint operator A admits the flow

$$Fl : (t, \psi) \in \mathbb{R} \times \mathcal{H} \mapsto Fl_t(\psi) \equiv \exp(tA)(\psi) \in \mathcal{H}.$$

Proposition

Given two skew-self-adjoint operators A and B, we have that the Lie bracket of their associated vector fields satisfy

$$[X^A, X^B] = -X^{[A,B]}.$$

Vector fields and Quantum Mechanics

$$\left[X^A, X^B\right]_\psi = \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \exp\left(-tB\right)\exp\left(-tA\right)\exp\left(tB\right)\exp\left(tA\right)\left(\psi\right)$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \left(\sum_{n_1=0}^{\infty}\frac{(-tB)^{n_1}}{n_1!}\right)\left(\sum_{n_2=0}^{\infty}\frac{(-tA)^{n_2}}{n_2!}\right)$$

$$\left(\sum_{n_3=0}^{\infty}\frac{(tB)^{n_3}}{n_3!}\right)\left(\sum_{n_4=0}^{\infty}\frac{(tA)^{n_4}}{n_4!}\right)(\psi)$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \left(t^2[B,A]\right)(\psi) = [B,A](\psi) = -X^{[A,B]}_\psi.$$

We can get the value of Lie brackets of the form $[X^A, X^B]$ in terms of commutators between operators.

Geometric Schrödinger equation

Given a t-dependent quantum Hamiltonian, i.e. self-adjoint family of operators $H(t) : \mathcal{H} \to \mathcal{H}$, the associated Schrödinger equation reads

$$i\frac{\partial\psi}{\partial t} = H(t)\psi \Longrightarrow \frac{\partial\psi}{\partial t} = -iH(t)\psi.$$

Obviously, the above equation describes the integral curves of the t-dependent vector field $X^{A(t)}$ with $A(t) = -iH(t)$.

Which Schrödinger equations can be reduced related to solutions on a Lie system on a Lie group?

Definition

We say that a time-dependent Hermitian Hamiltonian $H(t)$ is a **Quantum Lie system** if we can cast it into the form

$$H(t) = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha,$$

where the H_α are self-adjoint operators such that the iH_α span a Lie algebra V of skew-self-adjoint operators, i.e. they satisfy the relations

$$[iH_\alpha, iH_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} iH_\gamma, \qquad \alpha, \beta = 1, \ldots, r,$$

for a certain set of real constants $c_{\alpha\beta\gamma}$.

If $H(t)$ is a Quantum Lie system, its corresponding Schrödinger equation describes the integral curves of the time-dependent vector field $X^{A(t)}$, with $A(t) = -iH(t)$, i.e.

$$\frac{\partial \psi}{\partial t} = -iH(t)\psi = X_\psi^{A(t)}.$$

But, hence,

$$X_\psi^{A(t)} = \sum_{\alpha=1}^{r} b_\alpha(t)(-iH_\alpha\psi) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\psi^\alpha,$$

where X^α is the vector field associated with the skew-self-adjoint operator $B_\alpha = -iH_\alpha$. Moreover,

$$[X^\alpha, X^\beta]_\psi = -X_\psi^{[B_\alpha, B_\beta]} = -[iH_\alpha, iH_\beta]\psi = -\sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} iH_\gamma \psi.$$

And, consequently,

$$[X^\alpha, X^\beta]_\psi = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X^\gamma_\psi \Longrightarrow [X^\alpha, X^\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X^\gamma,$$

for $\alpha, \beta = 1, \ldots, r$.

The Lie algebra of skew-self-adjoint operators V induces an isomorphic Lie algebra of vector fields on $\mathcal{H}$.

We have copied the usual structure of Lie systems on a Hilbert space in order to describe certain Quantum systems.

The Schrödinger equation corresponding to a quantum Lie system describes the integral curves of a time-dependent vector field $X^{A(t)}$ associated with a skew-self-adjoint time-dependent operator $A(t)$ such that

$$X^{A(t)} = \sum_{\alpha=1}^{r} b_\alpha(t) X^\alpha,$$

for a given family of vector fields X^α associated with skew-self-adjoint operators $-iH_\alpha$ closing on a finite-dimensional Lie algebra of vector fields.

Lemma

Let $H(t)$ be a quantum Lie system associated with a Lie algebra of skew-self-adjoint operators V and G a connected Lie group with Lie algebra $T_eG \simeq V$, there exists an effective unitary action $\Phi : G \times \mathcal{H} \to \mathcal{H}$ whose fundamental vector fields are those associated with the operators in V.

Indeed, consider a basis of $\{iH_\alpha \,|\, \alpha = 1, \dots, r\}$ of V and a basis $\{\mathrm{a}_\alpha \,|\, \alpha = 1, \dots, r\}$ of T_eG closing on the same commutation relations. In a neighborhood U such that $e \in U \subset G$, we can write every element in a unique way, i.e. $g = \exp(c_1\mathrm{a}_1) \times \dots \times \exp(c_r\mathrm{a}_r)$. In this way, the conditions

$$\Phi(\exp(c_\alpha \mathrm{a}_\alpha), \psi) = \exp(ic_\alpha H_\alpha)\psi, \qquad \alpha = 1, \dots, r.$$

induce the existence of the required action.

Proposition

Given a quantum Lie system $H(t)$, its associated action $\Phi : G \times \mathcal{H} \to \mathcal{H}$ allows us to relate the Schrödinger equation associated with $H(t)$ to an equation of the form

$$\dot{g} = -\sum_{\alpha=1}^{n} b_\alpha(t) X_\alpha^R(g), \qquad g(0) = e,$$

on G. Moreover, the general solution for the Schrödinger equation is

$$\psi_t = \Phi(g(t), \psi_0).$$

Spin Hamiltonians

Consider the t-dependent Hamiltonian

$$H(t) = B_x(t)S_x + B_y(t)S_y + B_z(t)S_z,$$

with S_x, S_y and S_z being the spin operators. Note that the t-dependent Hamiltonian $H(t)$ is a quantum Lie system associated with a Lie algebra V isomorphic to $\mathfrak{su}(2,\mathbb{R})$. Moreover, its Schrödinger equation is

$$\frac{d\psi}{dt} = -iB_x(t)S_x(\psi) - iB_y(t)S_y(\psi) - iB_z(t)S_z(\psi),$$

that can be seen as the differential equation determining the integral curves for the t-dependent vector field

$$X(t) = B_x(t)X_1 + B_y(t)X_2 + B_z(t)X_3,$$

where $(X_1)_\psi = -iS_x(\psi),\ (X_2)_\psi = -iS_y(\psi),\ (X_3)_\psi = -iS_z(\psi)$.
Therefore our Schrödinger equation is a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{su}(2)$.
The solution of the above Schrödinger equation can be reduced to solving the equation on $SU(2)$ of the form

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha^R(g), \qquad g(0) = e.$$

Recently, the study of integrability conditions for this equation allowed us to explain the integrability of the Hamiltonian

$$H(t) = B(\sin(\theta)\cos(\omega)tS_x + \sin\theta\sin(\omega t)S_y + \cos\theta S_z).$$

Another example of Quantum Lie systems is given by the family of time dependent Hamiltonians

$$H(t) = a(t)P^2 + b(t)Q^2 + c(t)(QP + PQ) + d(t)Q + e(t)P + f(t)I,$$

or the family of time dependent Hamiltonians

$$H(t) = a(t)P^2 + b(t)\left(Q^2 + \frac{k}{Q^2}\right).$$

Standard superposition rules do not admit any time-dependent dependence. Obviously, systems admitting a time-dependent superposition rule contain those systems admitting a time-independent one.

Question:

What kind of non-autonomous systems admit a time-dependent superposition rule? Solution: ALL!!

Indeed, the flow of each time-dependent vector field X is a map $\bar{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that the general solution, $x(t)$, can be cast into the form

$$x(t) = \bar{F}(t; k_1, \ldots, k_n).$$

Final result:

The time-dependent superposition rule associated with the flow of a unique time-dependent vector field is, generally, useless.

Recently, it was found that various families of systems could admit a common time-dependent superposition rule depending on one or more particular solutions.

Definition

We say that a family of non-autonomous systems $\{X_c(t,x)\}_{c\in\Lambda}$ admits a **common time-dependent superposition rule**, if there exists a function $\bar{F}:\mathbb{R}\times\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that the general solution, $x(t)$, for any particular instance X_c of the family can be written as

$$x(t)=\bar{F}(t,x_{(1)}(t),\ldots,x_{(m)}(t);k_1,\ldots,k_n),$$

in terms of each generic family $x_{(1)}(t),\ldots,x_{(m)}(t)$ of particular solutions of X_c and a set of constants $k_1,\ldots,k_n$. Families admitting such a superposition are called **Lie families**.

Question:

What kind of families of first-order differential equations admit a common time-dependent superposition rule?

Consider the family of tiem-dependent vector fields $\{X_c\}_{c\in\Lambda}$ admitting a common time-dependent superposition rule $\bar{F}:\mathbb{R}\times\mathbb{R}^{nm}\times\mathbb{R}^n\to\mathbb{R}^n$. The general solution, $x(t)$, of any instance X_c of the above family reads

$$x(t)=\bar{F}(t,x_{(1)}(t),\ldots,x_{(m)}(t);k_1,\ldots,k_n),$$

in terms of any generic family of particular solutions of X_c and a set of constants. The Implicit Function Theorem implies that there exists a map $\bar{\Psi}:\mathbb{R}\times\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that

$$k=\bar{\Psi}(t,x_{(0)},x_{(1)},\ldots,x_{(m)}),$$

with k the only point of $\mathbb{R}^n$ so that $x_{(0)}=\bar{F}(t,x_{(1)},\ldots,x_{(m)};k)$.

Consequently, given a $(m+1)$-tuple of particular solutions, we get the time-dependent constants of the motion

$$k = \bar{\Psi}(t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)).$$

Differentiating in terms of the time, we obtain

$$\frac{d}{dt}\bar{\Psi}^i(t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) = 0, \qquad i = 1, \ldots, n,$$

and

$$\frac{\partial \bar{\Psi}^i}{\partial t} + \sum_{j=1}^{n}\sum_{a=0}^{m} \frac{dx^j_{(a)}}{dt}\frac{\partial \bar{\Psi}^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n.$$

Consequently,

$$\frac{\partial \bar{\Psi}^i}{\partial t} + \sum_{j=1}^{n} \sum_{a=0}^{m} X^j(t, x_{(a)}(t)) \frac{\partial \bar{\Psi}^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n.$$

As the above equality holds for every $(m+1)$-tuple of particular solutions, we get

$$\frac{\partial \bar{\Psi}^i}{\partial t} + \sum_{j=1}^{n} \sum_{a=0}^{m} X^j(t, x_{(a)}) \frac{\partial \bar{\Psi}^i}{\partial x^j_{(a)}} = 0, \qquad i = 1, \ldots, n. \tag{9}$$

Common time-dependent superposition rules are related to time-dependent constants of the motion.

Definitions

Given the time-dependent vector field $X=\sum_{j=1}^n X^j(t,x)\partial/\partial x^j$, we call its **pure-prolongation, hereby X^P, to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$** the vector field on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ of the form

$$X^P=\sum_{j=1}^n\sum_{a=0}^m X^j(t,x_{(a)})\frac{\partial}{\partial x^j_{(a)}},$$

and its **time-prolongation, hereby $\widetilde{X}$, to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$** the vector field on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ of the form

$$\widetilde{X}=\frac{\partial}{\partial t}+\sum_{j=1}^n\sum_{a=0}^m X^j(t,x_{(a)})\frac{\partial}{\partial x^j_{(a)}}.$$

Taking into account the above definitions and (9), we get $\widetilde{X}_c\bar{\Psi}^i = 0$, with $i = 1, \ldots, n$. The vector fields $\{\widetilde{X}_c\}_{c\in\Lambda}$ span a distribution of vector fields on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ tangent to the level sets of the function $\bar{\Psi}$.

Consider the set of vector fields

$$\widetilde{X}_c, [\widetilde{X}_{c'}, \widetilde{X}_c], [\widetilde{X}_{c''}, [\widetilde{X}_{c'}, \widetilde{X}_c]] \ldots, \qquad c, c', c'', \ldots \in \Lambda.$$

Obviously, they span an involutive distribution. As they are all tangent to the level sets of $\bar{\Psi}$, this distribution gives rise to a foliation $\bar{\mathcal{F}}_0$ with codimension greater or equal to n. In this way, we can always get obtain a n-codimensional foliation $\bar{\mathcal{F}}$.

Given a point $(t, x_{(1)}, \ldots, x_{(m)})$ and a leaf $\bar{\mathcal{F}}_k$ there exists just a $x_{(0)}$ such that $(t, x_{(0)}, \ldots, x_{(m)}) \in \bar{\mathcal{F}}_k$. Hence, the projection

$$\overline{\mathrm{pr}}(t, x_{(0)}, x_{(1)}, \ldots, x_{(m)}) = (t, x_{(1)}, \ldots, x_{(m)}),$$

induces diffeomorphisms among the leaves $\bar{\mathcal{F}}_k$ and $\mathbb{R}\times\mathbb{R}^{nm}$.

Each common time-dependent superposition rule in terms of m particular solutions defines a local n-codimensional foliation $\bar{\mathcal{F}}$ on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ horizontal with respect to $\overline{\rm pr}$ and such that the vector fields $\{\widetilde{X}_c\}_{c\in\Lambda}$ are tangent to its leaves.

Conversely, given a n-codimensional foliation $\bar{\mathcal{F}}$ on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ horizontal with respect to $\overline{\rm pr}$ provides a common time-dependent superposition rule $\bar{F}$. Indeed, if we take a set of particular solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$ and a leaf $\bar{\mathcal{F}}_k$ of the foliation $\bar{\mathcal{F}}$, there exists just a curve $x_{(0)}(t)$ such that $(t,x_{(0)}(t),x_{(1)}(t),\ldots,x_{(m)}(t))\in\bar{\mathcal{F}}_k$. Therefore, we can define

$$x_{(0)}(t)=\bar{F}(t,x_{(1)}(t),\ldots,x_{(m)}(t);k).$$

Moreover, it can be seen that $x_{(0)}(t)$ is another solution of the Lie system.

Each local n-codimensional foliation $\bar{\mathcal{F}}$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ horizontal with respect to $\overline{\mathrm{pr}}$ and with the vector fields $\{\widetilde{X}_c\}_{c \in \Lambda}$ being tangent to its leaves, defines a common time-dependent superposition rule in terms of m particular solutions for the systems $\{X_c\}_{c \in \Lambda}$.

Proposition

Giving a common time-dependent superposition rule for a family of system of differential equations on $\mathbb{R}^n$ is equivalent to giving a n-codimensional foliation on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ horizontal with respect to the projection $\overline{\mathrm{pr}} : \mathbb{R} \times \mathbb{R}^{(m+1)n} \to \mathbb{R} \times \mathbb{R}^{nm}$, for which the time-prolongations $\{\widetilde{X}_c\}_{c \in \Lambda}$ of the systems $\{X_c\}_{c \in \Lambda}$ are tangent to its leaves.

Lemma

Let $\widetilde{X}_j$, with $j = 1, \ldots, r$, be the time-prolongations to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ of a family of time-dependent vector fields X_j on $\mathbb{R}^n$ such that $\overline{\mathrm{pr}}_*(\widetilde{X}_j)$ are linearly independent at a generic point of $\mathbb{R} \times \mathbb{R}^{nm}$, therefore, a vector field on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ of the form $X = \sum_{j=1}^r b_j \widetilde{X}_j$, with $b_j \in C^\infty(\mathbb{R} \times \mathbb{R}^{n(m+1)})$ and $\sum_{j=1}^r b_j = 0$, is a pure-prolongation if and only if the functions b_j are time-dependent only.

Note

There exists an equivalent proposition with $\sum_{j=1}^r b_j = 1$ and X being a time-prolongation.

Generalized Lie Theorem

A family of systems $\{X_c\}_{c\in\Lambda}$ admits a common time-dependent superposition rule if and only if each element $X_c(t,x)$ of the family admits it autonomization $\bar{X}_c$ to be cast into the form

$$\bar{X}_c(t,x) = \sum_{\alpha=1}^{r} b_{c\alpha}(t)\bar{X}_\alpha(t,x),$$

where the X_α are time-dependent vector fields on $\mathbb{R}^n$ such that that there exist r^3 functions $f_{\alpha\beta\gamma}(t)$ satisfying that

$$[\bar{X}_\alpha, \bar{X}_\beta](t,x) = \sum_{\gamma=1}^{r} f_{\alpha\beta\gamma}(t)\bar{X}_\gamma(t,x), \qquad \alpha,\beta = 1,\ldots,r.$$

Abel equations

Abel equations

Consider the family of Abel equations

$$\frac{dx}{dt} = (t + x) + b(t)(1 + t + x)^3,$$

with $b(t)$ any arbitrary time-dependent function. The general solution of any instance of the above equation can be written as

$$x(t) = ((x_1(t) + t + 1)^{-2} + ke^{-2t})^{-1/2} - t - 1$$

in terms of a particular solution $x_{(1)}(t)$ of such an instance and a real constant k.

Abel equations II

The considered Abel equations are described by the family of time-dependent vector fields

$$X_{b(t)} = \left((t+x) + b(t)(1+t+x)^3\right)\frac{\partial}{\partial x},$$

which, according to Generalized Lie Theorem, holds that

$$\bar{X}_{b(t)} = (1-b(t))\bar{X}_1 + b(t)\bar{X}_2,$$

and

$$X_1 = (t+x)\frac{\partial}{\partial x}, \qquad X_2 = \left((1+t+x)^3 + t + x\right)\frac{\partial}{\partial x},$$

which satisfy that $[\bar{X}_1, \bar{X}_2] = 2\bar{X}_2 - 2\bar{X}_1$.