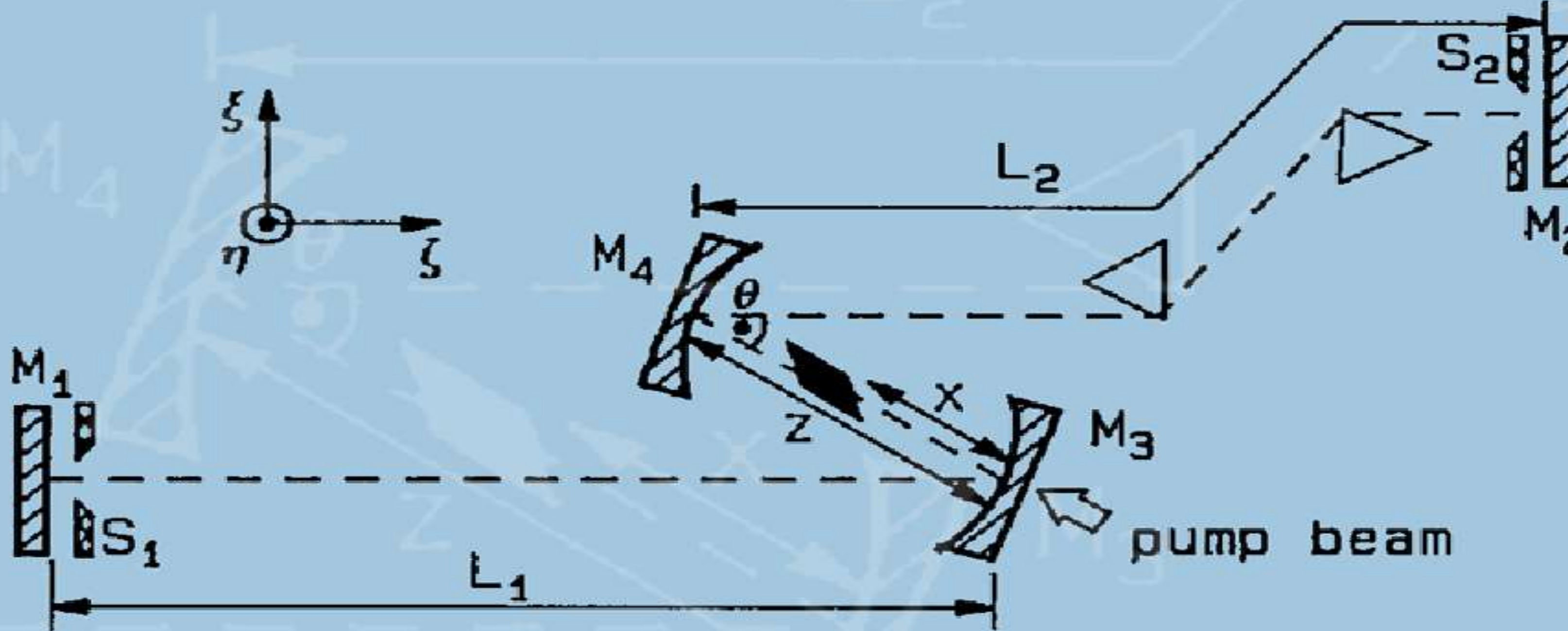
ULTRAFAST OPTICS



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Lecture 2. Short Light Pulses

Description of pulses

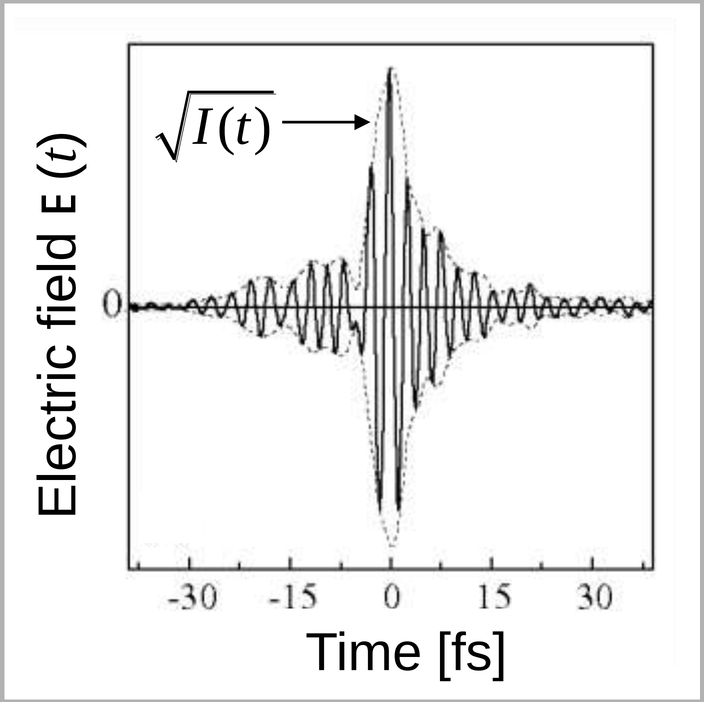
Intensity and phase

The instantaneous frequency and group delay

Zeroth and first-order phase

The linearly chirped Gaussian pulse

An ultrashort laser pulse has an intensity and phase vs. time.



Neglecting the spatial dependence for now, the pulse electric field is given by:

$$\mathrm{E}(t) = \tfrac{1}{2}\sqrt{I(t)}\,\exp\{i[\omega_0 t - \phi(t)]\} + c.c.$$

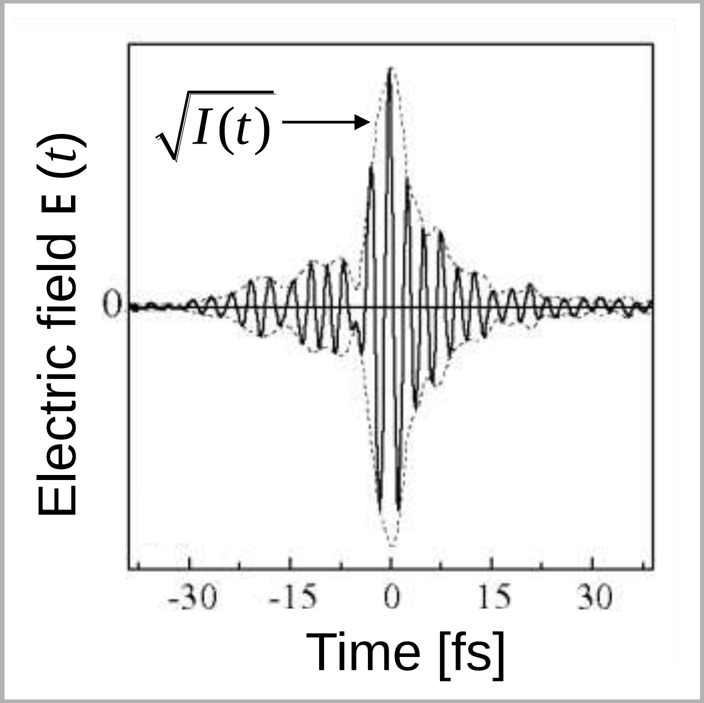
Intensity

Carrier frequency

Phase

A sharply peaked function for the intensity yields an ultrashort pulse. The phase tells us the color evolution of the pulse in time.

The real and complex pulse amplitudes



Removing the 1/2, the c.c., and the exponential factor with the carrier frequency yields the ***complex amplitude, $E(t)$,*** of the pulse:

$$E(t) = \sqrt{I(t)}\,\exp\{-i\phi(t)\}$$

This removes the rapidly varying part of the pulse electric field and yields a complex quantity, which is actually easier to calculate with.

$\sqrt{I(t)}$ is often called the ***real amplitude, $A(t)$,*** of the pulse.

The Gaussian pulse

For almost all calculations, a good first approximation for any ultrashort pulse is the ***Gaussian pulse*** (with zero phase).

$$E(t) = E_0 \exp\left[-(t/\tau_{HW1/e})^2\right]$$

$$= E_0 \exp\left[-2\ln 2\,(t/\tau_{FWHM})^2\right]$$

$$= E_0 \exp\left[-1.38\,(t/\tau_{FWHM})^2\right]$$

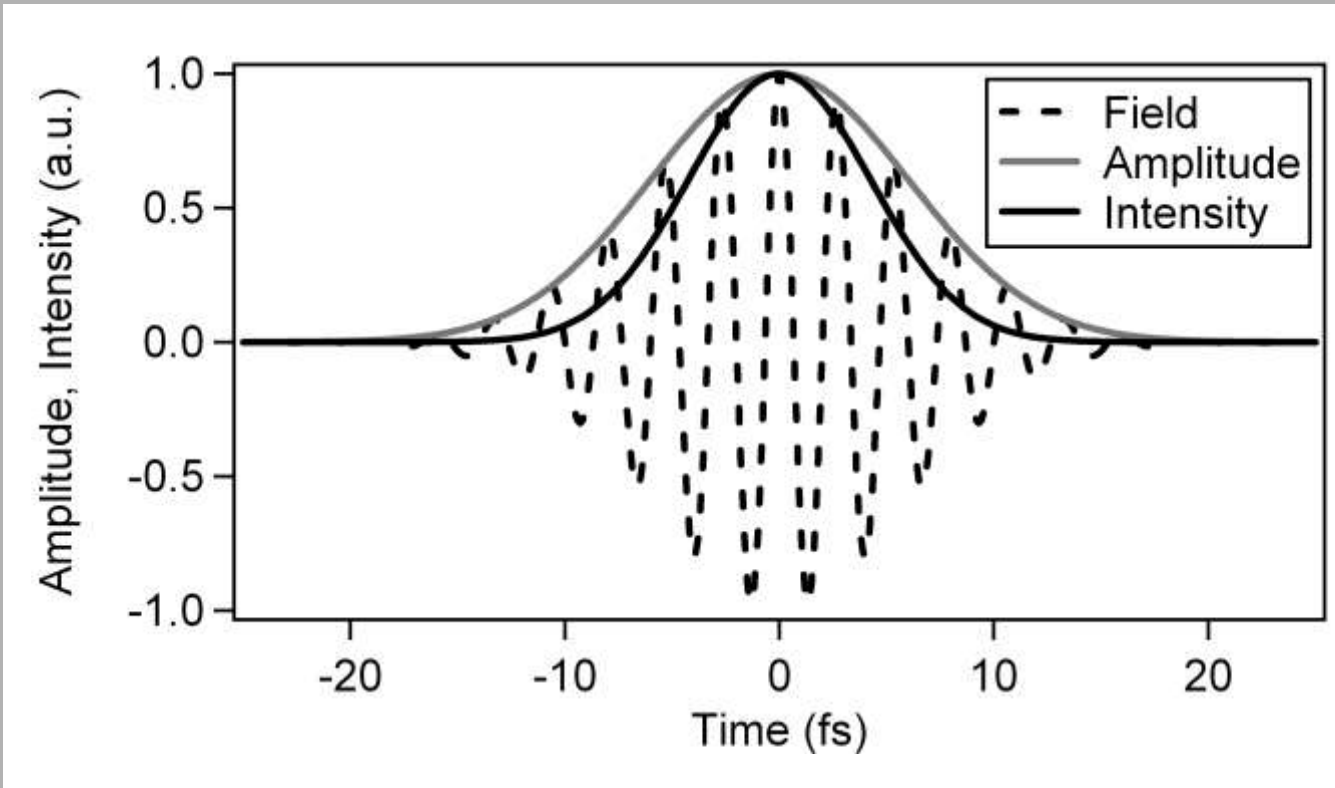
where $\tau_{HW1/e}$ is the field half-width-half-maximum, and τ_{FWHM} is intensity the full-width-half-maximum.

The intensity is:

$$I(t) = |E_0|^2 \exp\left[-4\ln 2\,(t/\tau_{FWHM})^2\right]$$

$$= |E_0|^2 \exp\left[-2.76\,(t/\tau_{FWHM})^2\right]$$

Intensity vs. amplitude



The phase of this pulse is constant, $\phi(t) = 0$, and is not plotted.

The intensity of a Gaussian pulse is $\sqrt{2}$ shorter than its real amplitude. This factor varies from pulse shape to pulse shape.

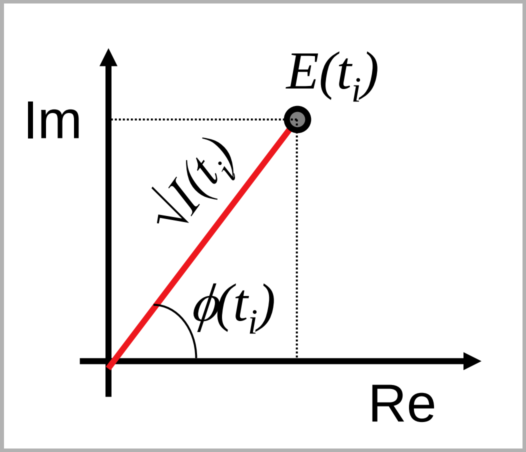
Calculating the intensity and the phase

It's easy to go back and forth between the electric field and the intensity and phase.

The intensity: $I(t) = |E(t)|^2$

The phase:

$$\phi(t) = \arctan\left\{\frac{\mathrm{Im}[E(t)]}{\mathrm{Re}[E(t)]}\right\}$$



The Fourier Transform

To think about ultrashort laser pulses, the Fourier Transform is essential.

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\exp(-i\omega t)\,dt$$

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega)\exp(i\omega t)\,d\omega$$

We always perform Fourier transforms on the real or complex pulse electric field, and not the intensity, unless otherwise specified.

The frequency-domain electric field

The frequency-domain equivalents of the intensity and phase are the spectrum and spectral phase.

Fourier-transforming the pulse electric field:

$$\mathrm{E}(t) = \tfrac{1}{2}\sqrt{I(t)}\,\exp\{i[\omega_0 t - \phi(t)]\} + c.c.$$

yields:

Note that ϕ and φ are different!

$$\tilde{\mathrm{E}}(\omega) = \tfrac{1}{2}\sqrt{S(\omega-\omega_0)}\,\exp\{-i[\varphi(\omega-\omega_0)]\} + \tfrac{1}{2}\sqrt{S(-\omega-\omega_0)}\,\exp\{+i[\varphi(-\omega-\omega_0)]\}$$

The frequency-domain electric field has positive- and negative-frequency components.

The complex frequency-domain pulse field

Since the negative-frequency component contains the same information as the positive-frequency component, we usually neglect it.

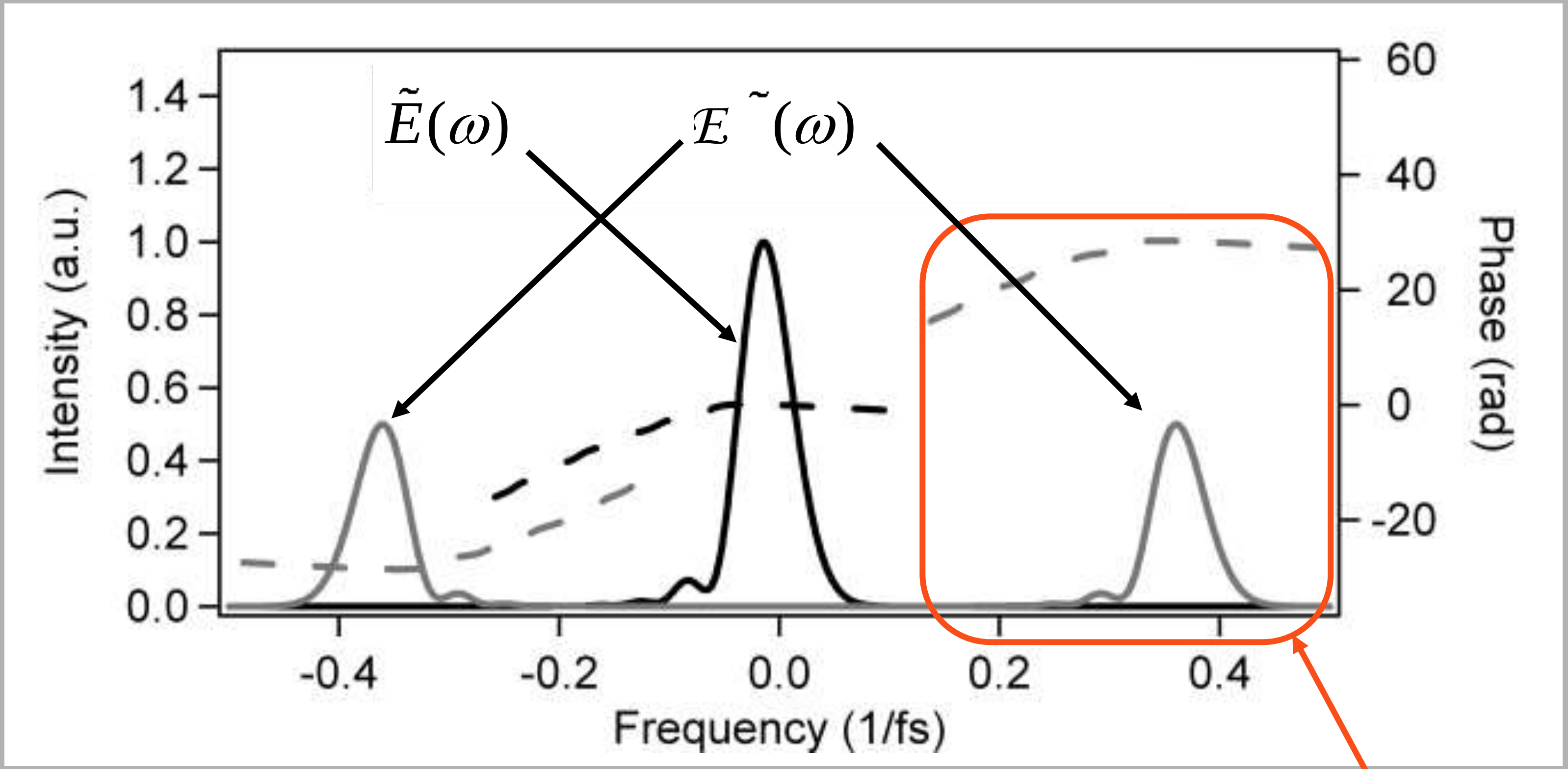
We also center the pulse on its actual frequency, not zero. Thus, the most commonly used complex frequency-domain pulse field is:

$$\tilde{E}(\omega) \equiv \sqrt{S(\omega)} \exp\{-i\,\varphi(\omega)\}$$

Thus, the frequency-domain electric field also has an intensity and phase. S is the spectrum, and φ is the spectral phase.

The spectrum with and without the carrier frequency

Fourier transforming $\mathcal{E}(t)$ and $E(t)$ yield different functions.



We usually use just this component.

The spectrum and spectral phase

The spectrum and spectral phase are obtained from the frequency-domain field the same way as the intensity and phase are from the time-domain electric field.

$$S(\omega) = \left|\tilde{E}(\omega)\right|^2$$

$$\varphi(\omega) = -\arctan\left\{\frac{\mathrm{Im}[\tilde{E}(\omega)]}{\mathrm{Re}[\tilde{E}(\omega)]}\right\}$$

Intensity and phase of a Gaussian

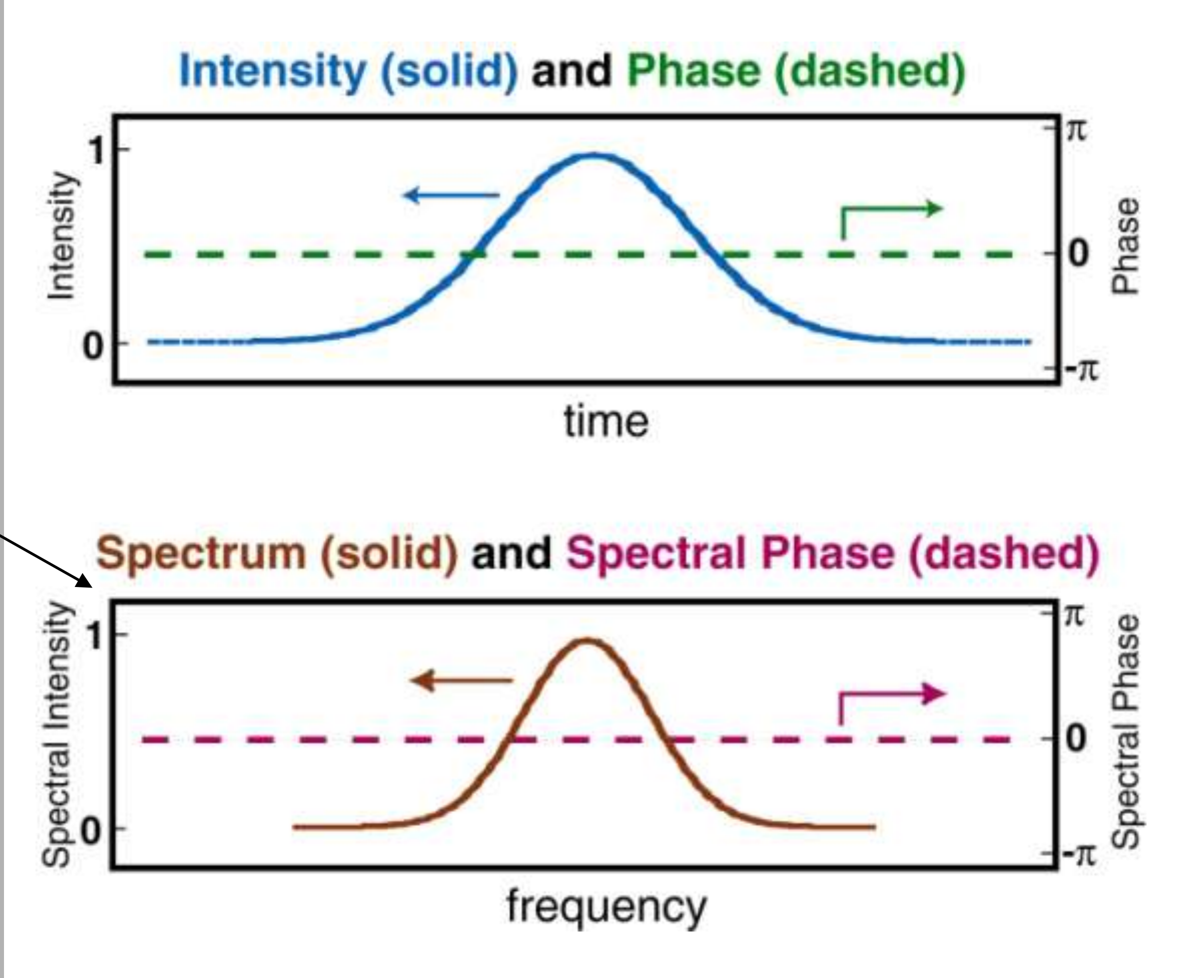
The Gaussian is real, so its phase is zero.

Time domain:

A Gaussian transforms to a Gaussian

Frequency domain:

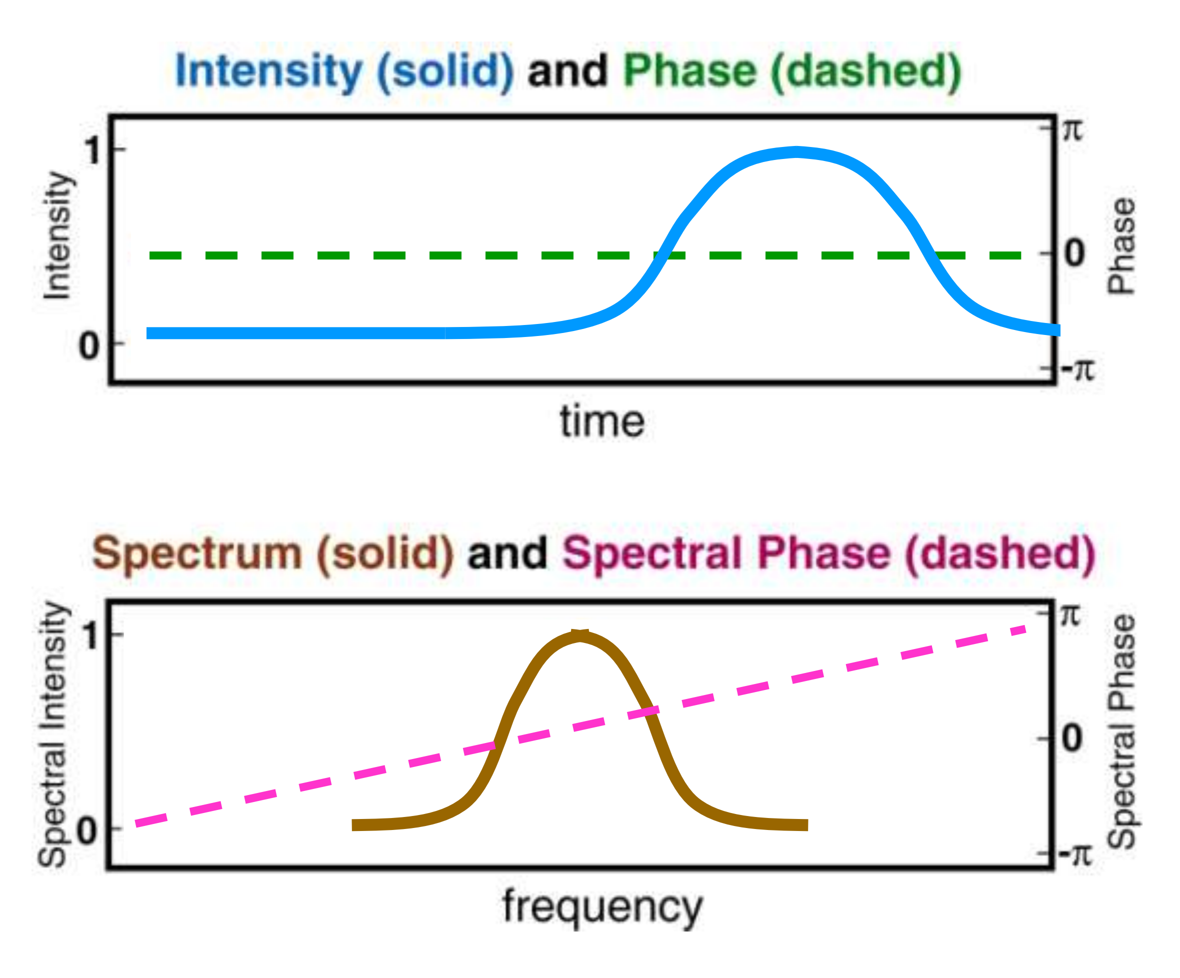
So the spectral phase is zero, too.



The spectral phase of a time-shifted pulse

Recall the Shift Theorem: $\mathcal{F}\left\{ f(t-a) \right\} = \exp(-i\omega a) F(\omega)$

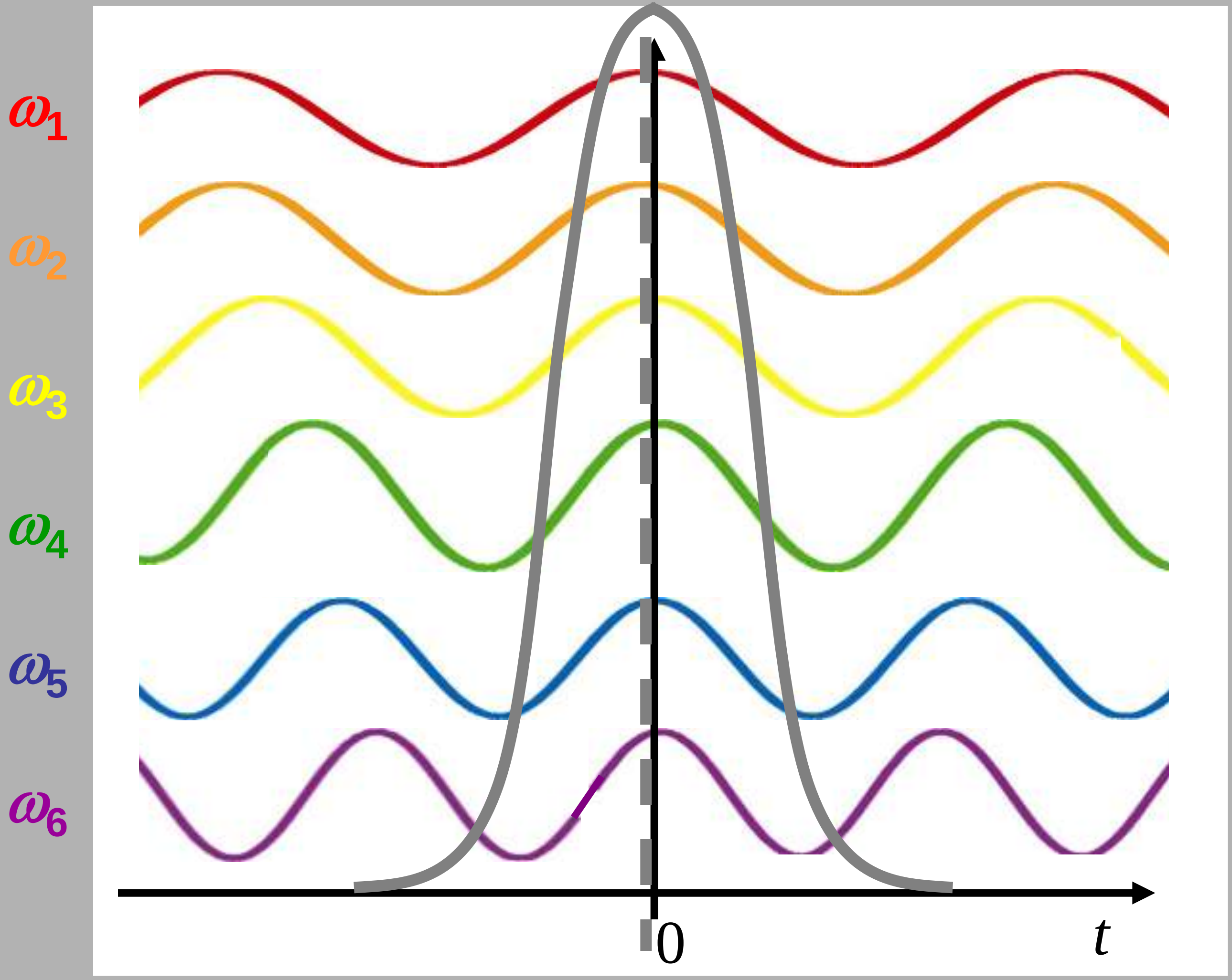
Time-shifted Gaussian pulse (with a flat phase):



So a time-shift simply adds some linear spectral phase to the pulse!

What is the spectral phase?

The spectral phase is the phase of each frequency in the wave-form.



All of these frequencies have zero phase. So this pulse has:

$$\varphi(\omega) = 0$$

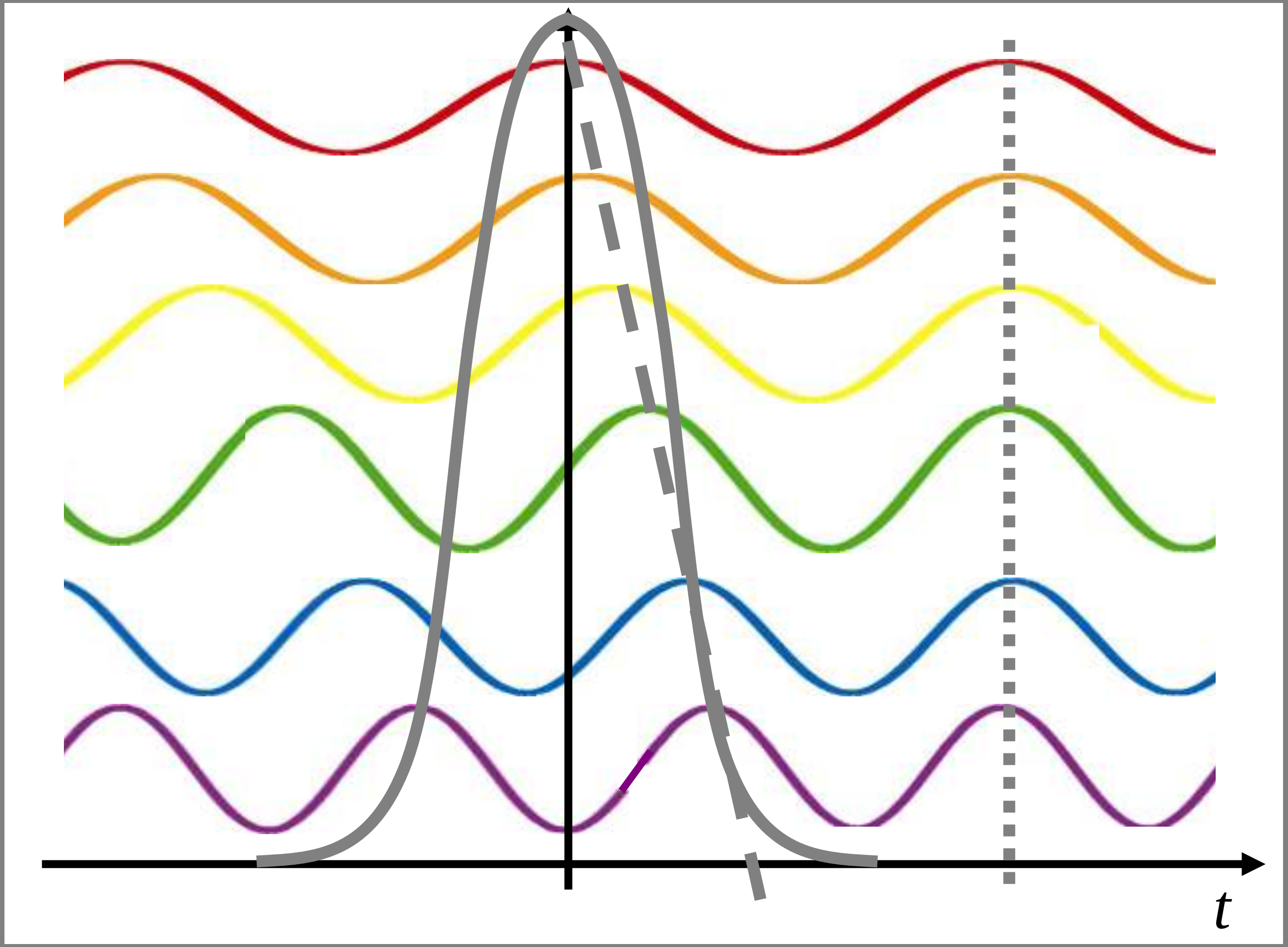
Note that this wave-form sees constructive interference, and hence peaks, at $t = 0$.

And it has cancellation everywhere else.

Now try a linear spectral phase: $\varphi(\omega) = a\omega$.

By the Shift Theorem, a linear spectral phase is just a delay in time.

And this is what occurs!



$\varphi(\omega_1) = 0$

$\varphi(\omega_2) = 0.2\ \pi$

$\varphi(\omega_3) = 0.4\ \pi$

$\varphi(\omega_4) = 0.6\ \pi$

$\varphi(\omega_5) = 0.8\ \pi$

$\varphi(\omega_6) = \pi$

Transforming between wavelength and frequency

The spectrum and spectral phase vs. frequency differ from the spectrum and spectral phase vs. wavelength.

$$\omega = \frac{2\pi c}{\lambda}$$

The spectral phase is easily transformed:

$$\boxed{\varphi_\lambda(\lambda) \; = \; \varphi_\omega(2\pi c / \lambda)}$$

To transform the spectrum, note that, because the energy is the same, whether we integrate the spectrum over frequency or wavelength:

$$\int_{-\infty}^{\infty} S_\lambda(\lambda)\, d\lambda \; = \; \int_{-\infty}^{\infty} S_\omega(\omega)\, d\omega$$

Changing variables:

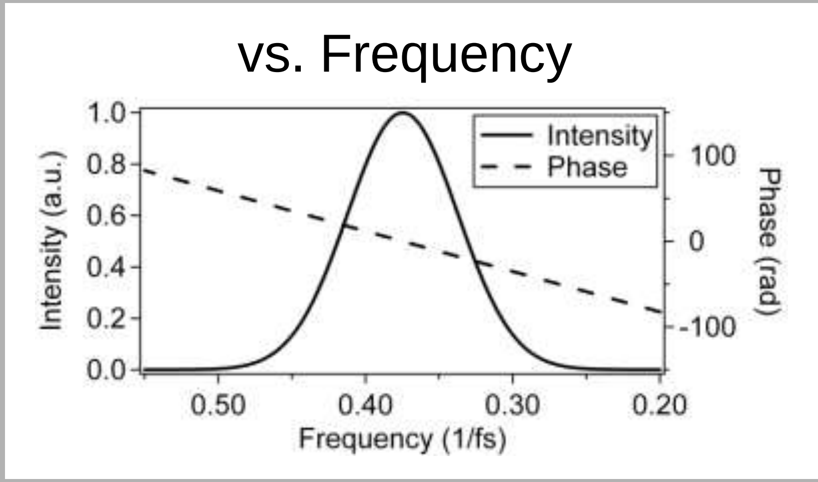
$$= \int_{\infty}^{-\infty} S_\omega(2\pi c / \lambda)\, \frac{-2\pi c}{\lambda^2}\, d\lambda \; = \; \int_{-\infty}^{\infty} S_\omega(2\pi c / \lambda)\, \frac{2\pi c}{\lambda^2}\, d\lambda$$

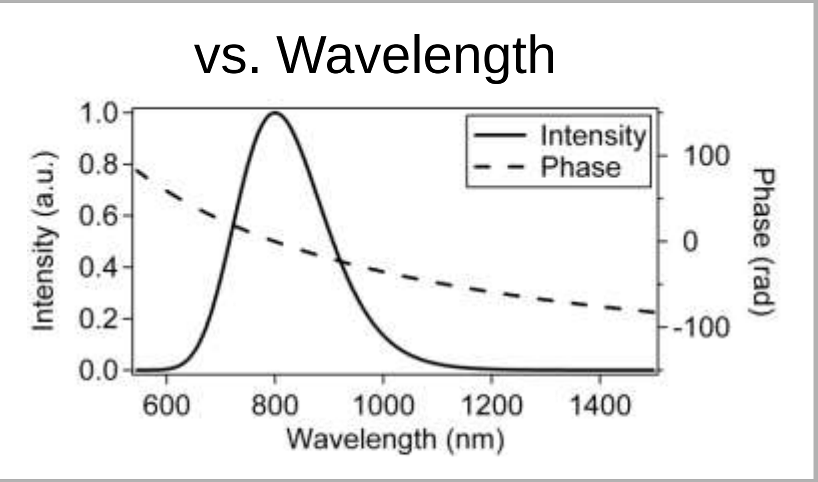
$$\frac{d\omega}{d\lambda} = \frac{-2\pi c}{\lambda^2}$$

$$\Rightarrow \quad \boxed{S_\lambda(\lambda) \; = \; S_\omega(2\pi c / \lambda)\, \frac{2\pi c}{\lambda^2}}$$

The spectrum and spectral phase vs. wavelength and frequency

Example: A Gaussian spectrum with a linear phase vs. frequency





Note the different shapes of the spectrum when plotted vs. wavelength and frequency.

Bandwidth in various units

By the Uncertainty Principle, a 1-ps pulse has a bandwidth, $\delta\nu$, of ~1/2 THz. But what is this in s^{-1}? In cm^{-1}? And in nm?

In angular frequency units, $\delta\omega = 2\pi\ \delta\nu$, so it's π x 10^{12} s^{-1}.

In wave numbers, (cm^{-1}), we can write:

$$\nu = \frac{c}{\lambda} \qquad \frac{d\nu}{d(1/\lambda)} = c \qquad \delta(1/\lambda) = \delta\nu \Big/ \frac{d\nu}{d(1/\lambda)} = \delta\nu / c$$

So $\delta(1/\lambda) = (0.5 \text{ x } 10^{12} \text{ /s}) / 3 \text{ x } 10^{10} \text{ cm/s} = 17 \text{ cm}^{-1}$

In nm, we can write:

$$\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} \qquad \delta\lambda = \delta\nu \Big/ \frac{d\nu}{d\lambda} = \delta\nu \frac{\lambda^2}{c}$$

Assuming an 800-nm wavelength:

$$\delta\lambda = 0.5 \text{x} 10^{12} \text{/s} \ \frac{(800 \text{ nm})(.8 \text{x} 10^{-4} \text{cm})}{3 \text{x} 10^{10} \text{cm/s}} = 1 \text{ nm}$$

The Instantaneous frequency

The temporal phase, $\phi(t)$, contains frequency-vs.-time information.

The pulse *instantaneous angular frequency*, $\omega_{inst}(t)$, is defined as:

$$\boxed{\omega_{inst}(t) \equiv \omega_0 - \frac{d\phi}{dt}}$$

This is easy to see. At some time, t, consider the total phase of the wave. Call this quantity ϕ_0:

$$\phi_0 = \omega_0 t - \phi(t)$$

Exactly one period, T, later, the total phase will (by definition) increase to $\phi_0 + 2\pi$:

$$\phi_0 + 2\pi = \omega_0 [t + T] - \phi(t + T)$$

where $\phi(t+T)$ is the slowly varying phase at the time, $t+T$. Subtracting these two equations:

$$2\pi = \omega_0 T - [\phi(t + T) - \phi(t)]$$

Instantaneous frequency (cont'd)

Dividing by T and recognizing that $2\pi/T$ is a frequency, call it $\omega_{inst}(t)$:

$$\omega_{inst}(t) = 2\pi/T = \omega_0 - [\phi(t+T) - \phi(t)] / T$$

But T is small, so $[\phi(t+T)-\phi(t)] / T$ is the derivative, $d\phi/dt$.

So we're done!

Usually, however, we'll think in terms of the *instantaneous frequency*, $\nu_{inst}(t)$, so we'll need to divide by 2π:

$$\boxed{\nu_{inst}(t) = \nu_0 - [d\phi/dt] / 2\pi}$$

While the instantaneous frequency isn't always a rigorous quantity, it's fine for ultrashort pulses, which have broad bandwidths.

Group delay

While the temporal phase contains frequency-vs.-time information, the spectral phase contains time-vs.-frequency information.

So we can define the *group delay vs. frequency*, $\tau_{gr}(\omega)$, given by:

$$\boxed{\tau_{gr}(\omega) = d\varphi \,/\, d\omega}$$

A similar derivation to that for the instantaneous frequency can show that this definition is reasonable.

Also, we'll typically use this result, which is a real time (the rad's cancel out), and never $d\varphi/d\nu$, which isn't.

Always remember that $\tau_{gr}(\omega)$ is *not* the inverse of $\omega_{inst}(t)$.

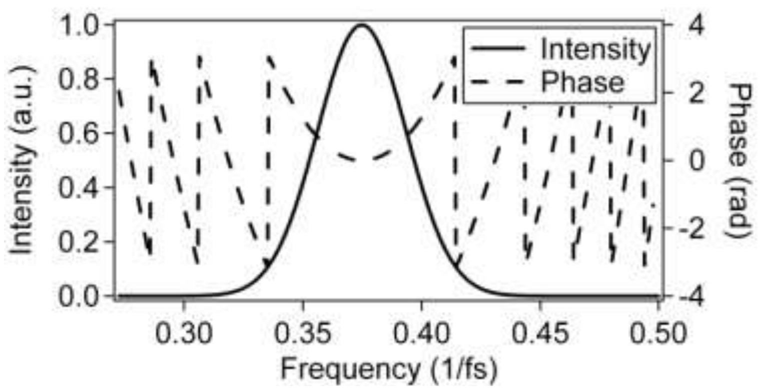
Phase wrapping and unwrapping

Technically, the phase ranges from $-\pi$ to π. But it often helps to "unwrap" it. This involves adding or subtracting 2π whenever there's a 2π phase jump.

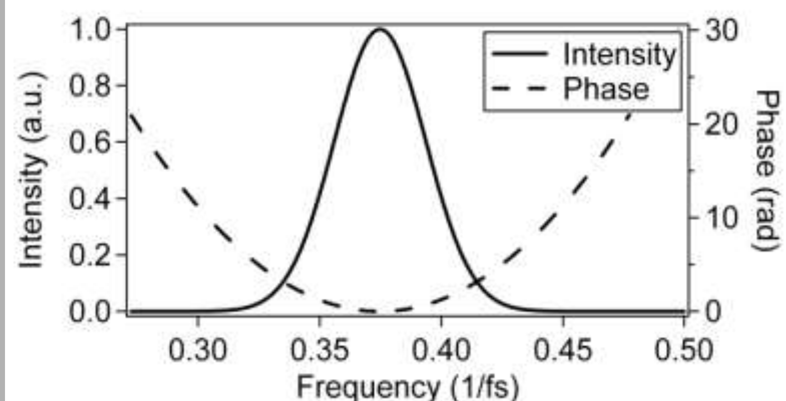
Example: a pulse with quadratic phase

Note the scale!

Wrapped phase



Unwrapped phase



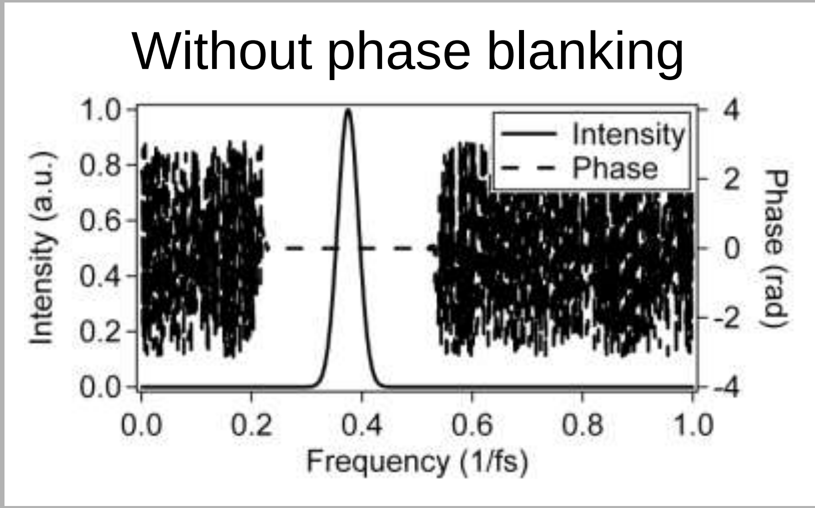
The main reason for unwrapping the phase is aesthetics.

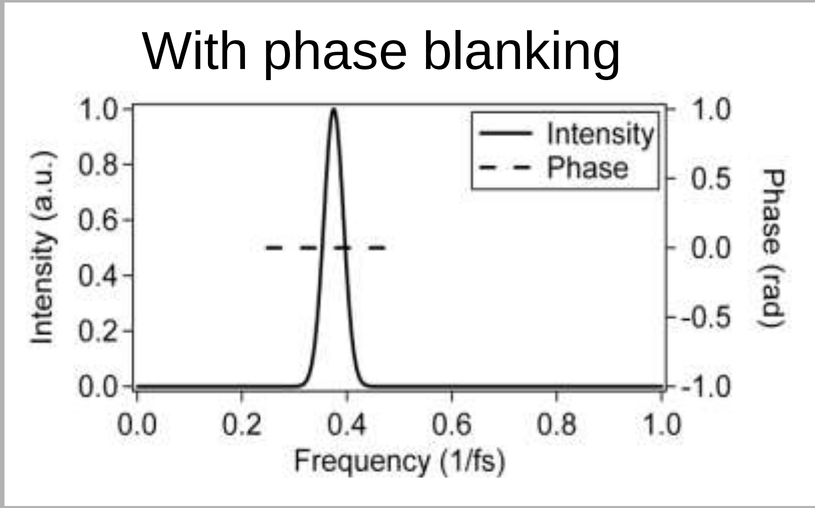
Phase-blanking

When the intensity is zero, the phase is meaningless.

When the intensity is nearly zero, the phase is nearly meaningless.

Phase-blanking involves simply not plotting the phase when the intensity is close to zero.





The only problem with phase-blanking is that you have to decide the intensity level below which the phase is meaningless.

Phase Taylor Series expansions

We can write a Taylor series for the phase, $\phi(t)$, about the time $t = 0$:

$$\phi(t) = \phi_0 + \phi_1 \frac{t}{1!} + \phi_2 \frac{t^2}{2!} + \ldots$$

where

$$\phi_1 = \left.\frac{d\phi}{dt}\right|_{t=0}$$ is related to the instantaneous frequency.

where only the first few terms are typically required to describe well-behaved pulses. Of course, we'll consider badly behaved pulses, which have higher-order terms in $\phi(t)$.

Expanding the phase in time is not common because it's hard to measure the intensity vs. time, so we'd have to expand it, too.

Frequency-domain phase expansion

It's more common to write a Taylor series for $\varphi(\omega)$:

$$\boxed{\varphi(\omega) = \varphi_0 + \varphi_1 \frac{\omega - \omega_0}{1!} + \varphi_2 \frac{(\omega - \omega_0)^2}{2!} + \ldots}$$

where

$$\varphi_1 = \left.\frac{d\varphi}{d\omega}\right|_{\omega=\omega_0}$$ is the group delay!

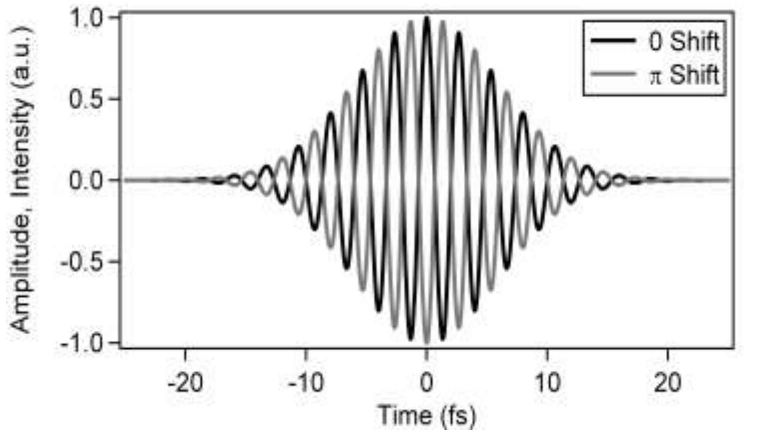
$$\varphi_2 = \left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega=\omega_0}$$ is called the "**group-delay dispersion**."

As in the time domain, only the first few terms are typically required to describe well-behaved pulses. Of course, we'll consider badly behaved pulses, which have higher-order terms in $\varphi(\omega)$.

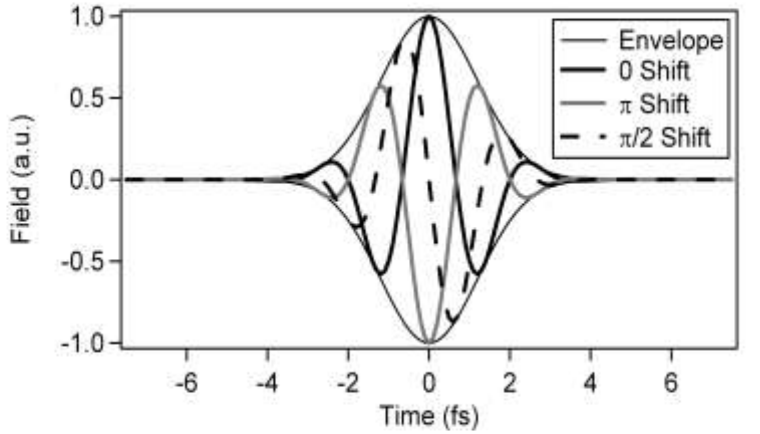
Zeroth-order phase: the absolute phase

An absolute phase of $\pi/2$ will turn a cosine carrier wave into a sine. It's usually irrelevant, unless the pulse is only a cycle or so long.

Different absolute phases for a four-cycle pulse



Different absolute phases for a single-cycle pulse

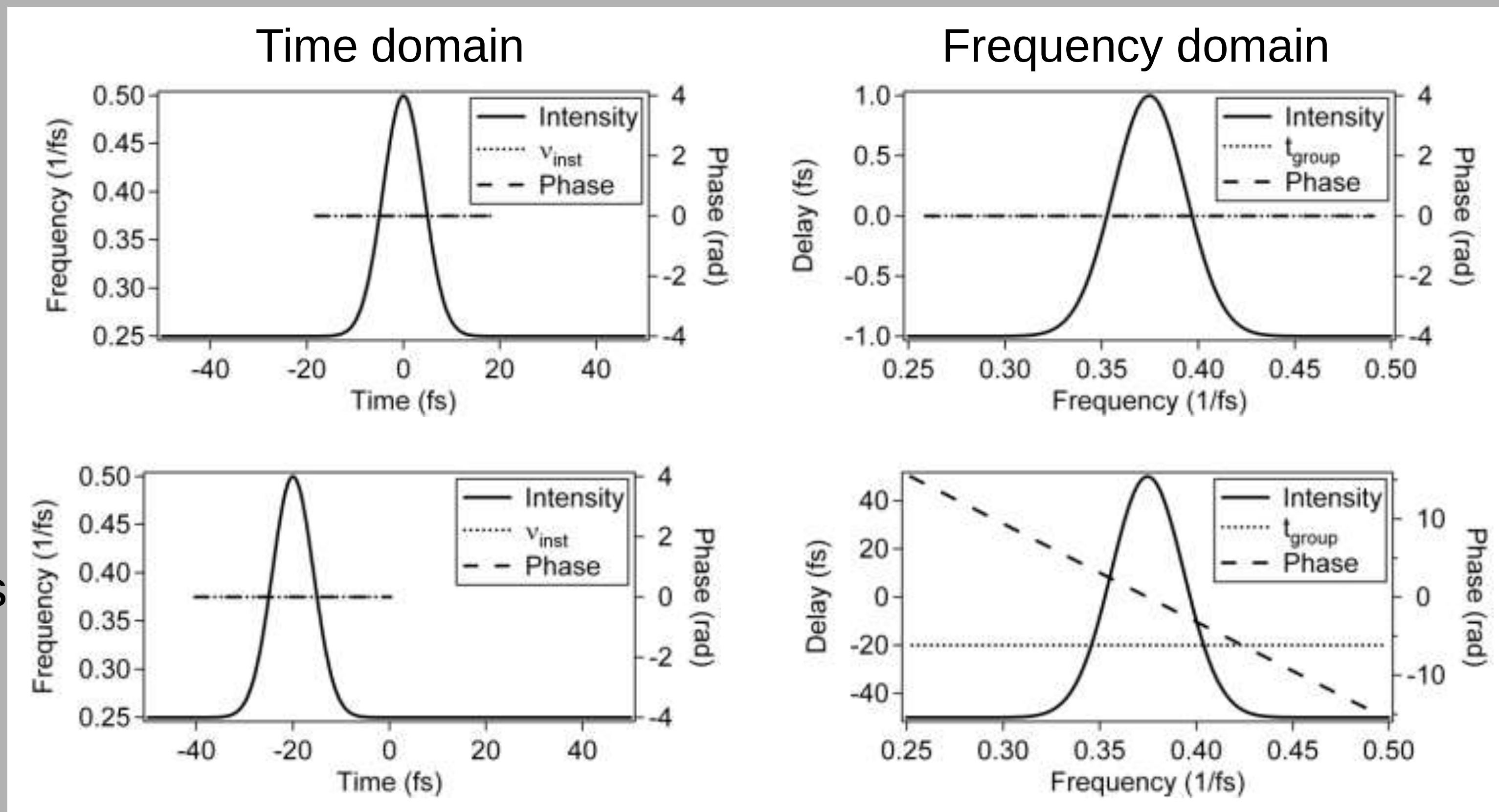


Notice that the two four-cycle pulses look alike, but the three single-cycle pulses are all quite different.

First-order phase in frequency: a shift in time

$\varphi_1 = 0$

$\varphi_1 = -20\,\text{fs}$

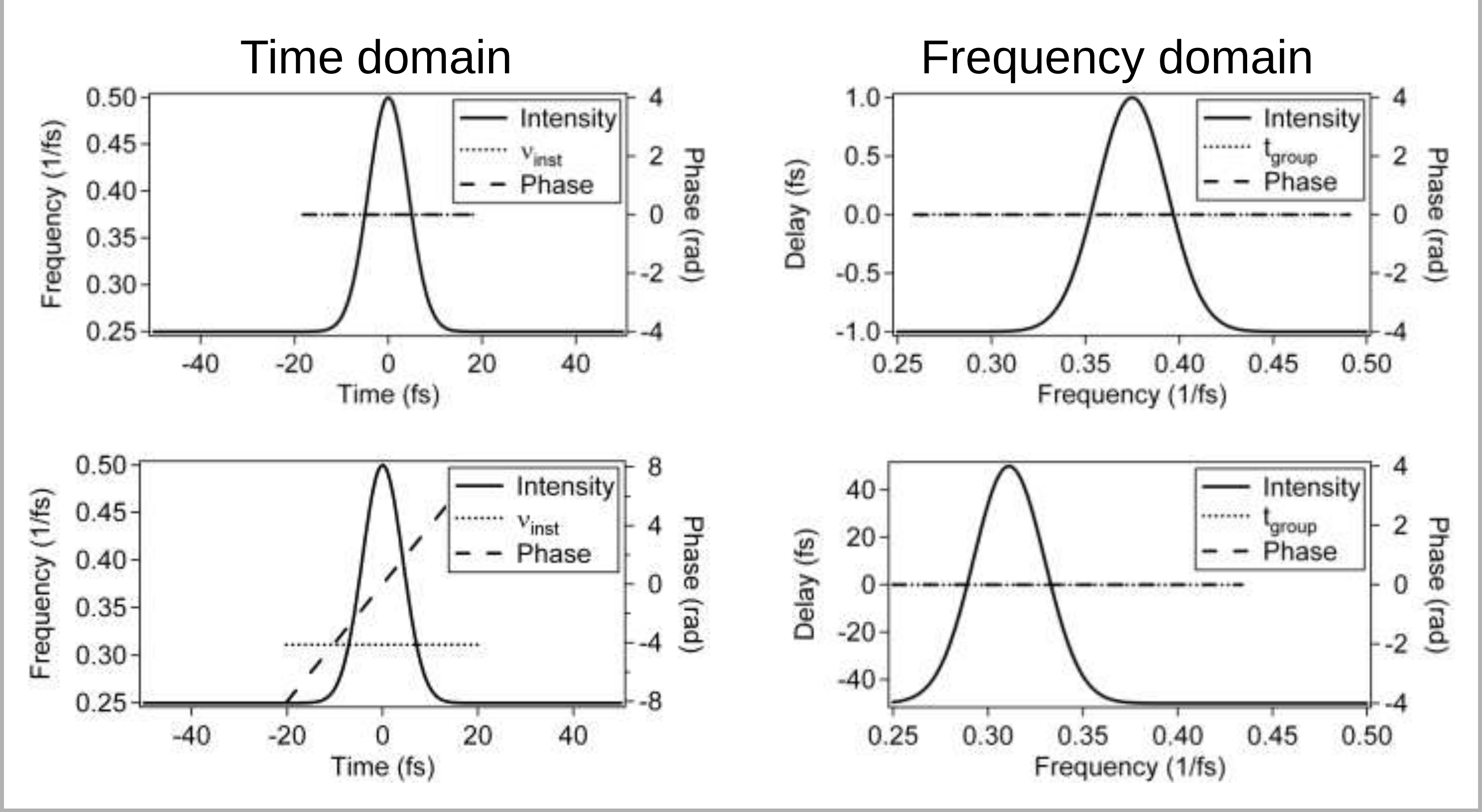


Note that φ_1 does not affect the instantaneous frequency, but the group delay = φ_1.

First-order phase in time: a frequency shift

$\phi_1 = 0/\text{fs}$

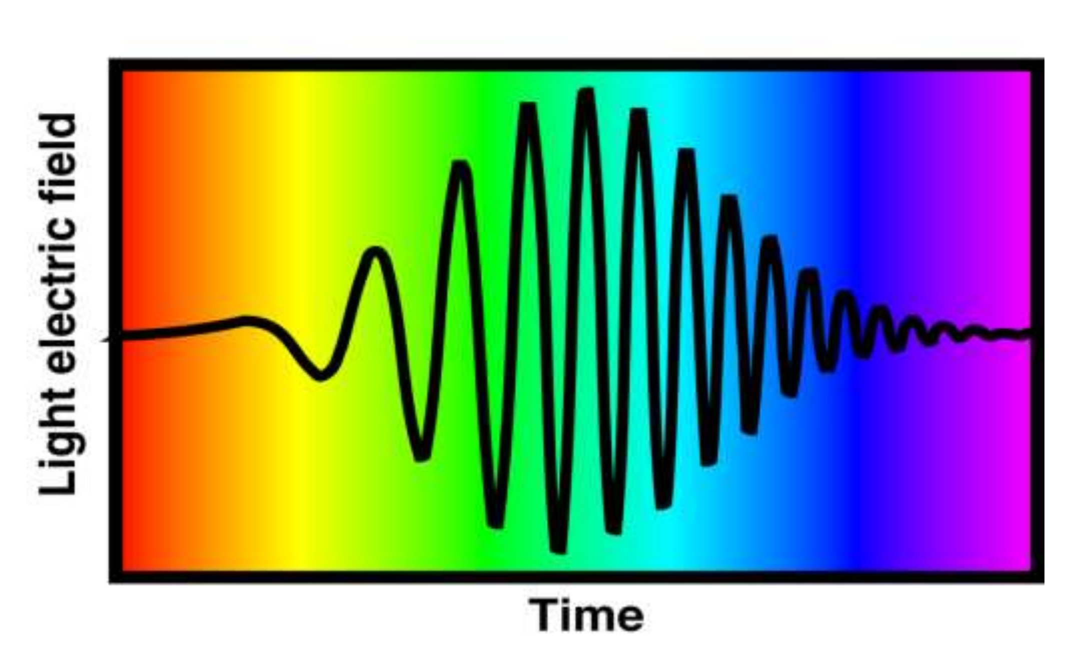
$\phi_1 = -.07/\text{fs}$



Note that ϕ_1 does not affect the group delay, but it does affect the instantaneous frequency = $-\phi_1$.

Second-order phase: the linearly chirped pulse

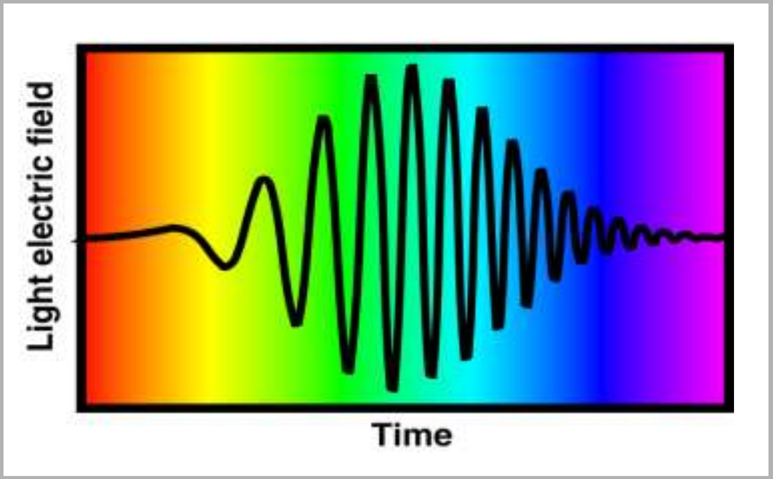
A pulse can have a frequency that varies in time.



This pulse increases its frequency linearly in time (from red to blue).

In analogy to bird sounds, this pulse is called a "chirped" pulse.

The linearly chirped Gaussian pulse

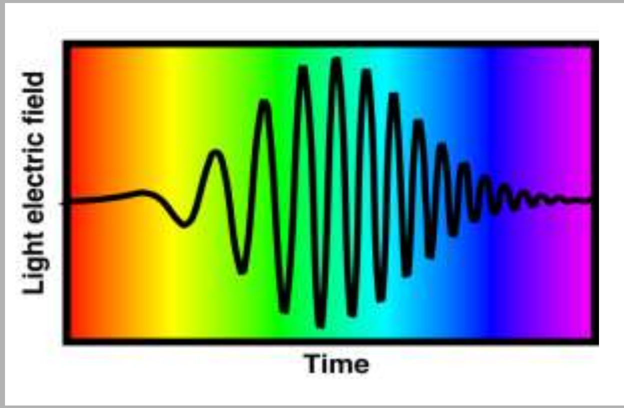


We can write a linearly chirped Gaussian pulse mathematically as:

$$E(t) = E_0 \underbrace{\exp\left[-(t/\tau_G)^2\right]}_{\text{Gaussian amplitude}} \exp\left[i\left(\underbrace{\omega_0 t}_{\text{Carrier wave}} + \underbrace{\beta t^2}_{\text{Chirp}}\right)\right]$$

Note that for $\beta > 0$, when $t < 0$, the two terms partially cancel, so the phase changes slowly with time (so the frequency is low). And when $t > 0$, the terms add, and the phase changes more rapidly (so the frequency is larger)

The instantaneous frequency vs. time for a chirped pulse



A chirped pulse has:

$$E(t) \propto \exp\left[i\left(\omega_0 t - \phi(t) \right) \right]$$

where:

$$\phi(t) = -\beta t^2$$

The instantaneous frequency is: $\omega_{inst}(t) \equiv \omega_0 - d\phi / dt$

which is:

$$\omega_{inst}(t) = \omega_0 + 2\beta t$$

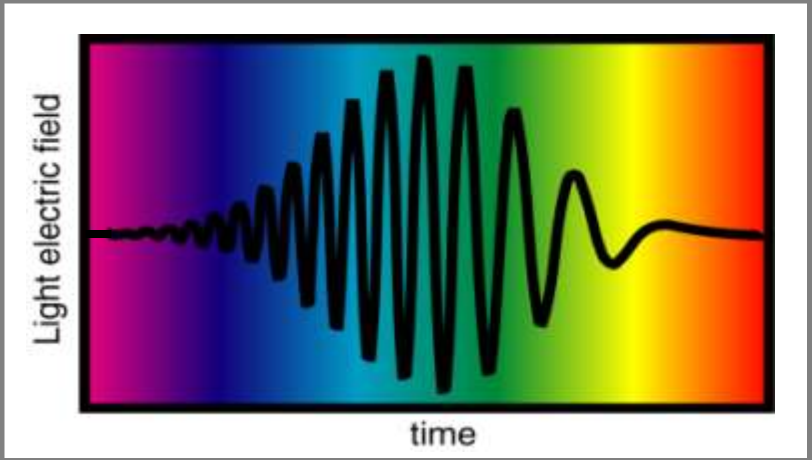
So the frequency increases linearly with time.

The negatively chirped pulse

We have been considering a pulse whose frequency *increases* linearly with time: a *positively* chirped pulse.

One can also have a *negatively* chirped (Gaussian) pulse, whose instantaneous frequency *decreases* with time.

We simply allow β to be negative in the expression for the pulse:



$$E(t) = E_0 \exp\left[-\left(t/\tau_G\right)^2\right]\exp\left[i\left(\omega_0 t + \beta t^2\right)\right]$$

And the instantaneous frequency will decrease with time:

$$\omega_{inst}(t) = \omega_0 + 2\beta t = \omega_0 - 2\left|\beta\right| t$$

Frequency-domain phase expansion

Recall the Taylor series for $\varphi(\omega)$:

$$\boxed{\varphi(\omega) = \varphi_0 + \varphi_1 \frac{\omega - \omega_0}{1!} + \varphi_2 \frac{(\omega - \omega_0)^2}{2!} + \ldots}$$

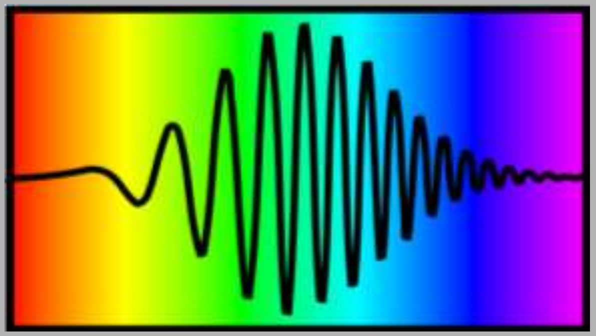
where

$$\varphi_1 = \left.\frac{d\varphi}{d\omega}\right|_{\omega=\omega_0}$$ is the group delay.

$$\varphi_2 = \left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega=\omega_0}$$ is called the "**group-delay dispersion**."

As in the time domain, only the first few terms are typically required to describe well-behaved pulses. Of course, we'll consider badly behaved pulses, which have higher-order terms in $\varphi(\omega)$.

The Fourier transform of a chirped pulse



Writing a linearly chirped Gaussian pulse as:

$$E(t) = E_0 \exp\left[-\alpha t^2\right] \exp\left[i\left(\omega_0 t + \beta t^2\right)\right]$$

or:

$$E(t) = E_0 \exp\left[-\left(\alpha - i\beta\right)t^2\right] \exp\left[i\left(\omega_0 t\right)\right]$$

A Gaussian with a complex width!

Fourier-Transforming yields:

$$\tilde{E}(\omega) = E_0 \exp\left[-\frac{1/4}{\alpha - i\beta}\left(\omega - \omega_0\right)^2\right]$$

A chirped Gaussian pulse Fourier-Transforms to itself!!!

Rationalizing the denominator and separating the real and imag parts:

$$\tilde{E}(\omega) = E_0 \exp\left[-\frac{\alpha/4}{\alpha^2 + \beta^2}\left(\omega - \omega_0\right)^2\right] \exp\left[-i\frac{\beta/4}{\alpha^2 + \beta^2}\left(\omega - \omega_0\right)^2\right]$$

The group delay vs. ω for a chirped pulse

The group delay of a wave is the derivative of the spectral phase:

$$\tau_g(\omega) \equiv d\varphi / d\omega$$

For a linearly chirped Gaussian pulse, the spectral phase is:

$$\varphi(\omega) = \frac{\beta / 4}{\alpha^2 + \beta^2}\left(\omega - \omega_0\right)^2$$

So:

$$\tau_g = \frac{\beta / 2}{\alpha^2 + \beta^2}\left(\omega - \omega_0\right)$$

And the delay vs. frequency is linear.

This is not the inverse of the instantaneous frequency, which is:

$$\omega_{inst}(t) = \omega_0 + 2\beta t$$

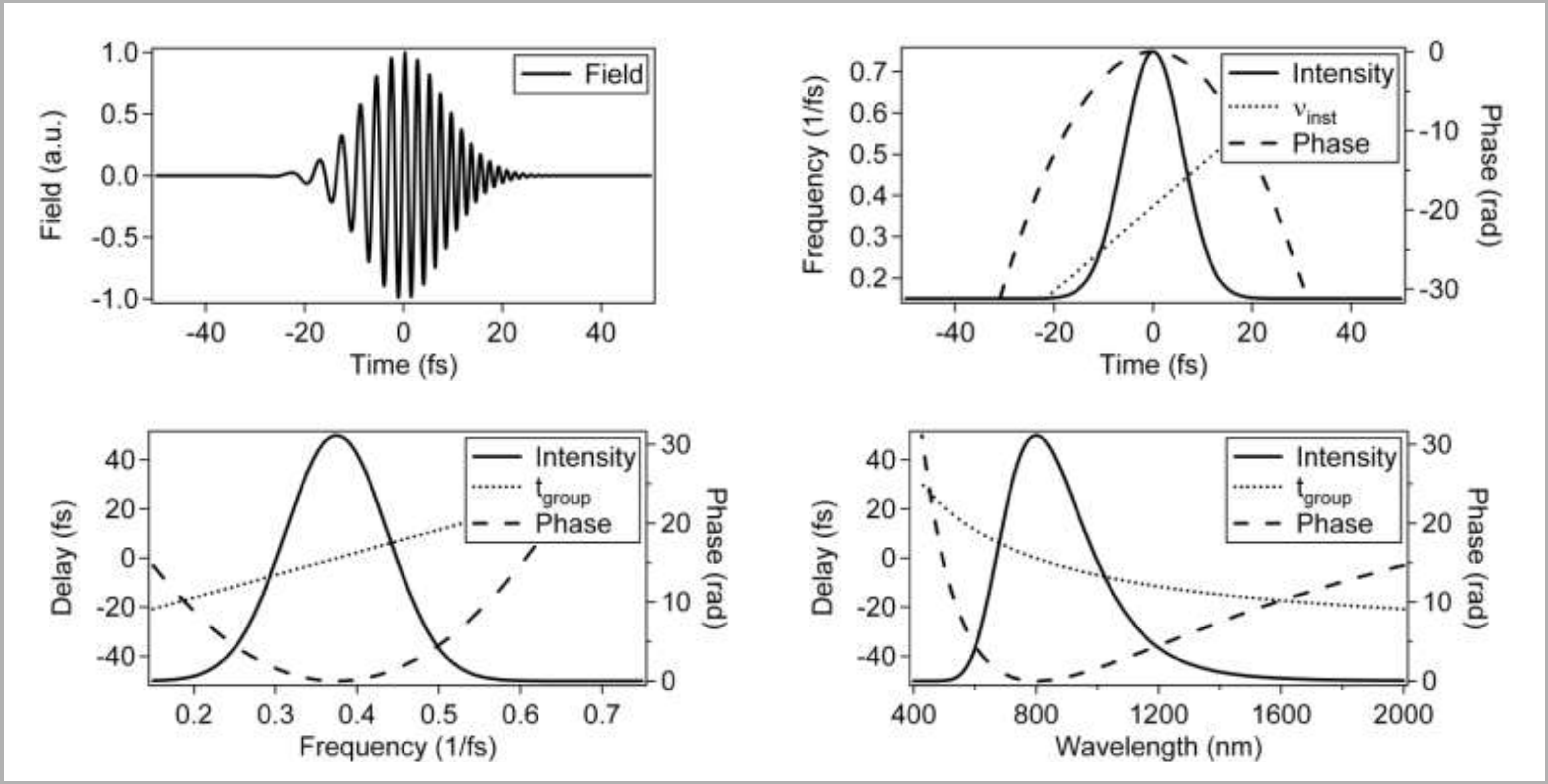
But when the pulse is long ($\alpha \to 0$):

$$\tau_g = \frac{1}{2\beta}\left(\omega - \omega_0\right)$$

which is the inverse of the instantaneous frequency vs. time.

2^{nd}-order phase: positive linear chirp

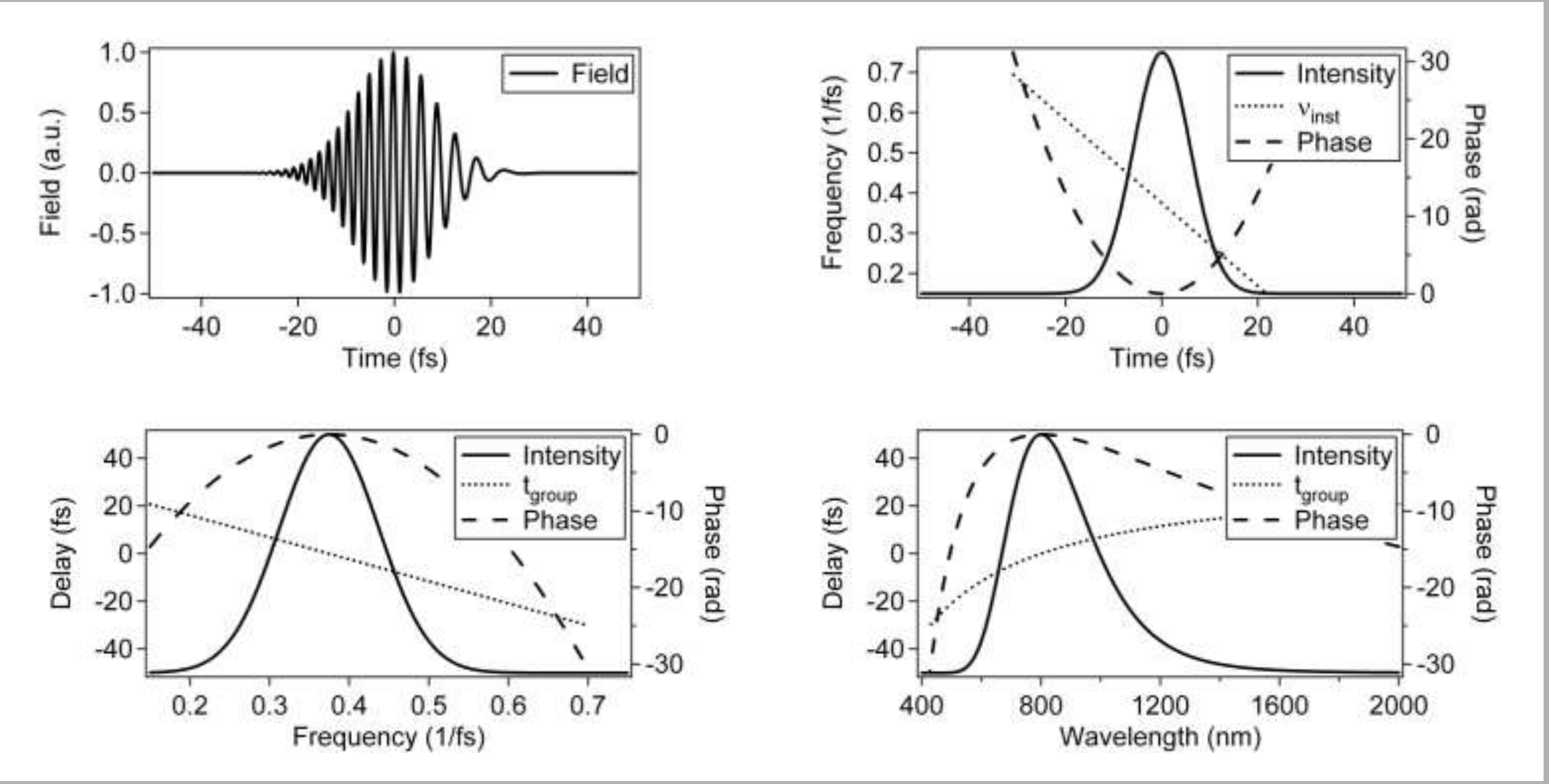
Numerical example: Gaussian-intensity pulse w/ positive linear chirp, ϕ_2 = –0.032 rad/fs^2 or φ_2 = 290 rad fs^2.



Here the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one.

2^{nd}-order phase: negative linear chirp

Numerical example: Gaussian-intensity pulse w/ negative linear chirp, $\phi_2 = 0.032$ rad/fs^2 or $\varphi_2 = -290$ rad fs^2.



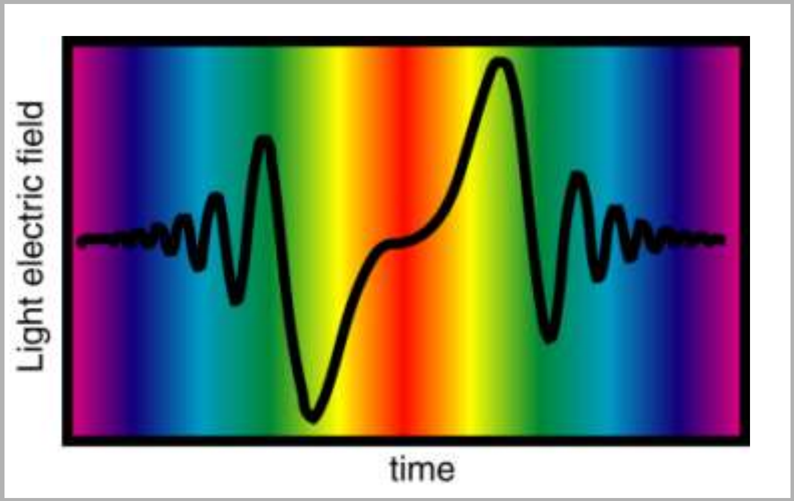
As with positive chirp, the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one.

Nonlinearly chirped pulses

The frequency of a light wave can also vary nonlinearly with time.

This is the electric field of a Gaussian pulse whose frequency varies quadratically with time:

$$\omega_{inst}(t) = \omega_0 + 3\gamma t^2$$



This light wave has the expression:

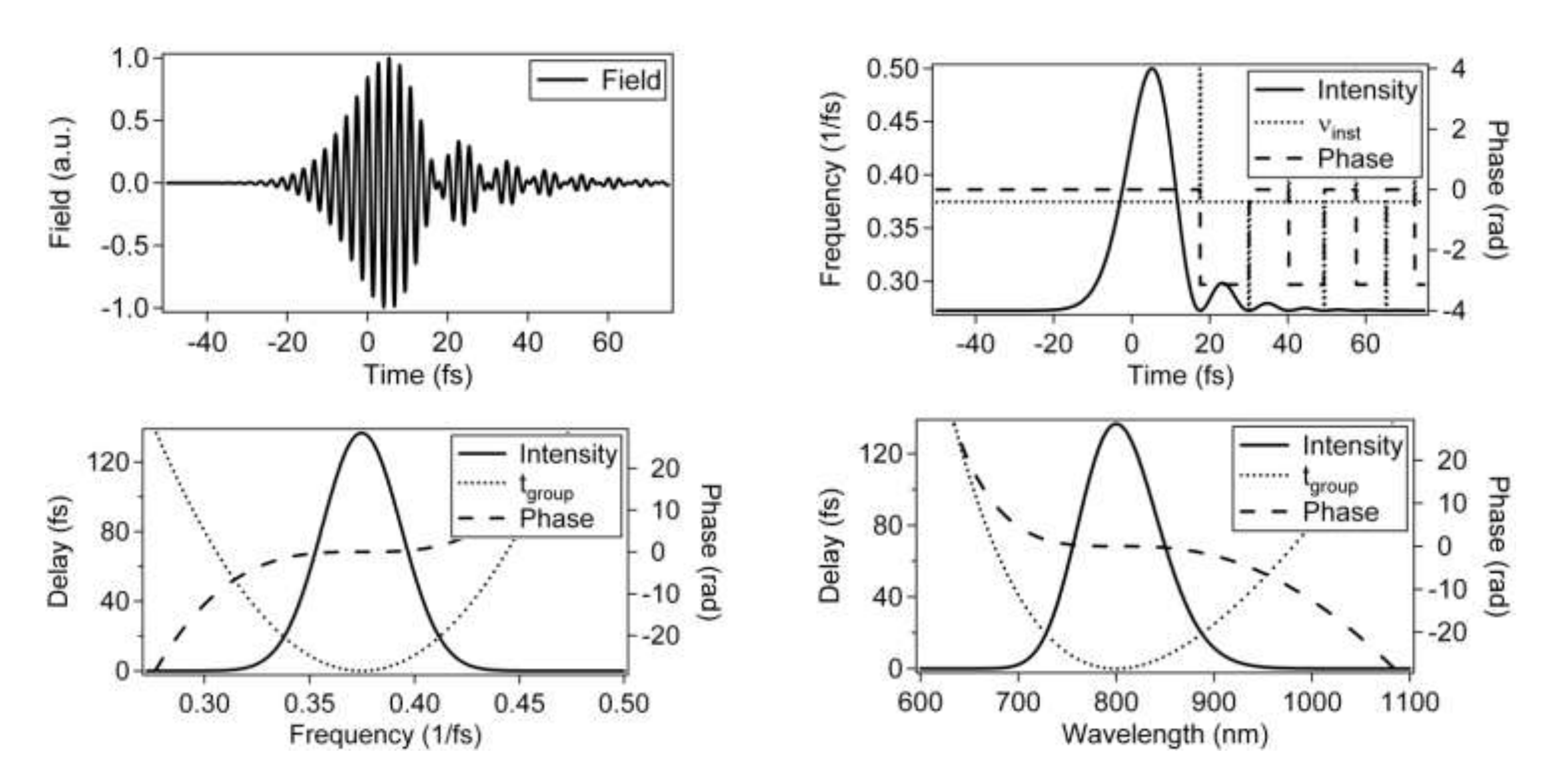
$$E(t) = \mathrm{Re}\, E_0 \exp\left[-\left(t/\tau_G\right)^2\right] \exp\left[i\left(\omega_0 t + \gamma t^3\right)\right]$$

Arbitrarily complex frequency-vs.-time behavior is possible.

But we usually describe phase distortions in the frequency domain.

3^{rd}-order spectral phase: quadratic chirp

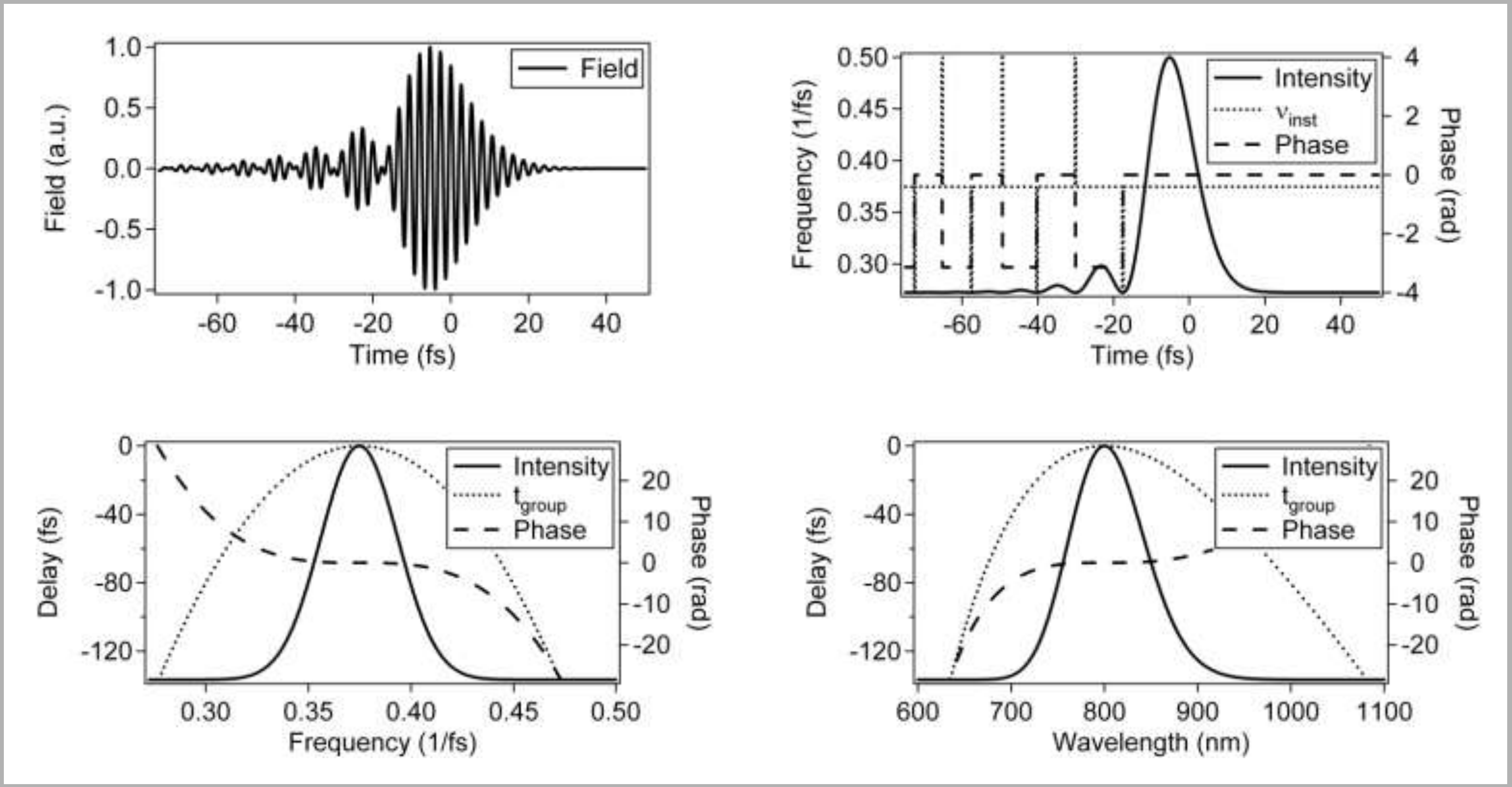
Numerical example: Gaussian spectrum and positive cubic spectral phase, with $\varphi_3 = 3\text{x}10^4$ rad fs^3



Trailing satellite pulses in time indicate positive spectral cubic phase.

Negative 3rd-order spectral phase

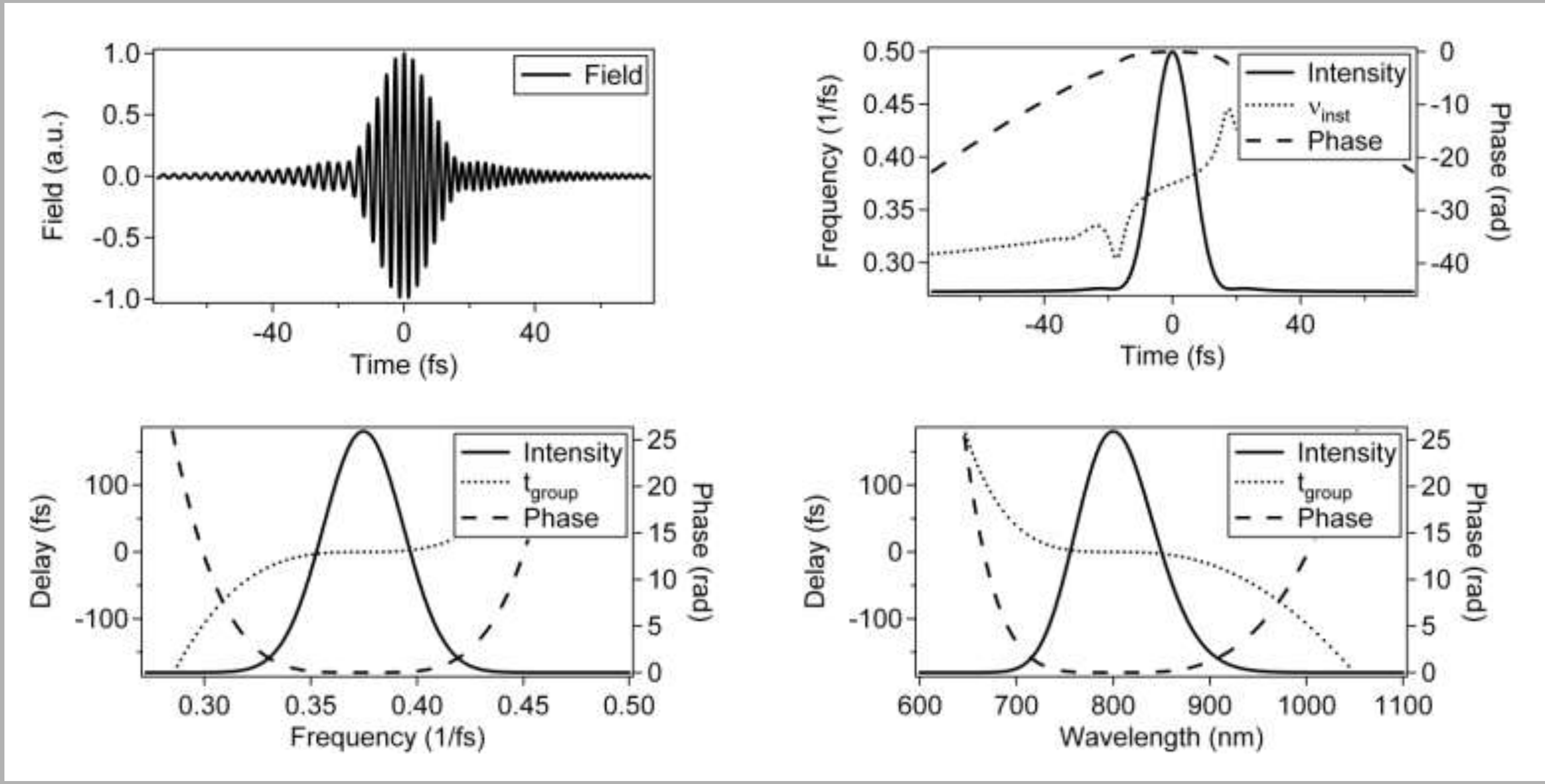
Another numerical example: Gaussian spectrum and negative cubic spectral phase, with $\varphi_3 = -3\text{x}10^4$ rad fs^3



Leading satellite pulses in time indicate negative spectral cubic phase.

4^{th}-order spectral phase

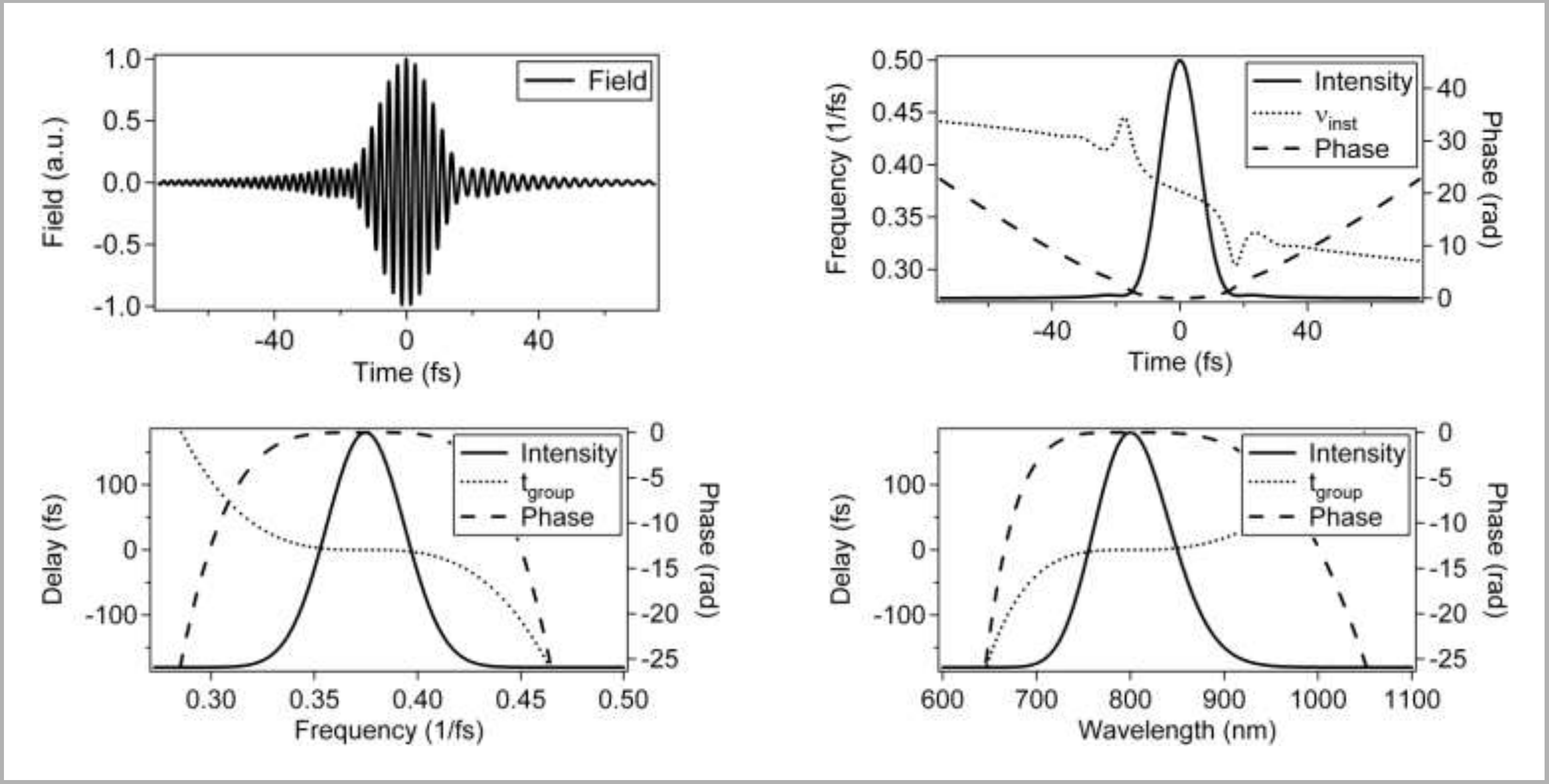
Numerical example: Gaussian spectrum and positive quartic spectral phase, $\varphi_4 = 4 \times 10^5$ rad fs^4.



Leading and trailing wings in time indicate quartic phase. Higher-frequencies in the trailing wing mean positive quartic phase.

Negative 4th-order spectral phase

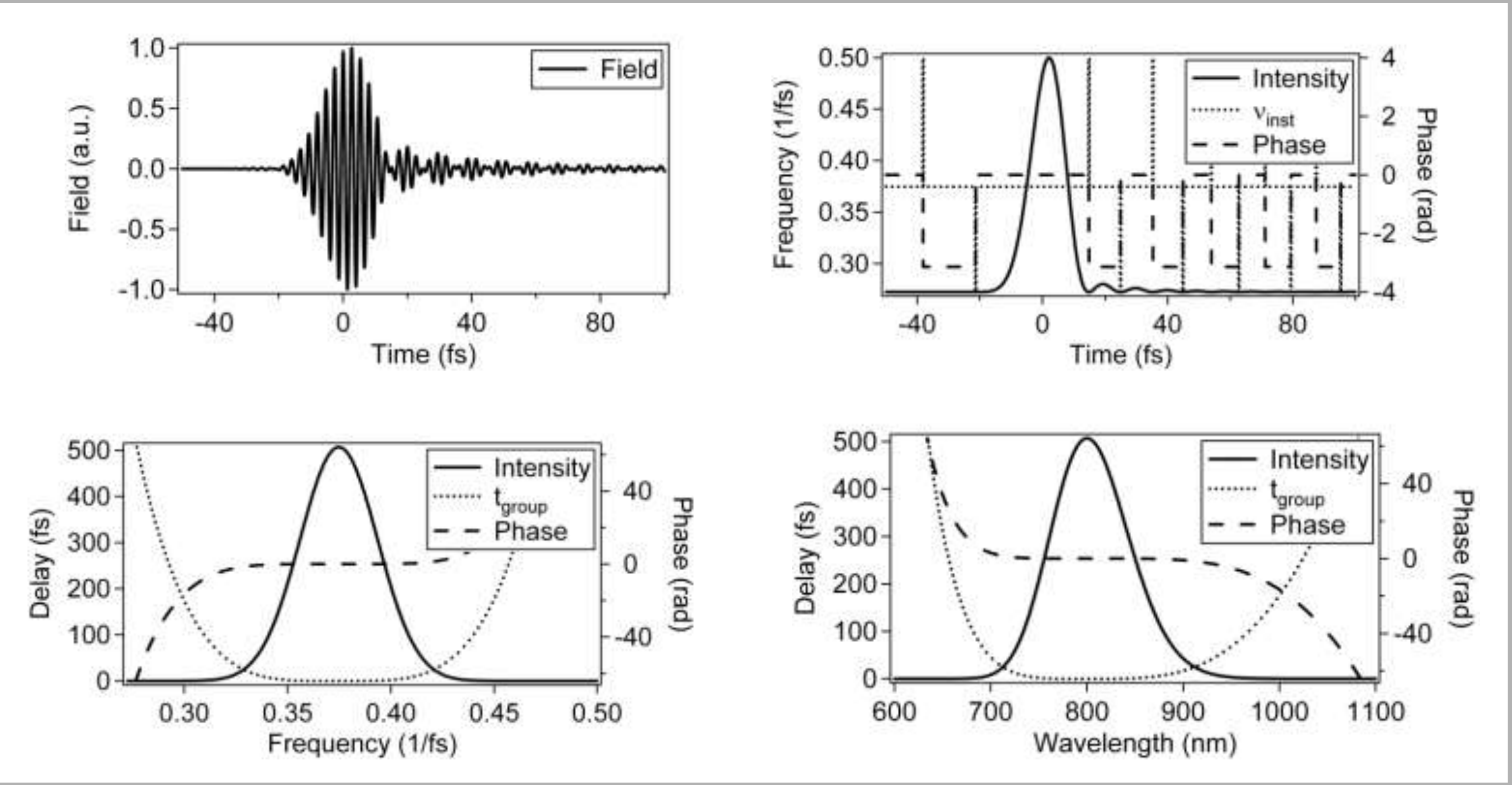
Numerical example: Gaussian spectrum and negative quartic spectral phase, $\varphi_4 = -4\times10^5$ rad fs^4.



Leading and trailing wings in time indicate quartic phase. Higher-frequencies in the leading wing mean negative quartic phase.

5^{th}-order spectral phase

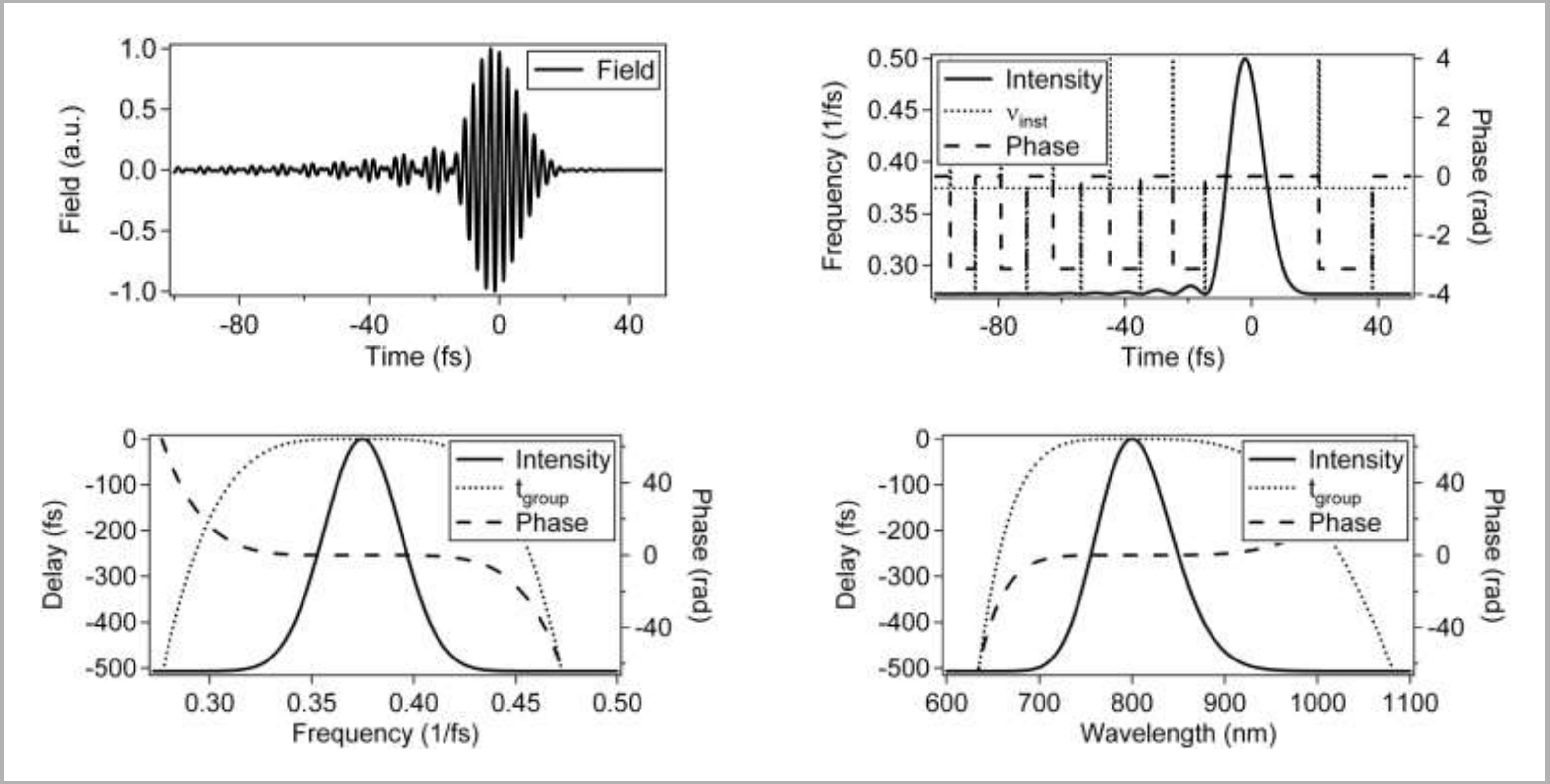
Numerical example: Gaussian spectrum and positive quintic spectral phase, $\varphi_5 = 7\text{x}10^6$ rad fs^5.



An oscillatory trailing wing in time indicates positive quintic phase.

Negative 5th-order spectral phase

Numerical example: Gaussian spectrum and negative quintic spectral phase, $\varphi_5 = -7\text{x}10^6$ rad fs^5.

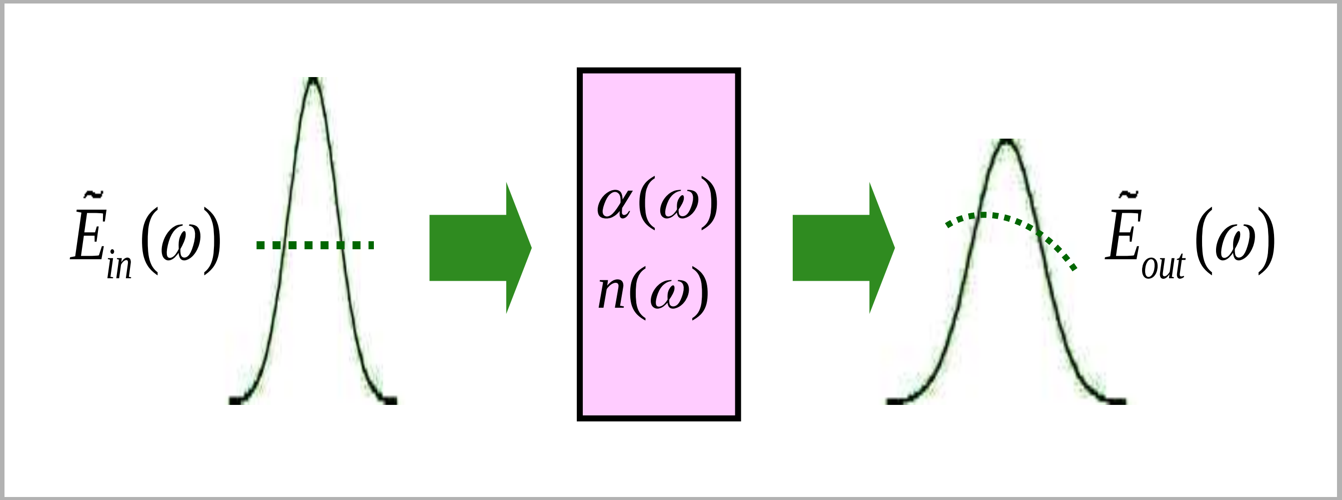


An oscillatory leading wing in time indicates negative quintic phase.

Pulse propagation

What happens to a pulse as it propagates through a medium?

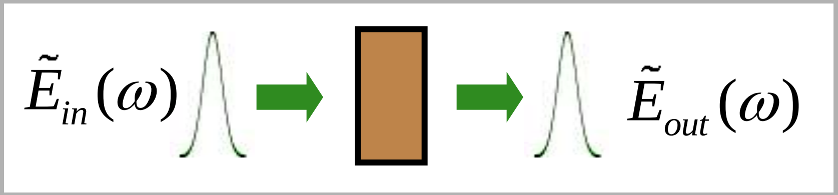
Always model (linear) propagation in the **frequency domain**. Also, you must know the entire field (i.e., the intensity and phase) to do so.



$$\boxed{\tilde{E}_{out}(\omega) \;=\; \tilde{E}_{in}(\omega)\exp[-\alpha(\omega)L/2]\exp[-i\,n(\omega)\,k\,L]}$$

In the time domain, propagation is a convolution—much harder.

Pulse propagation (continued)



Rewriting this expression:

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\exp[-\alpha(\omega)L/2]\exp[-i\,n(\omega)\,k\,L]$$

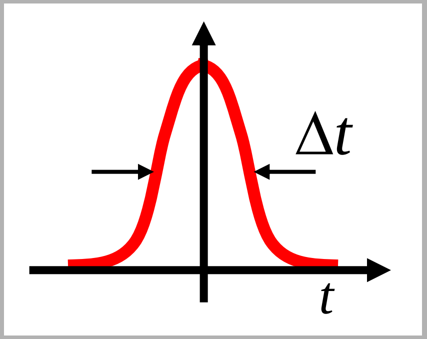
using $k = \omega/c$:

$$= \tilde{E}_{in}(\omega)\exp[-\alpha(\omega)L/2]\exp[-i\,n(\omega)\,\frac{\omega}{c}\,L]$$

Separating out the spectrum and spectral phase:

$$\boxed{\begin{aligned} S_{out}(\omega) &= S_{in}(\omega)\exp[-\alpha(\omega)L] \\ \varphi_{out}(\omega) &= \varphi_{in}(\omega) + n(\omega)\,\frac{\omega}{c}\,L \end{aligned}}$$

The pulse width



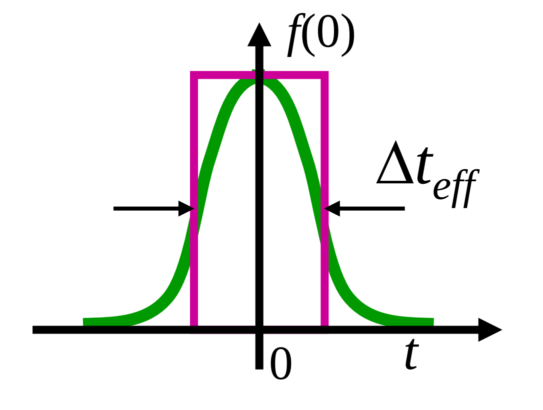
There are many definitions of the "width" or "length" of a wave or pulse.

The "**effective width**" is the width of a rectangle whose ***height*** and ***area*** are the same as those of the pulse.

Effective width ≡ Area / height:

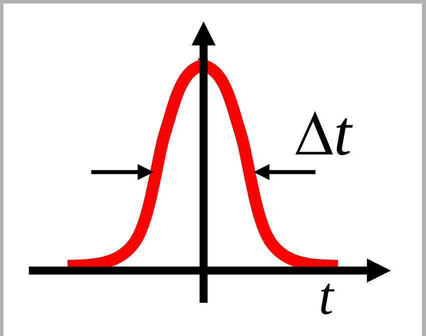
$$\Delta t_{eff} \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt$$

(Abs value is unnecessary for intensity.)



Advantage: It's easy to understand.
Disadvantages: The Abs value is inconvenient.
We must integrate to ± ∞.

The rms pulse width



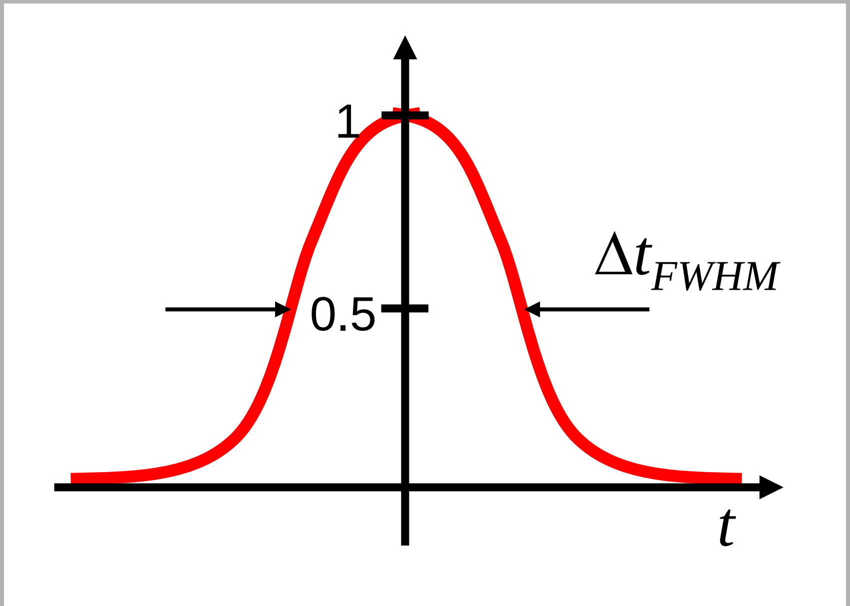
The "**root-mean-squared width**"
or "**rms width**:"

$$\Delta t_{rms} \equiv \left[\frac{\int_{-\infty}^{\infty} t^2 f(t)\, dt}{\int_{-\infty}^{\infty} f(t)\, dt} \right]^{1/2}$$

The rms width is the "second-order moment."

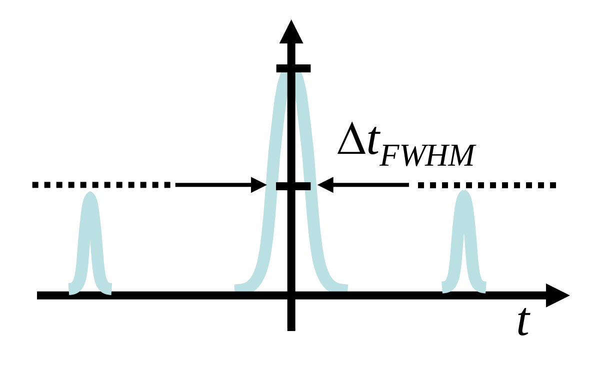
Advantages: Integrals are often easy to do analytically.
Disadvantages: It weights wings even more heavily,
so it's difficult to use for experiments, which can't scan to $\pm\infty$)

The Full-Width-Half-Maximum



"**Full-width-half-maximum**" is the distance between the half-maximum points.

Advantages: Experimentally easy.
Disadvantages: It ignores satellite pulses with heights < 49.99% of the peak!



Also: we can define these widths in terms of f(t) or of its intensity,$|f(t)|^2$.
Define *spectral* widths ($\Delta\omega$) similarly in the frequency domain ($t \rightarrow \omega$).

The Uncertainty Principle

The Uncertainty Principle says that the product of a function's widths in the time domain (Δt) and the frequency domain ($\Delta\omega$) has a minimum.

Define the widths assuming $f(t)$ and $F(\omega)$ peak at 0:

$$\Delta t \equiv \frac{1}{f(0)} \int_{-\infty}^{\infty} \left| f(t) \right| dt \qquad \Delta\omega \equiv \frac{1}{F(0)} \int_{-\infty}^{\infty} \left| F(\omega) \right| d\omega$$

$$\Delta t \geq \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) \exp(-i[0]t)\, dt = \frac{F(0)}{f(0)}$$

$$\Delta\omega \geq \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega)\, d\omega = \frac{1}{F(0)} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega[0])\, d\omega = \frac{2\pi f(0)}{F(0)}$$

Combining results:

$$\Delta\omega\, \Delta t \geq 2\pi \frac{f(0)}{F(0)} \frac{F(0)}{f(0)}$$

or:

$$\boxed{\Delta\omega\, \Delta t \geq 2\pi}$$

$$\boxed{\Delta\nu\, \Delta t \geq 1}$$

(Different definitions of the widths and the Fourier Transform yield different constants.)

The Time-Bandwidth Product

For a given wave, the product of the time-domain width (Δt) and the frequency-domain width ($\Delta \nu$) is the

Time-Bandwidth Product (TBP)

$$\boxed{\Delta\nu \;\; \Delta t \;\; \equiv} \; TBP$$

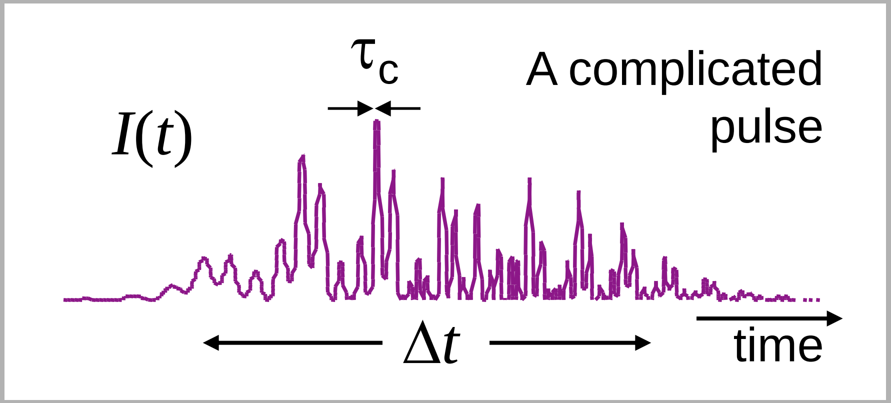
A pulse's TBP will always be greater than the theoretical minimum given by the Uncertainty Principle (for the appropriate width definition).

The TBP is a measure of how complex a wave or pulse is.

Even though every pulse's time-domain and frequency-domain functions are related by the Fourier Transform, a wave whose TBP is the theoretical minimum is called "*Fourier-Transform Limited*."

The Time-Bandwidth Product is a measure of the pulse complexity.

The coherence time ($\tau_c = 1/\Delta\nu$) indicates the smallest temporal structure of the pulse.



In terms of the coherence time:

$$TBP = \Delta\nu \; \Delta t = \Delta t \; / \; \tau_c$$

= about how many spikes are in the pulse

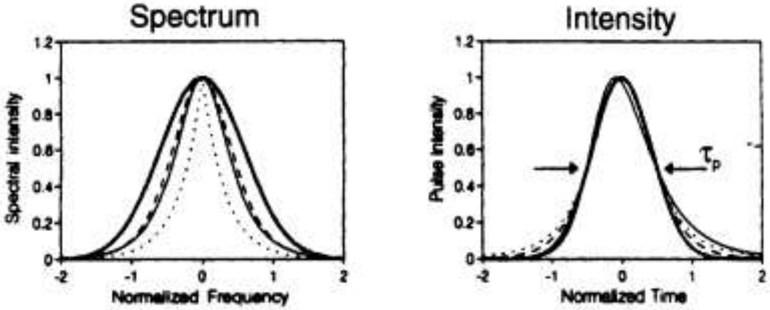
A similar argument can be made in the frequency domain, where the TBP is the ratio of the spectral width to the width of the smallest spectral structure.

Temporal and spectral shapes and TBPs of simple ultrashort pulses

Common pulse envelopes (with τ_p = Intensity FWHM):

Gaussian pulse	$\mathcal{E}(t)$	$\propto$	$\exp[-1.385(t/\tau_p)^2]$
sech - pulse	$\mathcal{E}(t)$	$\propto$	$\text{sech}[1.763(t/\tau_p)]$
Lorentzian pulse	$\mathcal{E}(t)$	$\propto$	$[1+1.656(t/\tau_p)^2]^{-1}$
asymm. sech pulse	$\mathcal{E}(t)$	$\propto$	$[\exp(t/\tau_p)+\exp(-3t/\tau_p)]^{-1}$

Spectra for pulses with the same pulse width

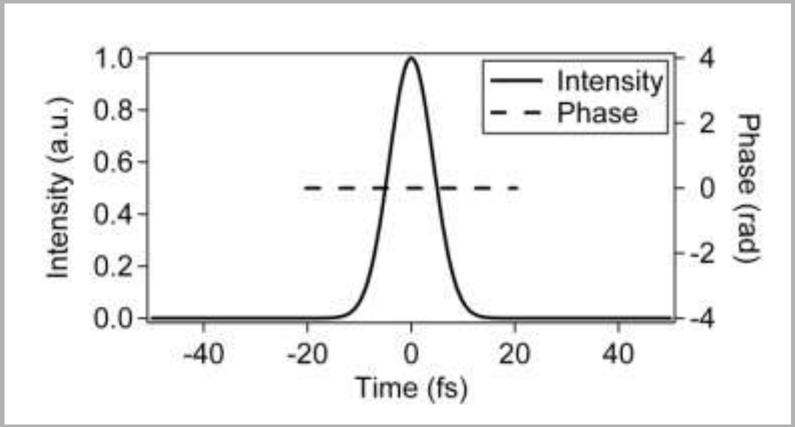


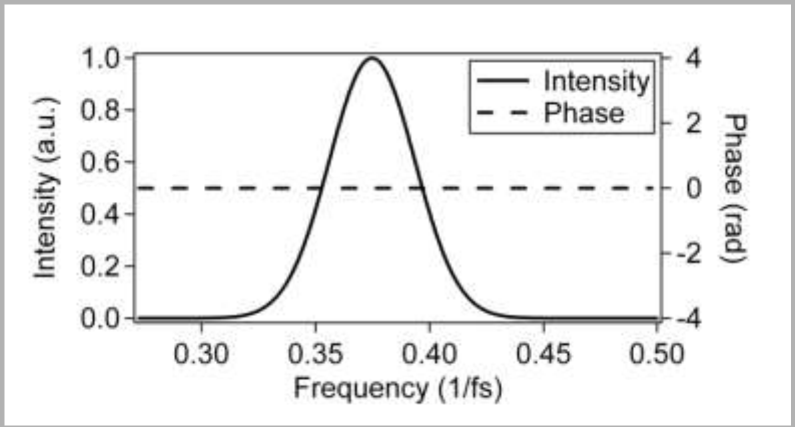
Field envelope	Intensity profile	τ_p (FWHM)	Spectral profile	$\Delta\omega_p$ (FWHM)	TBP
Gauss	$e^{-2(t/\tau_G)^2}$	$1.177\tau_G$	$e^{-(\omega\tau_G)^2/2}$	$2.355/\tau_G$	0.441
sech	$\text{sech}^2(t/\tau_s)$	$1.763\tau_s$	$\text{sech}^2(\pi\omega\tau_s/2)$	$1.122/\tau_s$	0.315
Lorentz	$[1+(t/\tau_L)^2]^{-2}$	$1.287\tau_L$	$e^{-2\lvert\omega\rvert\tau_L}$	$0.693/\tau_L$	0.142
asymm. sech	$\left[e^{t/\tau_a}+e^{-3t/\tau_a}\right]^{-2}$	$1.043\tau_a$	$\text{sech}(\pi\omega\tau_a/2)$	$1.677/\tau_a$	0.278
rectang.	1 for $\lvert t/\tau_r\rvert \le \frac{1}{2}$, 0 else	τ_r	$\text{sinc}^2(\omega\tau_r)$	$2.78/\tau_r$	0.443

Diels and Rudolph, Femtosecond Phenomena

Time-Bandwidth Product

Numerical example: A transform-limited pulse: A Gaussian-intensity pulse with constant phase and minimal TBP.





For the angular frequency and different definitions of the widths:

$TBP_{rms} = 0.5$

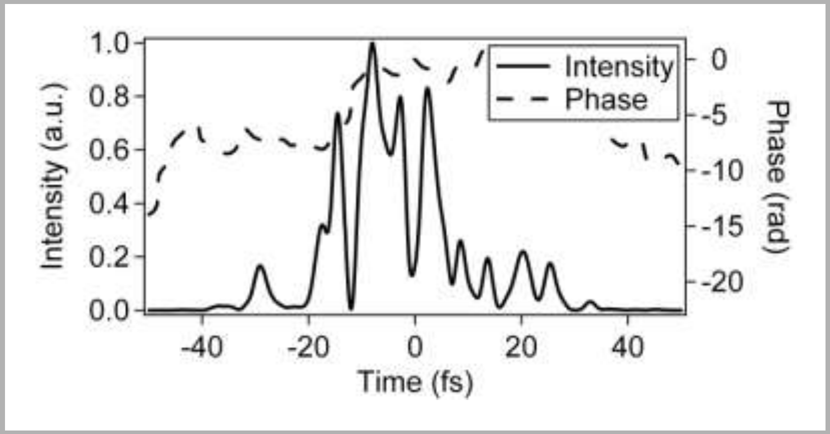
$TBP_{HW1/e} = 1$

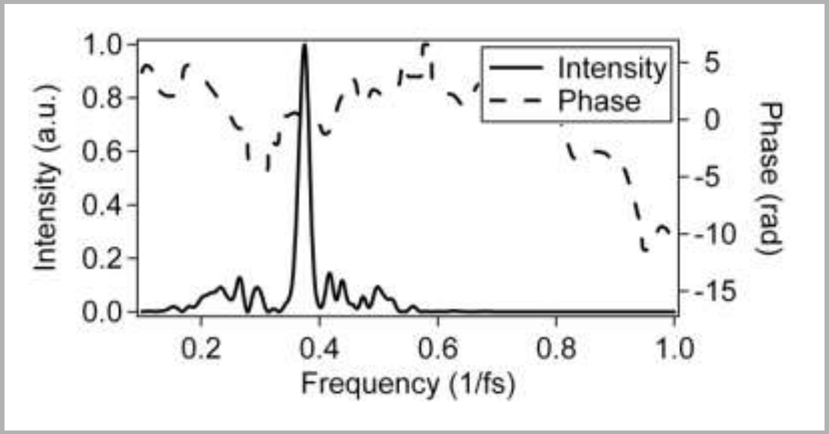
$TBP_e = 3.14$

$TBP_{FWHM} = 2.76$

Time-Bandwidth Product

Numerical example: A variable-phase, variable-intensity pulse with a fairly small TBP.





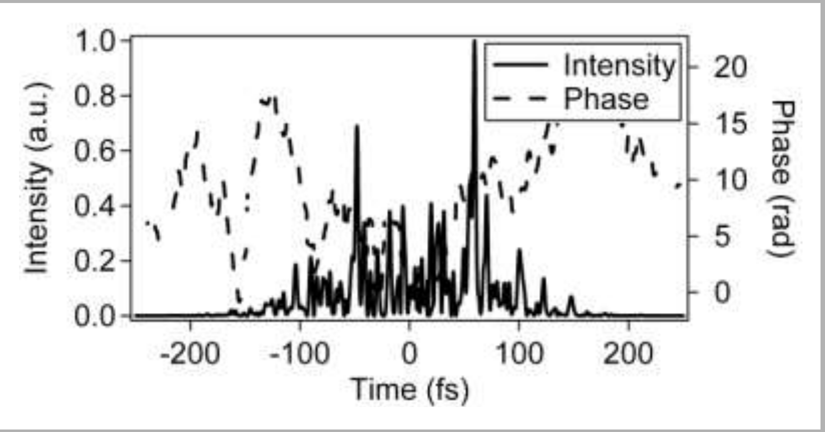
For the angular frequency and different definitions of the widths:

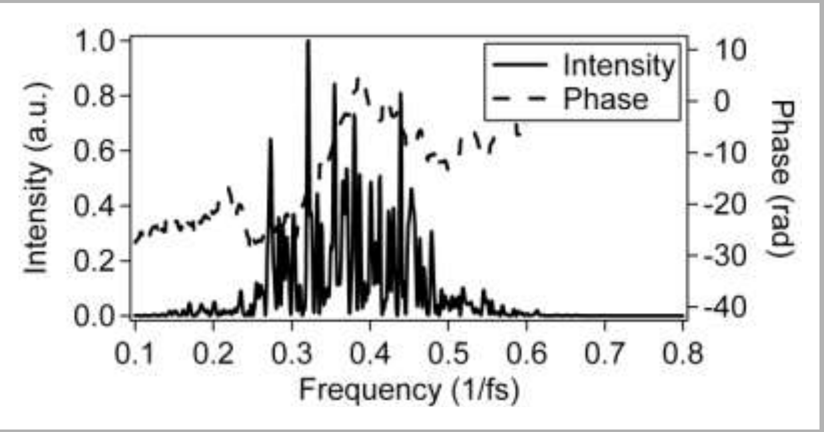
$TBP_{rms} = 6.09$ $TBP_{e} = 4.02$

$TBP_{HW1/e} = 0.82$ $TBP_{FWHM} = 2.57$

Time-Bandwidth Product

Numerical example: A variable-phase, variable-intensity pulse with a larger TBP.





For the angular frequency and different definitions of the widths:

$TBP_{rms} = 32.9$

$TBP_{HW1/e} = 35.2$

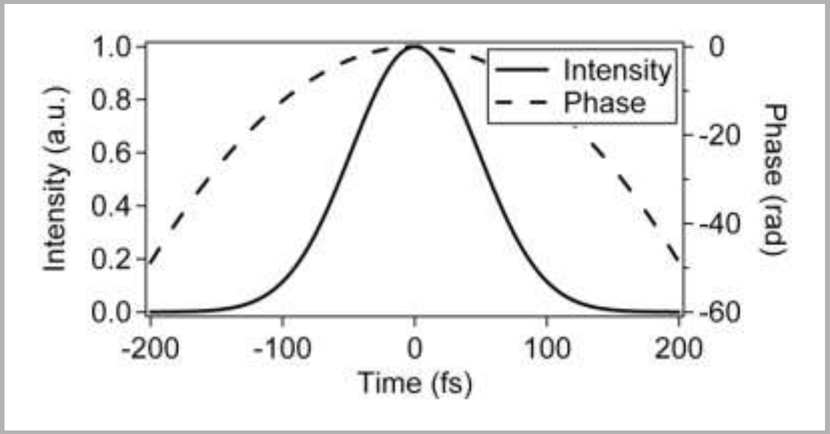
$TBP_{e} = 10.7$

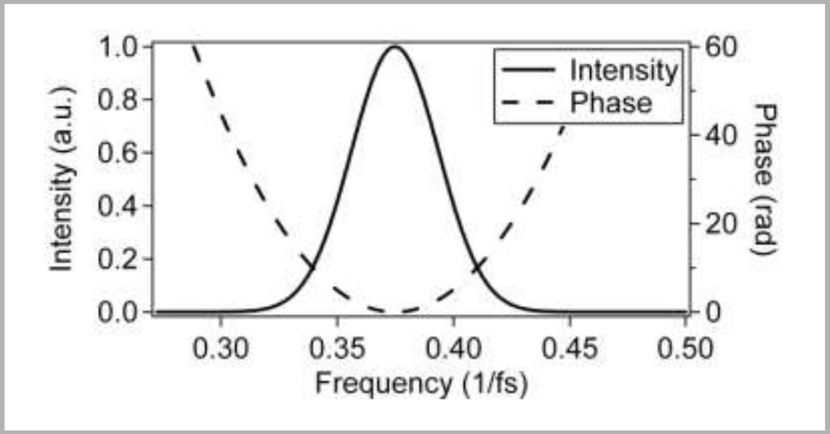
$TBP_{FWHM} = 116$

.

A linearly chirped pulse with no structure can also have a large time-bandwidth product.

Numerical example: A highly chirped, relatively long Gaussian-intensity pulse with a large TBP.





For the angular frequency and different definitions of the widths:

$TBP_{rms} = 5.65$

$TBP_{HW1/e} = 11.3$

$TBP_{e} = 35.5$

$TBP_{FWHM} = 31.3$