VECTOR ANALYSIS IN CURVELINEAR ORTHOGONAL (LAMÉ) COORDINATE SYSTEMS

We are used to work in the Cartesian coordinate system in which points of the space are identified by values of x, y and z. Associated with this system is the basis of three vectors

$$\mathbf{i}_x \equiv \mathbf{e}_x\,, \qquad \mathbf{i}_y \equiv \mathbf{e}_y\,, \qquad \mathbf{i}_z \equiv \mathbf{e}_z\,.$$

These three vectors have by definition unit lengths (we use the symbol $\mathbf{e}_i$ for unit length vectors) and are mutually orthogonal:

$$(\mathbf{e}_i|\mathbf{e}_j) \equiv \mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}\,.$$

They also satisfy the rule

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\,\mathbf{e}_k \equiv \mathbf{e}_k\,\epsilon_{kij}\,.$$

From these rules, the identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\,,$$

and the possibility of writing any vector $\mathbf{V}$ as a linear combination $\mathbf{V} = \mathbf{e}_i V^i$ all vector identities can easily be proved. For example

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{e}_i \times (\mathbf{e}_l \times \mathbf{e}_m) A^i B^l C^m \\
&= \mathbf{e}_i \times \mathbf{e}_k\,\epsilon_{klm}\,A^i B^l C^m \\
&= \mathbf{e}_j\,\epsilon_{jik}\,\epsilon_{klm}\,A^i B^l C^m \\
&= \mathbf{e}_j(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il})\,A^i B^l C^m \\
&= \mathbf{e}_j B^j (A^i C^i) - \mathbf{e}_j C^j (A^i B^i) \\
&\equiv \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})\,.
\end{aligned}$$

Usually in this type of calculations one does not write explicitly the unit vectors $\mathbf{e}_i$. This makes the notation more economical but is possible only either if the vectors are decomposed into the Cartesian unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, or (for vectors decomposed into unit vectors $\mathbf{e}_1(\xi)$, $\mathbf{e}_2(\xi)$, $\mathbf{e}_3(\xi)$ associated with some curvelinear coordinates ξ^1, ξ^2, ξ^3 - see below) if no differentiations are involved: for example, if in the example above $\mathbf{C}(\xi) = \mathbf{e}_i(\xi)C^i(x)$ and $\mathbf{B}(\xi) = \mathbf{e}_i(\xi)\nabla^i$, where ∇^i is a differential operator acting on everything standing to the right of it, then one cannot drop the vectors $\mathbf{e}_i(\xi)$ because they too get differentiated.

In numerous special problems of classical electrodynamics it proves more convenient to use coordinate systems ξ^i other than the Cartesian ones. Curvelinear systems are introduced by giving three functions

$$\begin{aligned}
x &= x(\xi^1,\,\xi^2,\,\xi^3)\,,\\
y &= y(\xi^1,\,\xi^2,\,\xi^3)\,,\\
z &= z(\xi^1,\,\xi^2,\,\xi^3)\,.
\end{aligned}$$

Associated with each point of the space are then three vectors

$$\mathbf{i}_i(\xi) \equiv \frac{\partial}{\partial \xi^i} \equiv \mathbf{e}_x \frac{\partial x}{\partial \xi^i} + \mathbf{e}_y \frac{\partial y}{\partial \xi^i} + \mathbf{e}_z \frac{\partial z}{\partial \xi^i}\,,$$

(the notation $\partial/\partial\xi^i$ used by differential geometers - różniczkowych omętrów zwanych gdzieniegdzie jeszcze różniczkowymi skoczybruzdami - should not terrify you as we will not use it). More precisely, with each point of the space (which should be thought of as a differential manifold) there is associated a vector space (the tangent space) in which vectors attached to this point live. The vectors $\mathbf{i}_i(\xi)$ form a basis of the vector space attached to the point labeled by ξ^1, ξ^2, ξ^3. The vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are not the same for different points and for an arbitrary choice of the coordinates ξ^i are not of unit length and even not mutually orthogonal. Their scalar product defines the *metric tensor* $g_{ij}(\xi)$

$$g_{ij}(\xi) \equiv (\mathbf{i}_i|\mathbf{i}_j) = \frac{\partial x}{\partial \xi^i}\frac{\partial x}{\partial \xi^j} + \frac{\partial y}{\partial \xi^i}\frac{\partial y}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j}\,.$$

Here we work in the Euclidean three dimensional space and the metric tensor $g_{ij}(\xi)$ can be computed directly because we assume that the three functions $x(\xi)$, $y(\xi)$, $z(\xi)$ are known.[1] As in the usual algebra, any vector $\mathbf{V}$ attached to the point labeled by ξ^i or a vector field $\mathbf{V}(\xi)$ can be written in the form

$$\mathbf{V}(\xi) = \mathbf{i}_k(\xi) V^k_{(\mathbf{i})}(\xi) \equiv \mathbf{i}_k(\xi) V^k(\xi)\,,$$

where $V^k_{(\mathbf{i})}$ is the notation borrowed from my Algebra notes (available from the web page of J. Wojtkiewicz) indicating explicitly that these are components of the vector $\mathbf{V}$ in the basis $\mathbf{i}_k$. The scalar product of two such vectors (vector fields) $\mathbf{V}$ and $\mathbf{W}$ is then given by

$$(\mathbf{V}|\mathbf{W}) = (\mathbf{i}_i|\mathbf{i}_k)\, V^i W^k = g_{ik} V^i W^k \equiv V_i W^i\,.$$

We have defined here *covariant* components $V_i \equiv g_{ij} V^j$ of the vector $\mathbf{V}$ (as opposed to its *contravariant* components V^i). Of course $V^i = g^{ij} V_j$ where g^{ij} is the matrix inverse with respect to the matrix g_{ij}. Mathematically V_k are components of a *linear form* $\hat{V}$ or, if V_i depend on ξ^j, components of a field of forms called also a *differential one-form* associated with the vector $\mathbf{V}$ (with the vector field $\mathbf{V}(\xi)$) which on all vectors attached to the point ξ acts through the scalar product

$$\hat{V}(\cdot) \equiv (\mathbf{V}(\xi)|\,\cdot\,)\,.$$

[1]In General Relativity we do not assume this and try instead to reconstruct all features of the space-time from the metric tensor which in turn is determined by the differential Einstein's equations; the space-time is then in most cases non-Euclidean, that is it has a nonvanishing curvature - a quantity which is independent of the choice of the coordinate system.

All such linear forms attached to the point ξ^i form a vector space (the adjoint vector space with respect to the vector space of vectors attached to this point) for which different bases can be chosen; the two natural bases will be defined below.

In the following we will be concerned with a special class of coordinate systems - the Lamé systems - singled out by the orthogonality (in each point of the space) of the three vectors $\mathbf{i}_i$. In such systems the metric tensor is diagonal:

$$g_{ij}(\xi) = h_i^2(\xi)\,\delta_{ij}\,, \qquad\qquad h_i = \sqrt{(\mathbf{i}_i|\mathbf{i}_i)} \equiv \|\mathbf{i}_i\|\,.$$

h_i are called Lamé coefficients. Of course, $g^{ij}(\xi) = h_i^{-2}(\xi)\,\delta^{ij}$. In the Lamé systems, to make vector analysis easier, i.e. to make it similar to the vector analysis in the Cartesian coordinates, one introduces three normalized vectors

$$\mathbf{e}_i \equiv \frac{\mathbf{i}_i}{\|\mathbf{i}_i\|} = h_i^{-1}\,\mathbf{i}_i\,,$$

(no summation over i here) such that

$$(\mathbf{e}_i|\mathbf{e}_j) = h_i^{-1}h_j^{-1}(\mathbf{i}_i|\mathbf{i}_j) = h_i^{-1}h_j^{-1}g_{ij} = h_i^{-1}h_j^{-1}h_ih_j\delta_{ij} = \delta_{ij}\,.$$

(Of course, these vectors still depend on ξ, because their orientation in the space varies from point to point). Any vector $\mathbf{V}$ can be then decomposed in two ways (and, of course, in many other ways too)

$$\begin{aligned}\mathbf{V} &= \mathbf{i}_k V^k_{(\mathbf{i})} \equiv \mathbf{i}_k V^k \\ &= \mathbf{e}_k V^k_{(\mathbf{e})} \equiv \mathbf{e}_k \bar{V}^k\,.\end{aligned}$$

From the relation between the vectors $\mathbf{i}_k$ and $\mathbf{e}_k$ it follows that

$$V^k_{(\mathbf{e})} \equiv \bar{V}^k = h_k V^k \equiv h_k V^k_{(\mathbf{i})}\,,$$

(no summation over k here). The scalar product of two vectors can be then written as

$$(\mathbf{V}|\mathbf{W}) = \bar{V}^k\bar{V}^k = \bar{V}_k\bar{V}^k\,,$$

i.e. it looks as in the Cartesian system. The barred covariant components $\bar{V}_k$ of the vector $\mathbf{V}$ are identical to the contravariant ones

$$\bar{V}_k = \bar{V}^k\,,$$

and are related to the unbarred covariant components V_k of $\mathbf{V}$ by

$$\bar{V}_k = h_k^{-1}V_k\,,$$

(again no sum over k here). Thus, the whole point of introducing "physical" components $\bar{V}^k$ is to get rid of the metric tensor in the scalar product.

Gradient

Consider a function S defined on the space (on the manifold). In coordinates ξ^i it is a function $S(\xi)$. At each point its total differential

$$dS = \frac{\partial S}{\partial \xi^i} \, d\xi^i \, .$$

is a linear form, or more precisely, a differential one-form (two- and three-forms will of course appear soon!). As every linear form, it is a machine with a hole into which one inserts a vector and obtains in return a number; moreover the action of such a form is linear. The differentials $d\xi^i$ form a basis in the space of one forms; their action on any vector follows from the rule

$$d\xi^k(\mathbf{i}_j) = \delta^k{}_j \, ,$$

and the linearity. The factors $\partial S/\partial \xi^i$ are simply components of the one-form dS in the natural basis $d\xi^i$ of one-forms associated with the coordinates ξ^i. On a vector $\delta\vec{\xi} = \mathbf{i}_k \delta\xi^k$ of a small displacement by $\delta\xi^i$ the total differential dS gives

$$dS(\delta\vec{\xi}) = \frac{\partial S}{\partial \xi^i} \, d\xi^i(\mathbf{i}_k \delta\xi^k) = \frac{\partial S}{\partial \xi^i} \, d\xi^i(\mathbf{i}_k) \, \delta\xi^k = \frac{\partial S}{\partial \xi^i} \, \delta\xi^i \approx S(\xi + \delta\xi) - S(\xi) \, ,$$

- the first approximation to the difference of S at ξ^i and the neighbouring point $\xi^i + \delta\xi^i$, that is what an average physicist, not mislead by mathematicians, would call dS.

In the Lamé systems one introduces also another basis $\hat{f}^i$ of one-forms singled out by their action on the $\mathbf{e}_i$ vectors:

$$\hat{f}^k(\mathbf{e}_j) = \delta^k{}_j \, .$$

From linearity it then follows that

$$\hat{f}^k = h_k \, d\xi^k \, ,$$

because then

$$\hat{f}^k(\mathbf{e}_j) = \hat{f}^k(h_j^{-1}\mathbf{i}_j) = h_j^{-1} \hat{f}^k(\mathbf{i}_j) = d\xi^k(\mathbf{i}_j) = \delta^k{}_j \, .$$

The action of a linear form ${}^{(1)}\hat{\omega} = \omega_k d\xi^k$ attached to the point ξ (or a field of one-forms ${}^{(1)}\hat{\omega}(\xi)$ defined for each point of the manifold, if the components ω_k are functions of ξ^i) on a vector $\mathbf{V}$ attached to the same point ξ (or a vector field defined on the manifold) is given by

$${}^{(1)}\hat{\omega}(\mathbf{V}) = \omega_k d\xi^k(\mathbf{i}_j V^j) = \omega_k V^k \equiv h_k \bar{\omega}_k \, h_k^{-1} \bar{V}^k = \bar{\omega}_k \bar{V}^k \, .$$

This shows that components of a one-form can be treated as (covariant) components of a vector and the action of the one-form ${}^{(1)}\hat{\omega}$ on a vector $\mathbf{V}$ can be represented by the scalar product of $\mathbf{V}$ with the vector $\mathbf{i}_i\,\omega^i = \mathbf{e}_i\,\bar{\omega}^i$ associated with the form ${}^{(1)}\hat{\omega}$.

In Lamé systems gradient (the "physical" gradient) of a function S is by definition the total differential dS referred to the basis $\hat{f}^k$:

$$dS = \frac{\partial S}{\partial \xi^k}\, d\xi^k = \left(\frac{1}{h_k}\,\frac{\partial S}{\partial \xi^k}\right) \hat{f}^k \equiv \overline{(\nabla S)}_k \hat{f}^k\,.$$

The gradient of $S : \xi^i \longrightarrow \mathbb{R}$, or in other words, the total derivative of S, is a liner function mapping the vectors living in the tangent space into $\mathbb{R}$:

$$dS(\mathbf{V}) = V^l\,\frac{\partial S}{\partial \xi^k}\, d\xi^k(\mathbf{i}_l) = V^k\,\frac{\partial S}{\partial \xi^k} = \bar{V}^k\,\overline{(\nabla S)}_k\,.$$

Of course, in physical calculations the bars over "physical" components are omitted (as components of vectors and forms in the bases $\mathbf{i}_i$ and $d\xi^j$ never appear in such calculations).

Divergence

Divergence of a vector field $\mathbf{V}(\xi) = \mathbf{i}_k V^k(\xi)$ is in the most general case defined as follows: We associate with the vector field $\mathbf{V}$ a one-form $\hat{V}$:

$$\hat{V} = V_i\, d\xi^i \equiv g_{ik} V^k\, d\xi^k\,,$$

and apply to it the Hodge star operator:

$$*\hat{V} \equiv \frac{1}{2}\,\sqrt{g}\,\epsilon_{ijk} V^k\, d\xi^i \wedge d\xi^j\,,$$

where $g \equiv \det(g_{ij})$ and finally take the exterior derivative of the resulting two-form:

$$\begin{aligned} d(*\hat{V}) &= \frac{1}{2}\,\epsilon_{ijk}\,\frac{\partial}{\partial \xi^l}\left(V^k\sqrt{g}\right) d\xi^l \wedge d\xi^i \wedge d\xi^j \\ &= \frac{\partial}{\partial \xi^k}\left(V^k\sqrt{g}\right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3\,. \end{aligned}$$

We have used here the relations

$$d\xi^l \wedge d\xi^i \wedge d\xi^j = \epsilon_{lij}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3\,, \qquad \text{and} \qquad \epsilon_{ijk}\epsilon_{lij} = 2\delta_{kl}\,.$$

"Physical" divergence is the three-form $d(*\hat{V})$ but referred to the canonical volume form $\hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3$:

$$d(*\hat{V}) = \frac{1}{h_1 h_2 h_3}\,\frac{\partial}{\partial \xi^k}\left(h_1 h_2 h_3 \frac{\bar{V}^k}{h_k}\right) \hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3\,.$$

i.e.

$$\mathrm{div}\mathbf{V} \equiv \nabla\cdot\mathbf{V} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial\xi^k}\left(h_1 h_2 h_3\,\frac{\bar{V}^k}{h_k}\right).$$

Curl

Curl of a vector field $\mathbf{V}(\xi) = \mathbf{i}_k V^k(\xi)$ is defined as follows: first associate with $\mathbf{V}$ the form $\hat{V} = V_i d\xi^i$. Then take its exterior derivative

$$d\hat{V} = \frac{\partial}{\partial\xi^k}\,\left(g_{ij}V^j\right) d\xi^k \wedge d\xi^i\,,$$

obtaining a two-form. Finally apply the Hodge star operation:

$$*(d\hat{V}) = \sqrt{g}\,g^{kl}g^{im}\frac{\partial}{\partial\xi^k}\,\left(g_{ij}V^j\right)\epsilon_{lmn}\,d\xi^n\,.$$

In a Lamé system, components of the resulting one-form in the basis $\hat{f}^i$ is just what is called the "physical" curl of $\mathbf{V}$:

$$*(d\hat{V}) = h_1 h_2 h_3\,h_k^{-2}h_i^{-2}\,\epsilon_{kin}\,\frac{\partial}{\partial\xi^k}\,\left(h_i\bar{V}^i\right)h_n^{-1}\,\hat{f}^n \equiv \overline{(\nabla\times\mathbf{V})}_n\,\hat{f}^n\,.$$

Simplifying a bit, the "physical" component of the curl of $\mathbf{V}$ is

$$\overline{(\nabla\times\mathbf{V})}_n = \frac{h_n}{h_1 h_2 h_3}\,\epsilon_{kin}\,\frac{\partial}{\partial\xi^k}\,\left(h_k\bar{V}^k\right).$$

Laplacian

The Laplacian acting on a function $S(\xi)$ is just the divergence of its gradient - it is a three-form:

$$d(*dS) = (\nabla^2 S)\,\hat{f}^1\wedge\hat{f}^2\wedge\hat{f}^3\,.$$

Explicitly:

$$\begin{aligned}
d\left(*\left(\frac{\partial S}{\partial\xi^i}\,d\xi^i\right)\right) &= \frac{1}{2}\,d\left(\frac{\partial S}{\partial\xi^i}\,\sqrt{g}\,g^{ik}\,\epsilon_{klm}\,d\xi^l\wedge d\xi^m\right)\\
&= \frac{1}{2}\,\frac{\partial}{\partial\xi^j}\left(\frac{\partial S}{\partial\xi^i}\,\sqrt{g}\,g^{ik}\,\epsilon_{klm}\right)d\xi^j\wedge d\xi^l\wedge d\xi^m\\
&= \frac{\partial}{\partial\xi^j}\left(\sqrt{g}\,g^{ij}\,\frac{\partial S}{\partial\xi^i}\right)d\xi^1\wedge d\xi^2\wedge d\xi^3\,.
\end{aligned}$$

We have used the same relations as in the derivation of the divergence. The "physical Laplacian" (in Lamé coordinate systems) is referred to the canonical volume three-form:

$$\nabla^2 S = \frac{1}{h_1 h_2 h_3}\,\frac{\partial}{\partial\xi^j}\left(\frac{h_1 h_2 h_3}{h_i^2}\,\frac{\partial S}{\partial\xi^i}\right).$$

Example

We illustrate all these considerations by considering the spherical coordinates $(\xi^i, \xi^2, \xi^3) \equiv (r, \theta, \phi)$ introduced through the well known relations

$$x = r \sin\theta \cos\phi\,,$$
$$y = r \sin\theta \sin\phi\,,$$
$$z = r \cos\theta\,.$$

One then has

$$\mathbf{i}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{i}_\theta = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \qquad \mathbf{i}_\phi = \begin{pmatrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

The Lamé coefficients read

$$h_r = (\mathbf{i}_r|\mathbf{i}_r) = 1\,, \qquad h_\theta = (\mathbf{i}_\theta|\mathbf{i}_\theta) = r\,, \qquad h_\phi = (\mathbf{i}_\phi|\mathbf{i}_\phi) = r\sin\theta\,,$$

and the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ have the form

$$\mathbf{e}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

The canonical volume three-form

$$\hat{f}^r \wedge \hat{f}^\theta \wedge \hat{f}^\phi = r^2 \sin\theta dr \wedge d\theta \wedge d\phi\,,$$

looks familiar for anybody who at least once has integrated something over a three dimensional domain using spherical coordinates, but what these "$\wedge$'s" serve for?! Be patient and look below how the integration of differential forms is defined.

Using the Lamé coefficients given above it is straightforward to write down "physical" components of the gradient of a function S

$$\overline{(\nabla S)}_r = \frac{\partial S}{\partial r}\,, \qquad \overline{(\nabla S)}_\theta = \frac{1}{r}\frac{\partial S}{\partial \theta}\,, \qquad \overline{(\nabla S)}_\phi = \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\,,$$

of the rotation of a vector field $\mathbf{V} = \mathbf{e}_r\bar{V}^r + \mathbf{e}_\theta\bar{V}^\theta + \mathbf{e}_\phi\bar{V}^\phi$

$$\overline{(\nabla \times \mathbf{V})}_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\bar{V}^\phi\sin\theta\right) - \frac{\partial\bar{V}^\theta}{\partial\phi}\right),$$
$$\overline{(\nabla \times \mathbf{V})}_\theta = \frac{1}{r\sin\theta}\frac{\partial\bar{V}^r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{V}^\phi\right),$$
$$\overline{(\nabla \times \mathbf{V})}_\phi = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\bar{V}^\theta\right) - \frac{\partial\bar{V}^r}{\partial\theta}\right),$$

as well as the “physical divergence” of $\mathbf{V}$:

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\bar{V}^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\bar{V}^\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\bar{V}^\phi}{\partial\phi},$$

and the “physical” Laplacian of a function $S(\xi)$:

$$\nabla^2 S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}.$$

Integration of p-forms over p-dimensional domains

A p-form ${}^{(p)}\hat{\omega} = \omega_{i_1\ldots i_p}(\xi)\, d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p}$ can be integrated over a p-dimensional domain (a p-dimensional submanifold) Ω_p of the d-dimensional space (d-dimensional manifold). The integral

$$\int_{\Omega_p}{}^{(p)}\hat{\omega} = \int_{\Omega_p}\omega_{i_1\ldots i_p}(\xi)\, d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p},$$

is defined as follows. We have first to parametrize the domain Ω_p with p parameters τ^1, …, τ^p:

$$\begin{aligned}\xi^1 &= \xi^1(\tau^1,\ldots,\tau^p),\\ \xi^2 &= \xi^2(\tau^1,\ldots,\tau^p),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \xi^d &= \xi^d(\tau^1,\ldots,\tau^p).\end{aligned}$$

One then has p vector fields $\mathbf{t}_{(1)}$, …, $\mathbf{t}_{(p)}$:

$$\mathbf{t}_{(i)} = \mathbf{i}_1\frac{\partial\xi^1}{\partial\tau^i} + \ldots + \mathbf{i}_d\frac{\partial\xi^d}{\partial\tau^i}, \qquad i = 1,\ldots,p.$$

all of which are tangent to the submanifold Ω_p. It is easy to see that $\mathbf{t}_{(i)}$ is tangent to the curve traced in Ω_p by varying the parameter τ^i keeping all other τ’s fixed.

By definition

$$\int_{\Omega_p}{}^{(p)}\hat{\omega} = \int d\tau^1\ldots\int d\tau^p\,\omega_{i_1\ldots i_p}(\xi(\tau))\, d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p}(\mathbf{t}_{(1)},\ldots,\mathbf{t}_{(p)}).$$

The domain of integration over the parameters τ follows of course from the parametrization of Ω_p. Since (see the definition of the action of a general p-form on p vectors)

$$\begin{aligned}d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p}(\mathbf{t}_{(1)},\ldots,\mathbf{t}_{(p)}) &= \sum_\pi \mathrm{sgn}(\pi)\, d\xi^{i_1}(\mathbf{t}_{\pi(1)})\ldots d\xi^{i_p}(\mathbf{t}_{\pi(p)})\\ &= \sum_\pi \mathrm{sgn}(\pi)\frac{\partial\xi^{k_1}}{\partial\tau^{\pi(1)}}\ldots\frac{\partial\xi^{k_p}}{\partial\tau^{\pi(p)}}\, d\xi^{i_1}(\mathbf{i}_{k_1})\ldots d\xi^{i_p}(\mathbf{i}_{k_p})\\ &= \sum_\pi \mathrm{sgn}(\pi)\frac{\partial\xi^{i_1}}{\partial\tau^{\pi(1)}}\ldots\frac{\partial\xi^{i_p}}{\partial\tau^{\pi(p)}} \equiv \frac{\partial(\xi^{i_1},\ldots,\xi^{i_p})}{\partial(\tau^{\pi(1)},\ldots,\tau^{\pi(p)})},\end{aligned}$$

so, the final practical formula for the integral reads

$$\int_{\Omega_p} {}^{(p)}\hat{\omega} = \int d\tau^1 \dots \int d\tau^p \, \omega_{i_1\dots i_p}(\xi(\tau)) \, \frac{\partial(\xi^{i_1}, \dots, \xi^{i_p})}{\partial(\tau^{\pi(1)}, \dots, \tau^{\pi(p)})} \, .$$

Stokes theorem

The fundamental Stokes theorem states that

$$\int_{\Omega_p} d({}^{(p-1)}\hat{\omega}) = \int_{\partial\Omega_p} {}^{(p-1)}\hat{\omega} \, ,$$

where $\partial\Omega_p$ is the $p-1$-dimensional boundary of the domain Ω_p.

Exterior derivative of a zero-form, i.e. of a function $S(\xi)$ is a one-form dS which can be integrated over a curve Γ_{AB} going from a point A to a point B. The Stokes theorem reduces then to the trivial statement that

$$\int_{\Gamma_{AB}} dS = \int_{\partial\Gamma_{AB}} S \equiv S(B) - S(A) \, ,$$

because the boundary of the curve Γ_{AB} consists of the points A and B.

What is the physical interpretation of an integral of a one form ${}^{(1)}\hat{\omega} = \omega_i d\xi^i$ over a curve Γ_{AB}? Let's see. To evaluate the integral we parametrize the curve with some parameter $\tau \in [\tau_A, \tau_B]$: $\xi^i = \xi(\tau)$, where $\xi(\tau_A) = \xi^i_A$ and $\xi(\tau_B) = \xi^i_B$. Then

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \, d\xi^i \left(\mathbf{i}_k \, \frac{d\xi^k}{d\tau} \right) = \int_{\tau_A}^{\tau_B} d\tau \, \omega_i(\xi(\tau)) \, \frac{d\xi^k}{d\tau} \, .$$

To get the physical interpretation let's however rewrite the integrand differently:

$$\begin{aligned} \omega_i d\xi^i \left(\mathbf{i}_k \, \frac{d\xi^k}{d\tau} \right) &= \bar{\omega}_i \, \hat{f}^i \left(\mathbf{e}_x \, \frac{\partial x}{\partial \xi^k} \frac{d\xi^k}{d\tau} + \mathbf{e}_y \, \frac{\partial y}{\partial \xi^k} \frac{d\xi^k}{d\tau} + \mathbf{e}_z \, \frac{\partial z}{\partial \xi^k} \frac{d\xi^k}{d\tau} \right) \\ &= \bar{\omega}_i \, \hat{f}^i \left(\mathbf{e}_x \, \frac{dx}{d\tau} + \mathbf{e}_y \, \frac{dy}{d\tau} + \mathbf{e}_z \, \frac{dz}{d\tau} \right) . \end{aligned}$$

We have used here the definition of the vectors $\mathbf{i}_k$ and the ordinary chain differentiation rule. On the other hand, in the Lamé systems one can also write

$$\bar{\omega}_i \, \hat{f}^i = \bar{\omega}_x \hat{f}^x + \bar{\omega}_y \hat{f}^y + \bar{\omega}_z \hat{f}^z \, ,$$

because both $(\bar{\omega}_x, \bar{\omega}_y, \bar{\omega}_z)$ and $(\hat{f}^x, \hat{f}^y, \hat{f}^z)$ are related to $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$ and $(\hat{f}^1, \hat{f}^2, \hat{f}^3)$ (associated with the coordinates ξ^i) by the same orthogonal transformation. Introducing a vector field $\mathbf{V}$ with the Cartesian components $V^x = \bar{\omega}_x$, $V^y = \bar{\omega}_y$, $V^z = \bar{\omega}_z$ we get

$$\int_{\Gamma_{AB}} {}^{(1)}\hat{\omega} = \int_{\tau_A}^{\tau_B} d\tau \, \frac{d\mathbf{r}(\tau)}{d\tau} \cdot \mathbf{V}(\mathbf{r}(\tau)) = \int_{\Gamma_{AB}} d\mathbf{l} \cdot \mathbf{V} \, ,$$

where $\mathbf{V}$ is a vector field associated with the one-form ${}^{(1)}\hat{\omega}$. The last equality is obvious from ordinary mechanics: $d\mathbf{l} \equiv d\tau\,(d\mathbf{r}(\tau)/d\tau)$ is just the vector $d\mathbf{r}(\tau)$ of the displacement along the curve Γ_{AB} corresponding to the change of the parameter from τ to $\tau + d\tau$; the integral of the scalar product of $d\mathbf{r}(\tau)$ and $\mathbf{V}(\tau)$ is just what one calls the integral of $\mathbf{V}$ along the curve Γ_{AB}.

Thus, to compute the integral of a vector field $\mathbf{V}$ over a curve Γ one takes this field decomposed into vectors $\mathbf{i}_k$ associated with some curvelinear coordinates ξ^i and integrates the form $\hat{V} = V_i\,d\xi^i \equiv g_{ij}V^j\,d\xi^i$.

And how to get a flux of a vector field $\mathbf{V}$ through a surface Σ? To get the hint let's look at the Stokes theorem and compare it with the ordinary Gauss theorem for a closed surface $\Sigma = \partial\Omega$ (Ω being a three-dimensional domain):

$$\int_\Omega \mathrm{div}\mathbf{V}\,d(\mathrm{Volume}) \equiv \int_\Omega d(*\hat{V}) = \text{ Stokes Th. } = \int_{\partial\Omega} *\hat{V}\,.$$

This shows that $*\hat{V}$ must be the right object to integrate over Σ, i.e.

$$\int_\Sigma *\hat{V}\,,$$

should give the flux of the vector field $\mathbf{V}$ through the surface Σ. Indeed,

$$\begin{aligned}\int_\Sigma *\hat{V} &= \int_\Sigma \frac{1}{2}\sqrt{g}\,\epsilon_{ijk}\,V^k\,d\xi^i \wedge d\xi^j \\ &= \iint d\tau^1 d\tau^2\,\frac{1}{2}\,h_1h_2h_3\,\epsilon_{ijk}\,h_k^{-1}\bar{V}^k\,\frac{1}{h_ih_j}\,\hat{f}^i \wedge f^j\left(\mathbf{t}_{(1)},\mathbf{t}_{(2)}\right).\end{aligned}$$

Due to the presence of the totally antisymmetric symbol ϵ_{ijk}, the three Lamé coefficients $h_k^{-1}h_i^{-1}h_j^{-1}$ must be simply $h_1^{-1}h_2^{-1}h_3^{-1}$ and they cancel out the factors $h_1h_2h_3$, so that

$$\int_\Sigma *\hat{V} = \iint d\tau^1 d\tau^2\,\bar{V}^k\,\frac{1}{2}\,\epsilon_{ijk}\left(\bar{t}^i_{(1)}\bar{t}^j_{(2)} - \bar{t}^i_{(2)}\bar{t}^j_{(1)}\right) = \iint d\tau^1 d\tau^2\,\bar{V}^k\,\epsilon_{ijk}\,\bar{t}^i_{(1)}\bar{t}^j_{(2)}.$$

The factor $\epsilon_{ijk}\,(d\tau^1\bar{t}^i_{(1)})(d\tau^2\bar{t}^j_{(2)})$ is nothing else than the vector perpendicular to the parallelogram spanned by the vectors $d\tau^1\mathbf{t}_{(1)}$ and $d\tau^2\mathbf{t}_{(2)}$ of the infinitesimal displacements corresponding to varying the two parameters from τ^1 and τ^2 to $\tau^1 + d\tau^1$ and $\tau^2 + d\tau^2$ respectively, and has length equal to the area of this parallelogram. It follows that the expression under the integral is just what one physically interprets as the flux of $\mathbf{V}$ through the small element of area of the surface Σ. This completes the demonstration.

Example

As an example let us compute the flux of the electric field $\mathbf{E}$ produced by a uniformly charged ball (of radius R and total charge Q) through a flat disc also of radius R, tangent to the ball.

Outside the ball the electric field has the form as if it was produced by the point charge Q located in the center of the ball. We will work with the spherical coordinates $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$ with the origin ($r = 0$) in the center of the ball. In these coordinates only the radial component of the electric field is nonzero: $E^r = \bar{E}^r = k_1 Q/r^2$ (because $h_r = 1$). According to the general considerations the flux is given by

$$\begin{aligned}\text{Flux} = \int_{\text{disc}} *\hat{E} &= \int_{\text{disc}} \frac{1}{2} \sqrt{g}\, \epsilon_{ijk}\, E^k\, xi^i \wedge d\xi^j \\ &= \int_{\text{disc}} r^2 \sin\theta \left(\frac{k_1 Q}{r^2} \right) d\theta \wedge d\phi\,.\end{aligned}$$

We have used $\sqrt{g} = h_r h_\theta h_\phi = r^2 \sin\theta$ and the specific form of the components of the electric field $\mathbf{E}$.

We parametrize the disc by the parameters $\alpha \in [0,\, \frac{\pi}{4}]$ and $\beta \in [0,\, 2\pi]$:

$$\begin{aligned} r &= R/\cos\alpha\,, \\ \theta &= \alpha\,, \\ \phi &= \beta\,, \end{aligned}$$

so that the tangent vectors read

$$\begin{aligned} \mathbf{t}_{(\alpha)} &= \mathbf{i}_r \frac{\partial r}{\partial \alpha} + \mathbf{i}_\theta \frac{\partial \theta}{\partial \alpha} + \mathbf{i}_\phi \frac{\partial \phi}{\partial \alpha} = \mathbf{i}_r \frac{R \sin\alpha}{\cos^2\alpha} + \mathbf{i}_\theta\,, \\ \mathbf{t}_{(\beta)} &= \mathbf{i}_r \frac{\partial r}{\partial \beta} + \mathbf{i}_\theta \frac{\partial \theta}{\partial \beta} + \mathbf{i}_\phi \frac{\partial \phi}{\partial \beta} = \mathbf{i}_\phi\,. \end{aligned}$$

Hence,

$$d\theta \wedge d\phi \left(\mathbf{t}_{(\alpha)},\, \mathbf{t}_{(\beta)}\right) = d\theta \left(\mathbf{t}_{(\alpha)}\right)\, d\phi \left(\mathbf{t}_{(\beta)}\right) - d\theta \left(\mathbf{t}_{(\beta)}\right)\, d\phi \left(\mathbf{t}_{(\alpha)}\right) = 1 - 0 = 1\,,$$

and

$$\text{Flux} = \int_0^{2\pi} d\beta \int_0^{\pi/4} d\alpha\, k_1 Q\, \sin\alpha = k_1 Q\, \pi (2 - \sqrt{2})\,.$$

Useful formulae

1. A p-form ${}^{(p)}\hat{\omega}$ is a p-linear *totally antisymmetric* mapping of p vectors into $\mathbb{R}$. p-forms form a vector space; for their basis one can take $\binom{d}{p}$ (d is the space dimension) antisymmetrized tensor products of p basic one-forms $d\xi^i$:

$$d\xi^{i_1} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} \equiv \sum_\pi \mathrm{sgn}(\pi)\, d\xi^{i_{\pi(1)}} \otimes d\xi^{i_{\pi(2)}} \otimes \ldots \otimes d\xi^{i_{\pi(p)}} \,.$$

π is a permutation and $\mathrm{sgn}(\pi)$ its sign. Action of a general p-form

$${}^{(p)}\hat{\omega} \equiv \omega_{i_1 i_2 \ldots i_p}\, d\xi^{i_1} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} \,,$$

on p vectors $\mathbf{V}_{(1)} = \mathbf{i}_{k_1} V^{k_1}_{(1)}$, …, $\mathbf{V}_{(p)} = \mathbf{i}_{k_p} V^{k_p}_{(p)}$:

$$\begin{aligned}
{}^{(p)}\hat{\omega}(\mathbf{V}_{(1)}, \ldots \mathbf{V}_{(p)}) &= V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{i_1 \ldots i_p}\, d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p}(\mathbf{i}_{k_1}, \ldots \mathbf{i}_{k_p}) \\
&= \sum_\pi \mathrm{sgn}(\pi)\, V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{i_1 i_2 \ldots i_p}\, d\xi^{i_1}(\mathbf{i}_{k_{\pi(1)}}) d\xi^{i_2}(\mathbf{i}_{k_{\pi(2)}}) \ldots d\xi^{i_p}(\mathbf{i}_{k\pi(p)}) \\
&= \sum_\pi \mathrm{sgn}(\pi)\, V^{k_1}_{(1)} \ldots V^{k_p}_{(p)}\, \omega_{k_{\pi(1)} \ldots k_{\pi(p)}} \,.
\end{aligned}$$

2. Exterior derivative of a p-form ${}^{(p)}\hat{\omega} = \omega_{i_1 \ldots i_p}(\xi)\, d\xi^{i_i} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p}$ is $p+1$-form:

$$d({}^{(p)}\hat{\omega}) = \frac{\partial \omega_{i_1 \ldots i_p}(\xi)}{\partial \xi^k}\, d\xi^k \wedge d\xi^{i_i} \wedge d\xi^{i_2} \wedge \ldots \wedge d\xi^{i_p} \,.$$

3. Action of the Hodge star operator on basic one-forms

$$*(d\xi^i) = \frac{1}{2} \sqrt{g}\, g^{ik}\, \epsilon_{klm}\, d\xi^l \wedge d\xi^m \,,$$

and on basic two-forms

$$*(d\xi^i \wedge d\xi^j) = \sqrt{g}\, g^{ik}\, g^{jl}\, \epsilon_{klm}\, d\xi^m \,.$$

Action on general one- and two-forms follows from linearity of the $\star$ operation.

We can also check that $** = \mathrm{Id}$:

$$\begin{aligned}
((d\xi^i)) &= * \left(\sqrt{g}\, g^{ik}\, \epsilon_{klm}\, d\xi^l \wedge d\xi^m \right) \\
&= \frac{1}{2} \sqrt{g}\, g^{ik}\, \epsilon_{klm}\, * \left(d\xi^l \wedge d\xi^m \right) \\
&= \frac{1}{2} \sqrt{g}\, g^{ik}\, \epsilon_{klm}\, \sqrt{g} g^{lj}\, g^{ms} \epsilon_{jsp}\, d\xi^p \\
&= \frac{1}{2}\, g \left(g^{ik}\, g^{lj}\, g^{ms}\, \epsilon_{klm}\right) \epsilon_{jsp}\, d\xi^p = d\xi^i
\end{aligned}$$

because the expression in the last bracket is just $\det(g^{kl})\,\epsilon^{ijs}$ (and $\det(g^{kl})$ is the inverse of $g = \det(g_{ij})$) and $\epsilon^{ijs}\epsilon_{jsp} = 2\delta^i{}_p$.

Similarly,

$$\begin{aligned}
((d\xi^i \wedge d\xi^j)) &= *\left(\sqrt{g}\, g^{ik}\, g^{jl}\, \epsilon_{klm}\, d\xi^m\right) \\
&= \sqrt{g}\, g^{ik}\, g^{jl}\, \epsilon_{klm}\, \frac{1}{2}\, \sqrt{g}\, g^{mp}\; \epsilon_{prs}\, d\xi^r \wedge d\xi^s \\
&= \frac{1}{2}\, g\, g^{ik}\, g^{jl}\, g^{mp}\, \epsilon_{prs}\, d\xi^r \wedge d\xi^s \\
&= \frac{1}{2}\, g\, g^{-1}\, \epsilon^{ijp}\, \epsilon_{prs}\, d\xi^r \wedge d\xi^s \\
&= \frac{1}{2}\left(\delta^i{}_r \delta^j{}_s - \delta^i{}_s \delta^j{}_r\right) d\xi^r \wedge d\xi^s \\
&= \frac{1}{2}\left(d\xi^i \wedge d\xi^j - d\xi^j \wedge d\xi^i\right) = d\xi^i \wedge d\xi^j\, .
\end{aligned}$$

4. Divergence referred to the canonical volume form $\sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \equiv \hat{f}^1 \wedge \hat{f}^2 \wedge \hat{f}^3$ is what in General Relativity is called the covariant divergence:

$$V^k{}_{;\,k}\, \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \equiv \left(\partial_k V^k + \Gamma^k{}_{kj}\, V^j\right) \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3$$

We recall the definition of the Christoffel symbols (Krzysztofelki po naszemu) $\Gamma^i{}_{kj}$ in terms of the metric tensor:

$$\Gamma^i{}_{kj} = \frac{1}{2}\, g^{il}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{kj}\right).$$

Hence,

$$\begin{aligned}
\Gamma^k{}_{kj} &= \frac{1}{2}\, g^{kl}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{kj}\right) = \frac{1}{2}\, g^{kl} \partial_j g_{lk} \\
&= \frac{1}{2}\, \mathrm{tr}\left(g^{-1}\partial_j g\right) = \frac{1}{2}\, \partial_j \ln(g) = \partial_j \ln(\sqrt{g}) \\
&= \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial \xi^j}\, \sqrt{g}\, .
\end{aligned}$$

This should be compared to $d(*\hat{V})$:

$$\begin{aligned}
d(*\hat{V}) &= \frac{\partial}{\partial \xi^i}\left(\sqrt{g}\, V^k\right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\sqrt{g}\, \partial_k V^k + V^k\, \partial_k \sqrt{g}\right) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\partial_k V^k + V^k\, \frac{1}{\sqrt{g}}\, \partial_k \sqrt{g}\right) \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \\
&= \left(\partial_k V^k + \Gamma^i{}_{ik} V^k\right) \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3\, .
\end{aligned}$$