

# Dispersion and Ultrashort Pulses

Angular dispersion and group-velocity dispersion

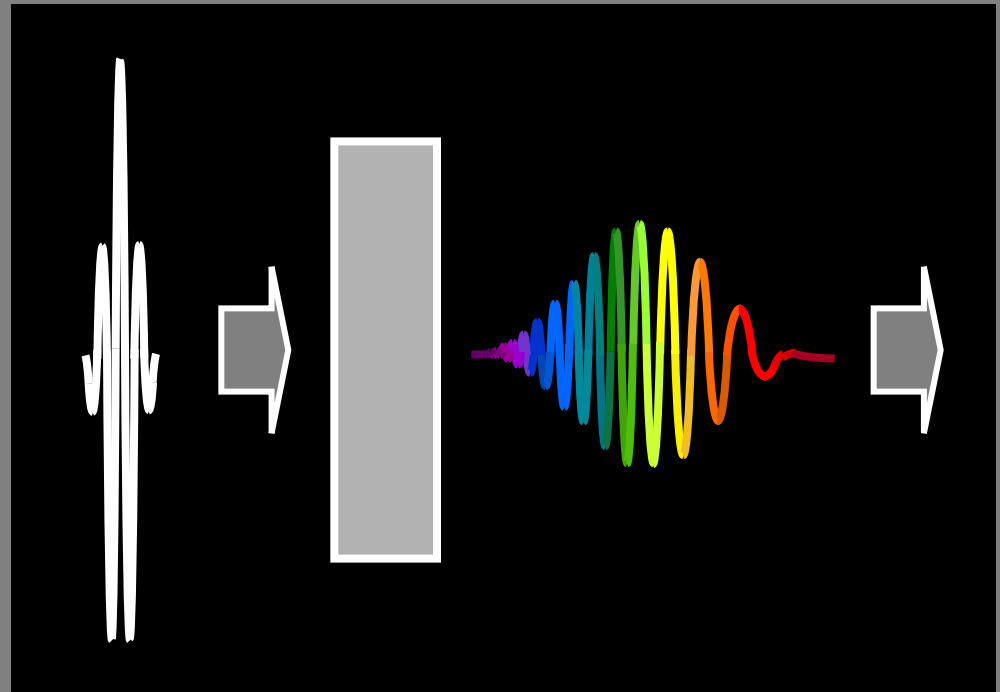
Phase and group velocities

Group-delay dispersion

Negative group-delay dispersion

Pulse compression

Chirped mirrors



# Refractive index dispersion

Sellmeiera formula

e.g. SF 10 glass from Schott

$$n^2(\lambda) = 1 + \frac{B_1 \lambda^2}{\lambda^2 - C_1} + \frac{B_2 \lambda^2}{\lambda^2 - C_2} + \frac{B_3 \lambda^2}{\lambda^2 - C_3}$$

$$B_1 = 1,61625977$$

$$B_2 = 0,259229334$$

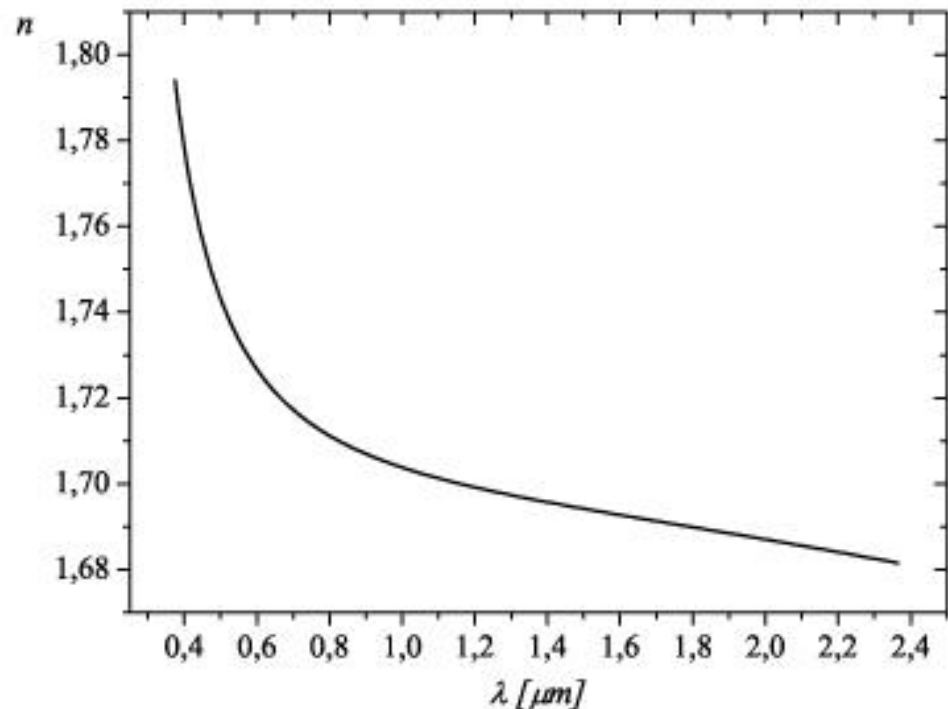
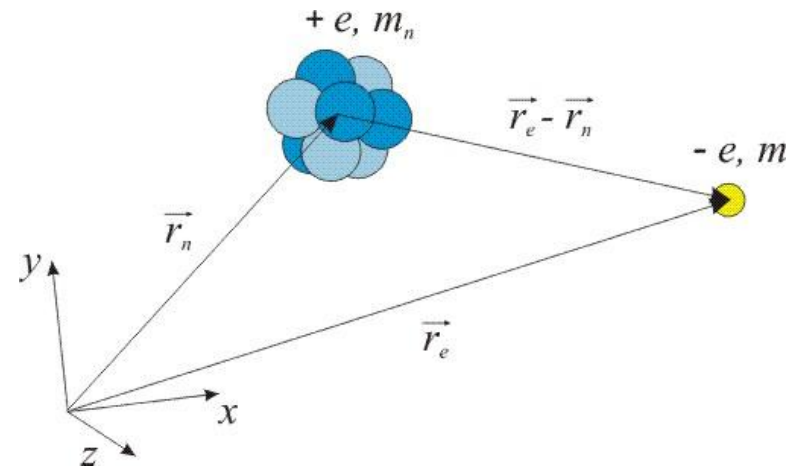
$$B_3 = 1,07762317$$

$$C_1 = 0,0127534559$$

$$C_2 = 0,0581983954$$

$$C_3 = 116,607680$$

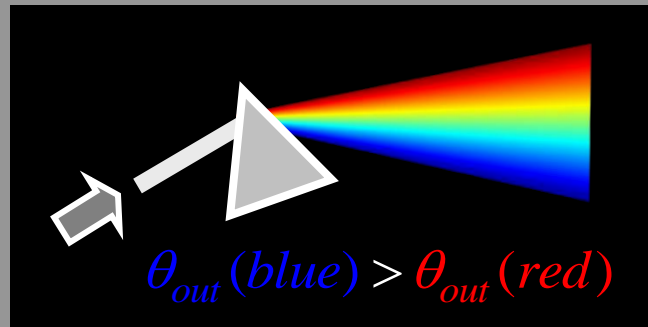
$$n(0,532 \mu m) = 1,73673$$



# Dispersion in Optics

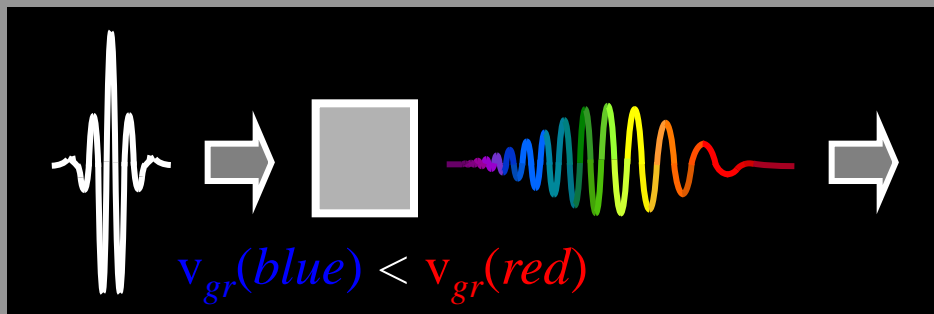
The dependence of the refractive index on wavelength has two effects on a pulse, one in space and the other in time.

Dispersion disperses a pulse in space (angle):



angular dispersion  
 $dn/d\lambda$

Dispersion also disperses a pulse in time:

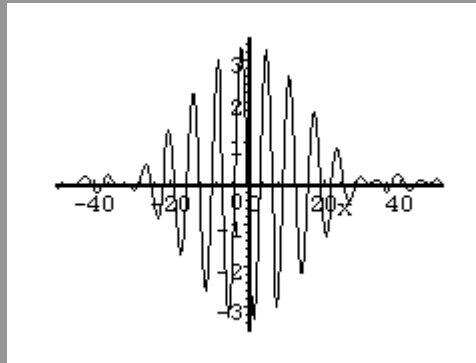


temporal dispersion  
 $d^2n/d\lambda^2$

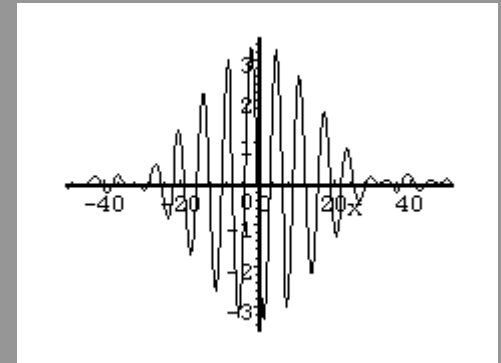
Both of these effects play major roles in ultrafast optics.

Group  
velocity  
vs.  
phase  
velocity

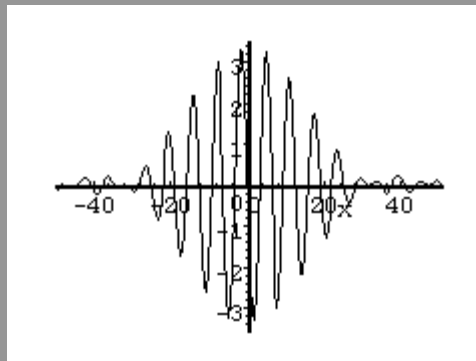
Movies!



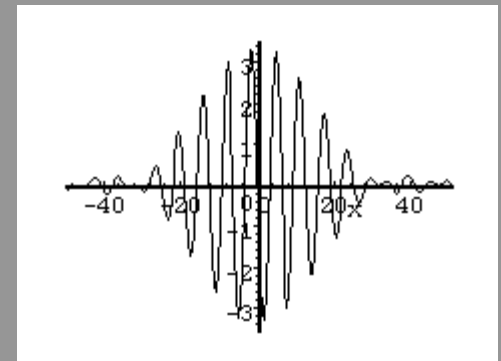
$$v_{\phi} = 0$$



$$v_g = v_{\phi}$$



$$v_g < v_{\phi}$$



$$v_g > v_{\phi}$$

# Calculating the group velocity

$$v_g \equiv d\omega / dk$$

Now,  $\omega$  is the same in or out of the medium, but  $k = k_0 n$ , where  $k_0$  is the k-vector in vacuum, and  $n$  is what depends on the medium. So it's easier to think of  $\omega$  as the independent variable:

$$v_g \equiv [dk / d\omega]^{-1}$$

Using  $k = \omega n(\omega) / c_0$ , calculate:  $dk / d\omega = (n + \omega dn/d\omega) / c_0$

$$v_g = c_0 / (n + \omega dn/d\omega) = (c_0/n) / (1 + \omega/n dn/d\omega)$$

Finally:

$$v_g = v_{phase} / \left( 1 + \frac{\omega}{n} \frac{dn}{d\omega} \right)$$

So the group velocity equals the phase velocity when  $dn/d\omega = 0$ , such as in vacuum. But  $n$  usually increases with  $\omega$ , so  $dn/d\omega > 0$ , and:

$$v_g < v_{phase}.$$

# Calculating group velocity vs. wavelength

We more often think of the refractive index in terms of wavelength, so let's write the group velocity in terms of the vacuum wavelength  $\lambda_0$ .

Use the chain rule: 
$$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0} \frac{d\lambda_0}{d\omega}$$

Now,  $\lambda_0 = 2\pi c_0 / \omega$ , so: 
$$\frac{d\lambda_0}{d\omega} = \frac{-2\pi c_0}{\omega^2} = \frac{-2\pi c_0}{(2\pi c_0 / \lambda_0)^2} = \frac{-\lambda_0^2}{2\pi c_0}$$

Recalling that: 
$$v_g = \left( \frac{c_0}{n} \right) / \left[ 1 + \frac{\omega}{n} \frac{dn}{d\omega} \right]$$

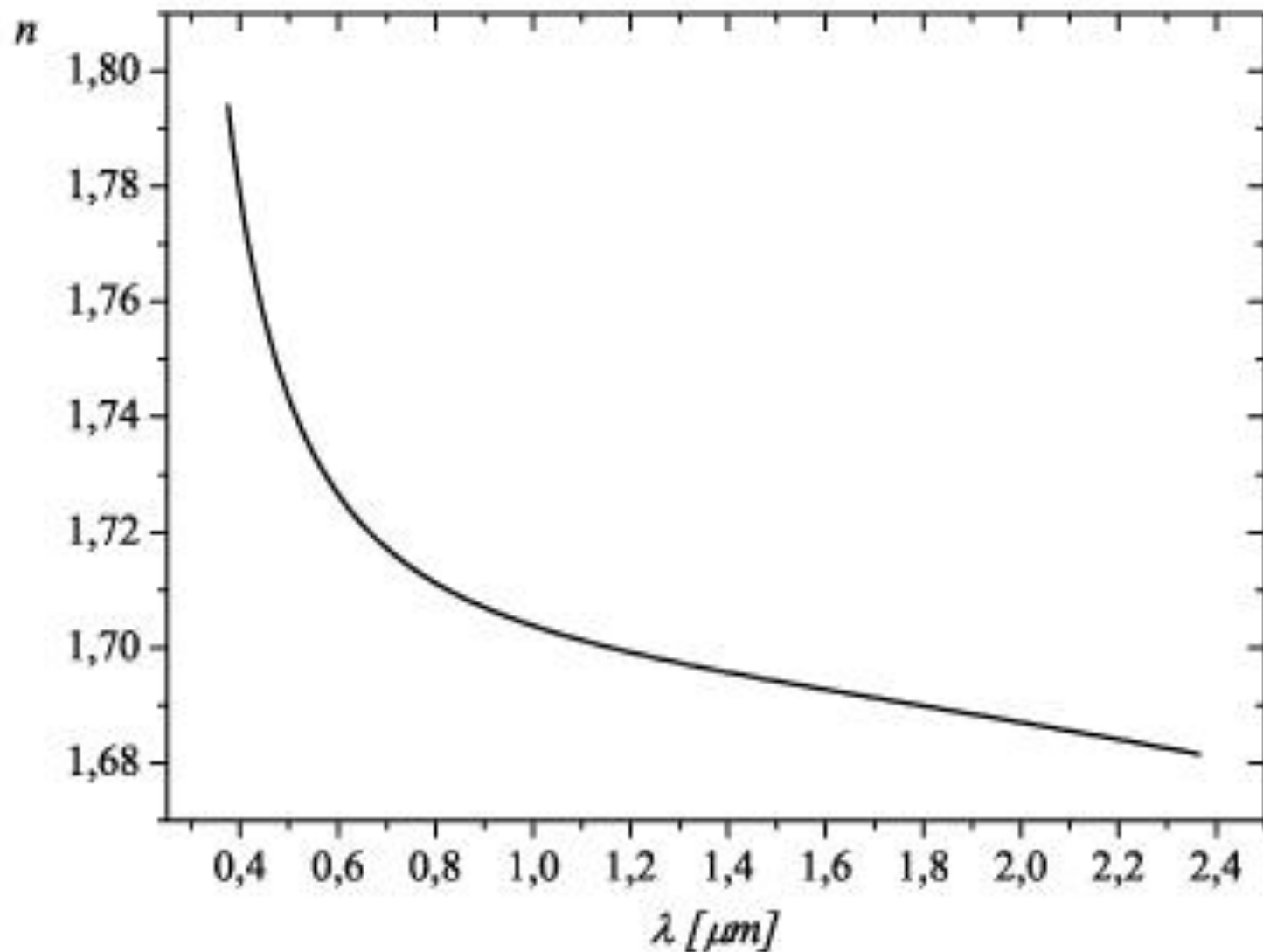
we have: 
$$v_g = \left( \frac{c_0}{n} \right) / \left[ 1 + \frac{\cancel{2\pi c_0}}{n\lambda_0} \left\{ \frac{dn}{d\lambda_0} \left( \frac{-\lambda_0^2}{\cancel{2\pi c_0}} \right) \right\} \right]$$

or :

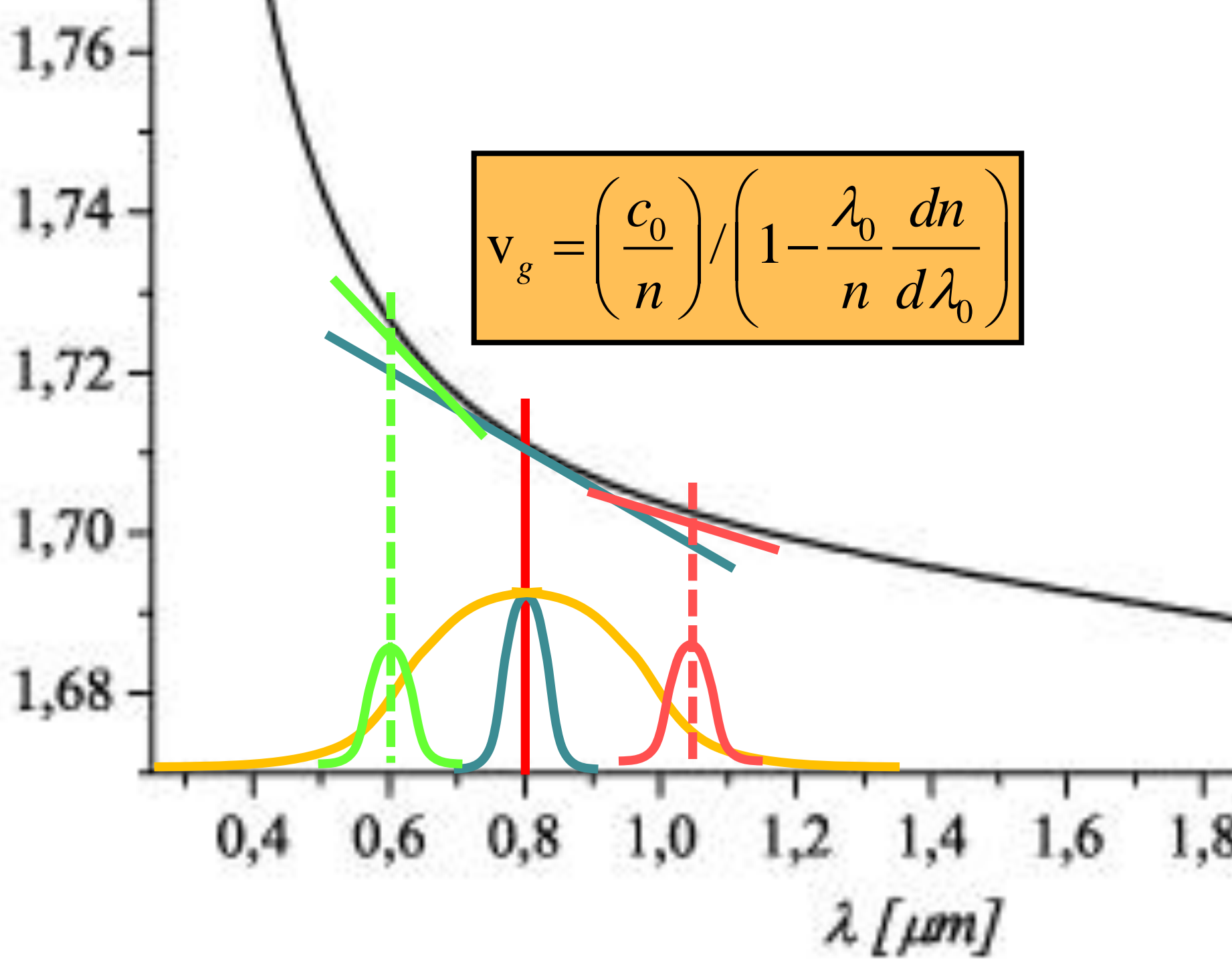
$$v_g = \left( \frac{c_0}{n} \right) / \left( 1 - \frac{\lambda_0}{n} \frac{dn}{d\lambda_0} \right)$$

$n(\omega)$

## Schott SF 10 glass



$$v_g = \left( \frac{c_0}{n} \right) / \left( 1 - \frac{\lambda_0}{n} \frac{dn}{d\lambda_0} \right)$$

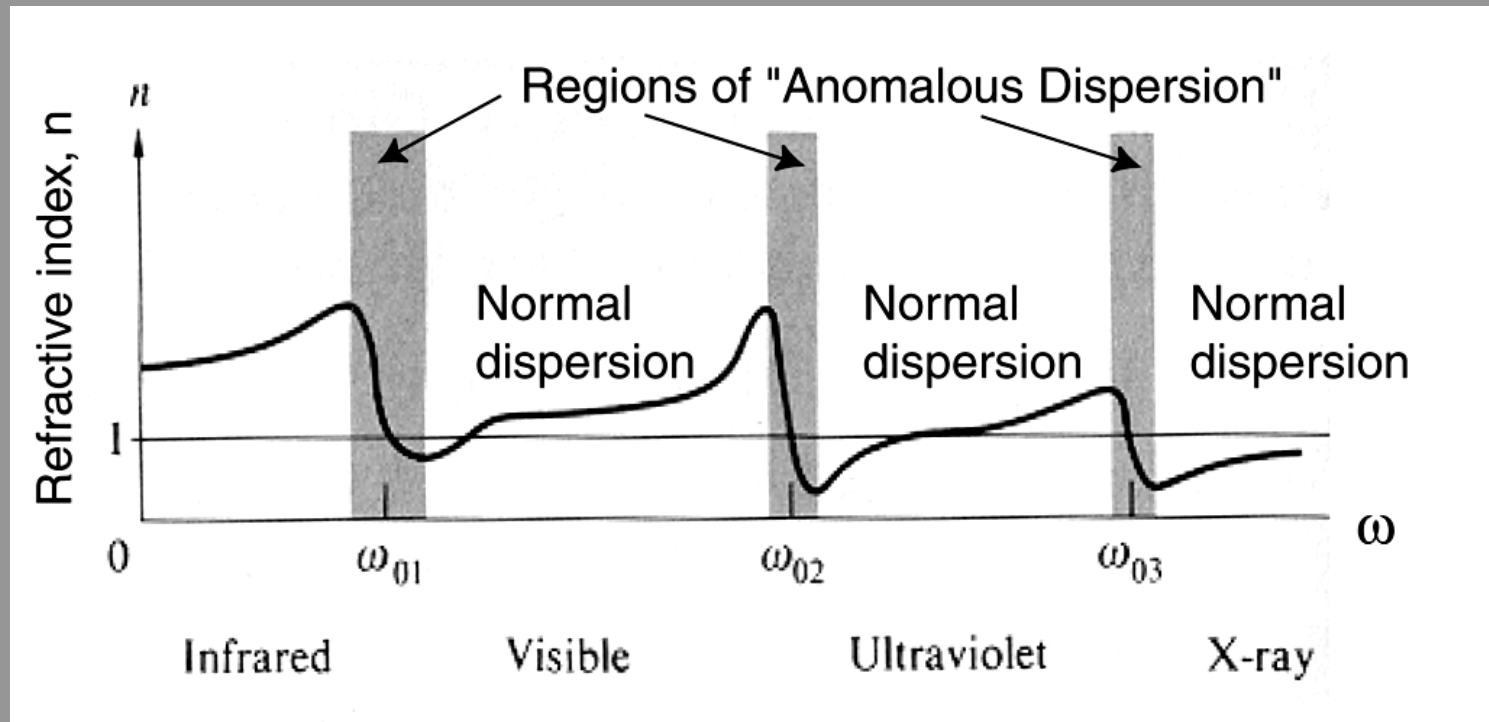




# The group velocity is lower than the phase velocity in non-absorbing regions.

$$v_g = c_0 / (n + \omega \, dn/d\omega)$$

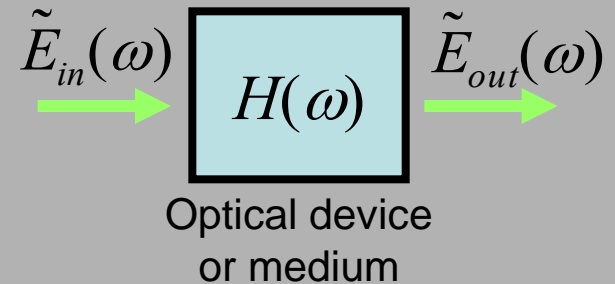
Except in regions of anomalous dispersion (near a resonance and which are absorbing),  $dn/d\omega$  is positive. So  $v_g < v_{phase}$  for most frequencies!



# Spectral Phase and Optical Devices

Recall that the effect of a linear passive optical device (i.e., windows, filters, etc.) on a pulse is to **multiply** the frequency-domain field by a transfer function:

$$\tilde{E}_{out}(\omega) = H(\omega) \tilde{E}_{in}(\omega)$$



where  $H(\omega)$  is the transfer function of the device/medium:

$$H(\omega) = B_H(\omega) \exp[-i\phi_H(\omega)]$$

$\exp[-\alpha(\omega)L/2]$  for a medium

Since we also write  $E(\omega) = \sqrt{S(\omega)} \exp[-i\phi(\omega)]$ , the spectral phase of the output light will be:

$$\phi_{out}(\omega) = \phi_H(\omega) + \phi_{in}(\omega)$$

**We simply add spectral phases.**

Note that we CANNOT add the temporal phases!

$$\phi_{out}(t) \neq \phi_H(t) + \phi_{in}(t)$$

# The Group-Velocity Dispersion (GVD)

The phase due to a medium is:  $\varphi_H(\omega) = n(\omega) k L = k(\omega) L$

To account for dispersion, expand the phase (k-vector) in a Taylor series:

$$k(\omega)L = k(\omega_0)L + k'(\omega_0)[\omega - \omega_0]L + \frac{1}{2}k''(\omega_0)[\omega - \omega_0]^2 L + \dots$$
$$k(\omega_0) = \frac{\omega_0}{v_\phi(\omega_0)} \quad k'(\omega_0) = \frac{1}{v_g(\omega_0)} \quad k''(\omega) = \frac{d}{d\omega} \left[ \frac{1}{v_g} \right]$$

The first few terms are all related to important quantities.

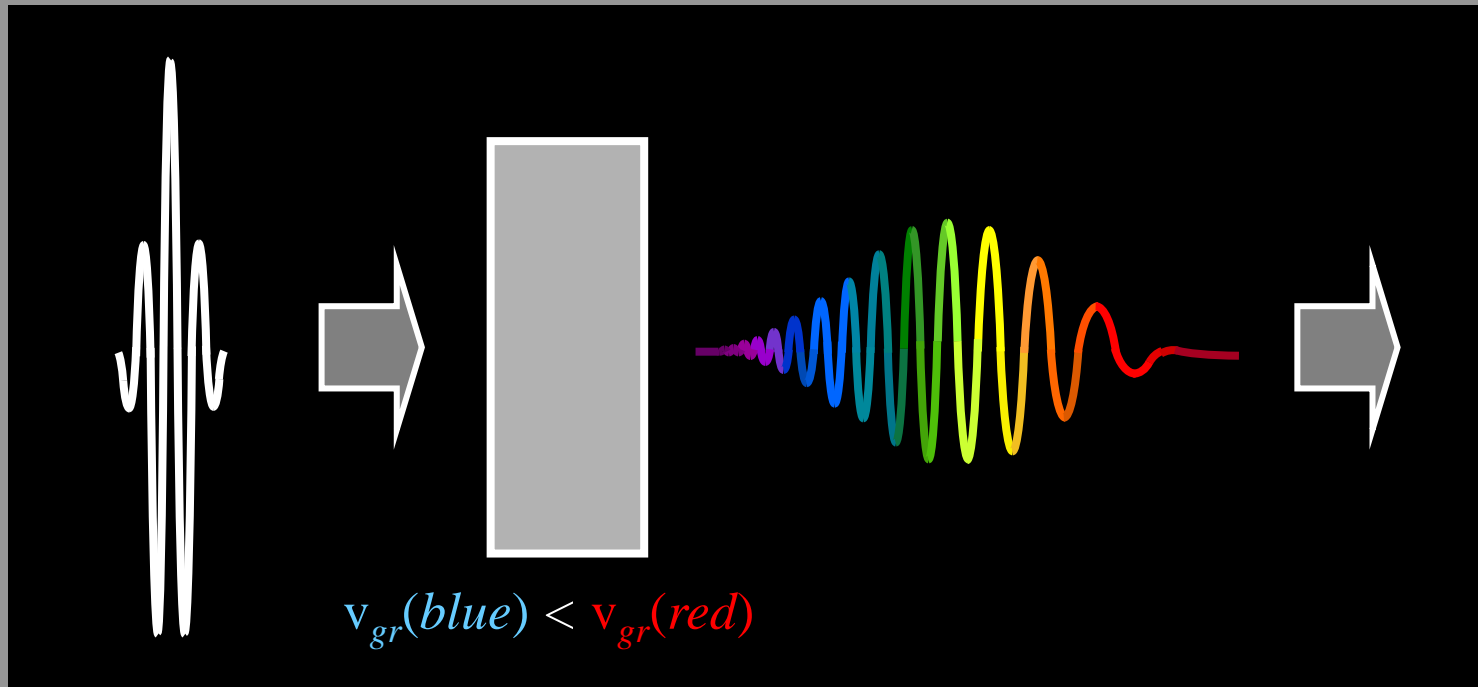
The third one is new: the variation in group velocity with frequency:

$$k''(\omega) = \frac{d}{d\omega} \left[ \frac{1}{v_g} \right]$$

is the “**group velocity dispersion.**”

# The effect of group velocity dispersion

GVD means that the group velocity will be different for different wavelengths in the pulse.



Because ultrashort pulses have such large bandwidths, GVD is a bigger issue than for cw light.

# Calculation of the GVD (in terms of wavelength)

Recall that:  $\frac{d\lambda_0}{d\omega} = \frac{-\lambda_0^2}{2\pi c_0}$        $\frac{d}{d\omega} = \frac{d\lambda_0}{d\omega} \frac{d}{d\lambda_0} = \frac{-\lambda_0^2}{2\pi c_0} \frac{d}{d\lambda_0}$

and  $v_g = c_0 / \left( n - \lambda_0 \frac{dn}{d\lambda_0} \right)$

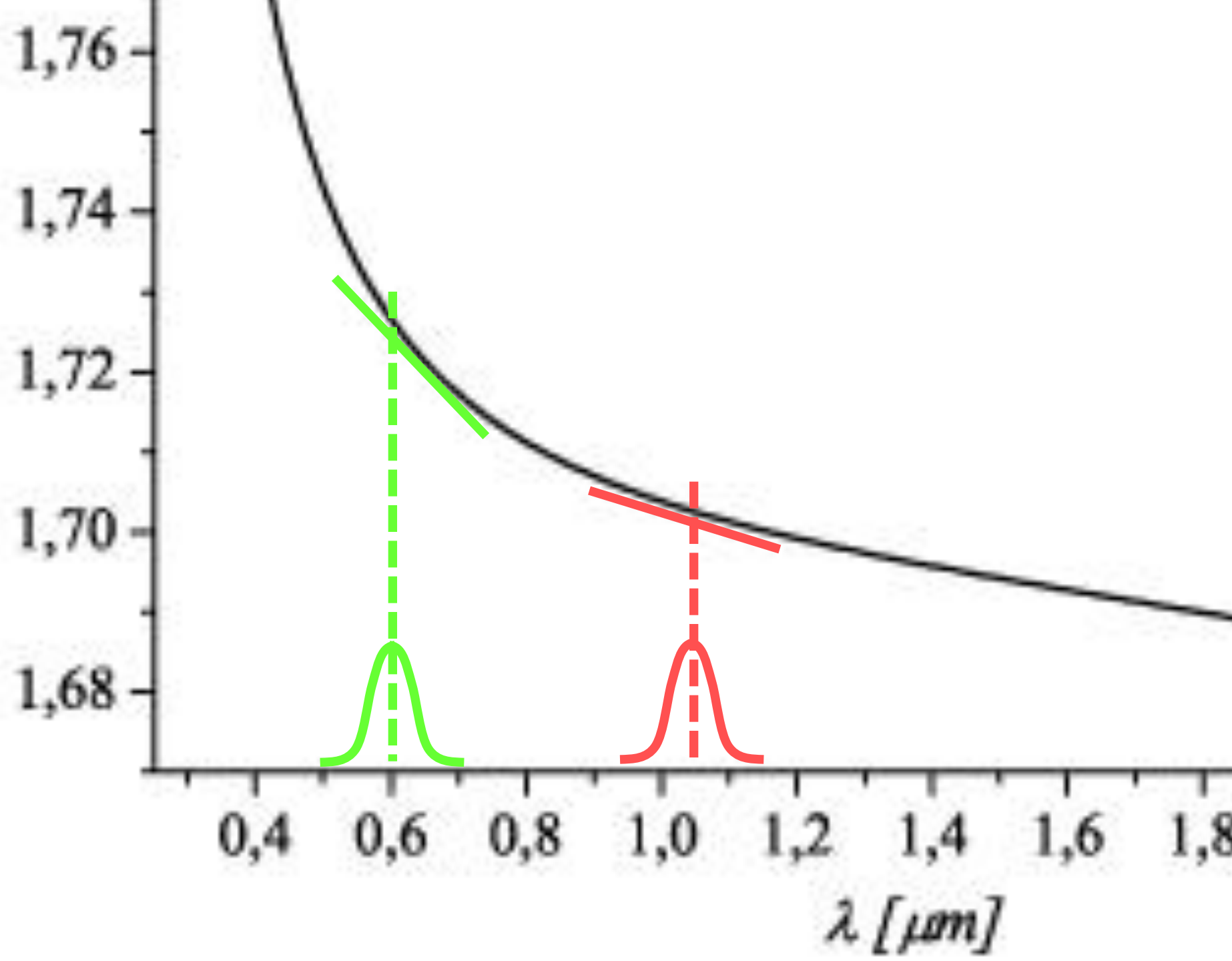
Okay, the GVD is:

$$\begin{aligned} \frac{d}{d\omega} \left[ \frac{1}{v_g} \right] &= \frac{-\lambda_0^2}{2\pi c_0} \frac{d}{d\lambda_0} \left[ \frac{1}{c_0} \left( n - \lambda_0 \frac{dn}{d\lambda_0} \right) \right] = \frac{-\lambda_0^2}{2\pi c_0^2} \frac{d}{d\lambda_0} \left[ n - \lambda_0 \frac{dn}{d\lambda_0} \right] \\ &= \frac{-\lambda_0^2}{2\pi c_0^2} \left[ \frac{dn}{d\lambda_0} - \lambda_0 \frac{d^2 n}{d\lambda_0^2} - \frac{dn}{d\lambda_0} \right] \end{aligned}$$

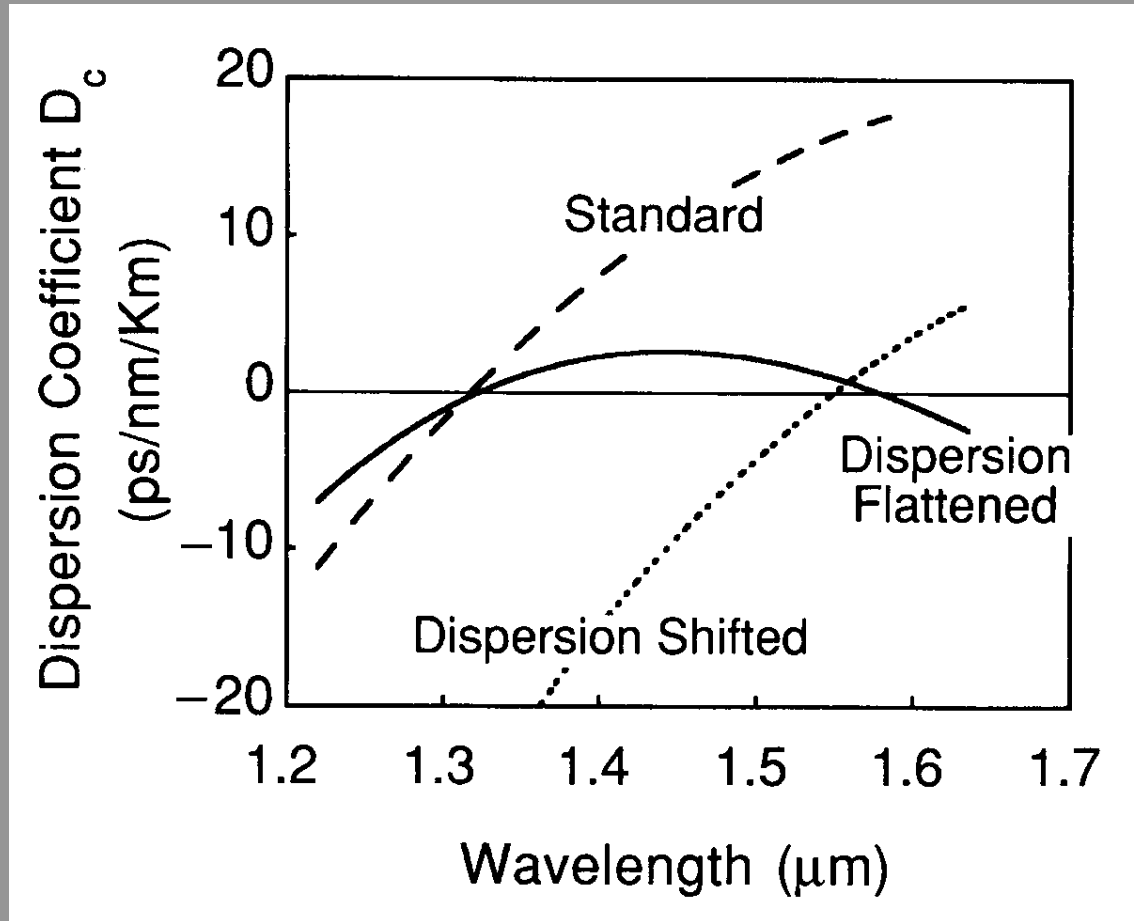
Simplifying:

$$GVD \equiv k''(\omega_0) = \frac{\lambda_0^3}{2\pi c_0^2} \frac{d^2 n}{d\lambda_0^2}$$

Units:  
ps<sup>2</sup>/km  
ps/THz·km



# GVD in optical fibers



Sophisticated cladding structures, i.e., index profiles, have been designed and optimized to produce a waveguide dispersion that modifies the bulk material dispersion

# GVD yields group delay dispersion (GDD).

Delays are just velocities times the medium length  $L$ .

The phase delay:

$$k(\omega_0) = \frac{\omega_0}{v_\phi(\omega_0)} \quad \text{so:} \quad t_\phi = \frac{L}{v_\phi(\omega_0)} = \frac{k(\omega_0)L}{\omega_0}$$

The group delay:

$$k'(\omega_0) = \frac{1}{v_g(\omega_0)} \quad \text{so:} \quad t_g(\omega_0) = \frac{L}{v_g(\omega_0)} = k'(\omega_0)L$$

The group delay dispersion (GDD):

$$GDD = GVD L$$

$$k''(\omega) = \frac{d}{d\omega} \left[ \frac{1}{v_g} \right]$$

so:

$$GDD = \frac{d}{d\omega} \left[ \frac{1}{v_g} \right] L = k''(\omega)L$$

Units: fs<sup>2</sup> or fs/Hz



material	$\lambda$ [nm]	$n(\lambda)$	$\frac{dn}{d\lambda} \cdot 10^{-2} \left[ \frac{1}{\mu m} \right]$	$\frac{d^2n}{d\lambda^2} \cdot 10^{-1} \left[ \frac{1}{\mu m^2} \right]$	$\frac{dn^3}{d\lambda^3} \left[ \frac{1}{\mu m^3} \right]$	$T_g \left[ \frac{fs}{mm} \right]$	$GDD \left[ \frac{fs^2}{mm} \right]$	$TOD \left[ \frac{fs^3}{mm} \right]$
BK7	400	1,5308	-13,17	10,66	-12,21	5282	120,79	40,57
	500	1,5214	-6,58	3,92	-3,46	5185	86,87	32,34
	600	1,5163	-3,91	1,77	-1,29	5136	67,52	29,70
	800	1,5108	-1,97	0,48	-0,29	5092	43,96	31,90
	1000	1,5075	-1,40	0,15	-0,09	5075	26,93	42,88
	1200	1,5049	-1,23	0,03	-0,04	5069	10,43	66,12
SF10	400	1,7783	-52,02	59,44	-101,56	6626	673,68	548,50
	500	1,7432	-20,89	15,55	-16,81	6163	344,19	219,81
	600	1,7267	-11,00	6,12	-4,98	5980	233,91	140,82
	800	1,7112	-4,55	1,58	-0,91	5830	143,38	97,26
	1000	1,7038	-2,62	0,56	-0,27	5771	99,42	92,79
	1200	1,6992	-1,88	0,22	-0,10	5743	68,59	107,51
Sapphire	400	1,7866	-17,20	13,55	-15,05	6189	153,62	47,03
	500	1,7743	-8,72	5,10	-4,42	6064	112,98	39,98
	600	1,7676	-5,23	2,32	-1,68	6001	88,65	37,97
	800	1,7602	-2,68	0,64	-0,38	5943	58,00	42,19
	1000	1,7557	-1,92	0,20	-0,12	5921	35,33	57,22
	1200	1,7522	-1,70	0,04	-0,05	5913	13,40	87,30
Quartz	300	1,4878	-30,04	34,31	-54,66	5263	164,06	46,49
	400	1,4701	-11,70	9,20	-10,17	5060	104,31	31,49
	500	1,4623	-5,93	3,48	-3,00	4977	77,04	26,88
	600	1,4580	-3,55	1,59	-1,14	4934	60,66	25,59
	800	1,4533	-1,80	0,44	-0,26	4896	40,00	28,43

# Manipulating the phase of light

Recall that we expand the spectral phase of the pulse in a Taylor Series:

$$\varphi(\omega) = \varphi_0 + \varphi_1 [\omega - \omega_0] + \varphi_2 [\omega - \omega_0]^2 / 2! + \dots$$

and we do the same for the spectral phase of the optical medium,  $H$ :

$$\varphi_H(\omega) = \varphi_{H0} + \varphi_{H1} [\omega - \omega_0] + \varphi_{H2} [\omega - \omega_0]^2 / 2! + \dots$$

phase                      group delay                      group delay dispersion (GDD)

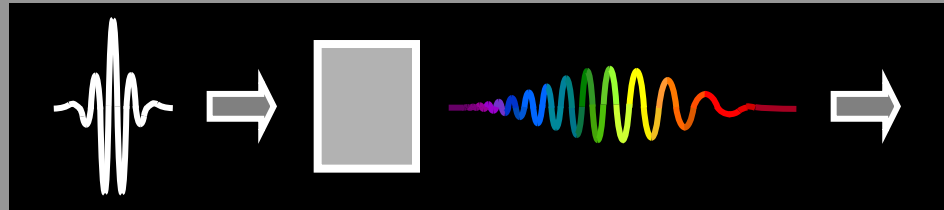
So, to manipulate light, we must add or subtract **spectral-phase terms**.

For example, to eliminate the linear chirp (second-order spectral phase), we must design an optical device whose second-order spectral phase cancels that of the pulse:

$$\varphi_2 + \varphi_{H2} = 0 \quad \text{i.e.,} \quad \left. \frac{d^2 \varphi}{d\omega^2} \right|_{\omega_0} + \left. \frac{d^2 \varphi_H}{d\omega^2} \right|_{\omega_0} = 0$$

# Propagation of the pulse manipulates it.

Dispersive pulse  
broadening  
is unavoidable.



If  $\varphi_2$  is the pulse 2<sup>nd</sup>-order spectral phase on entering a medium, and  $k''L$  is the 2<sup>nd</sup>-order spectral phase of the medium, then the resulting pulse 2<sup>nd</sup>-order phase will be the sum:  $\varphi_2 + k''L$ .

A linearly chirped input pulse has 2<sup>nd</sup>-order phase:  $\varphi_{2,in} = \frac{\beta / 2}{\alpha^2 + \beta^2}$  (This result pulls out the  $\frac{1}{2}$  in the Taylor Series.)

Emerging from a medium, its 2<sup>nd</sup>-order phase will be:

$$\varphi_{2,out} = \frac{\beta / 2}{\alpha^2 + \beta^2} + GDD = \frac{\beta / 2}{\alpha^2 + \beta^2} + \frac{\lambda_0^3}{2\pi c_0^2} \frac{d^2 n}{d\lambda_0^2} L \quad \leftarrow$$

This result, with the spectrum, can be inverse Fourier-transformed to yield the pulse.

A positively chirped pulse will broaden further;  
a negatively chirped pulse will shorten.

Too bad material GDD is always positive in the visible and near-IR...

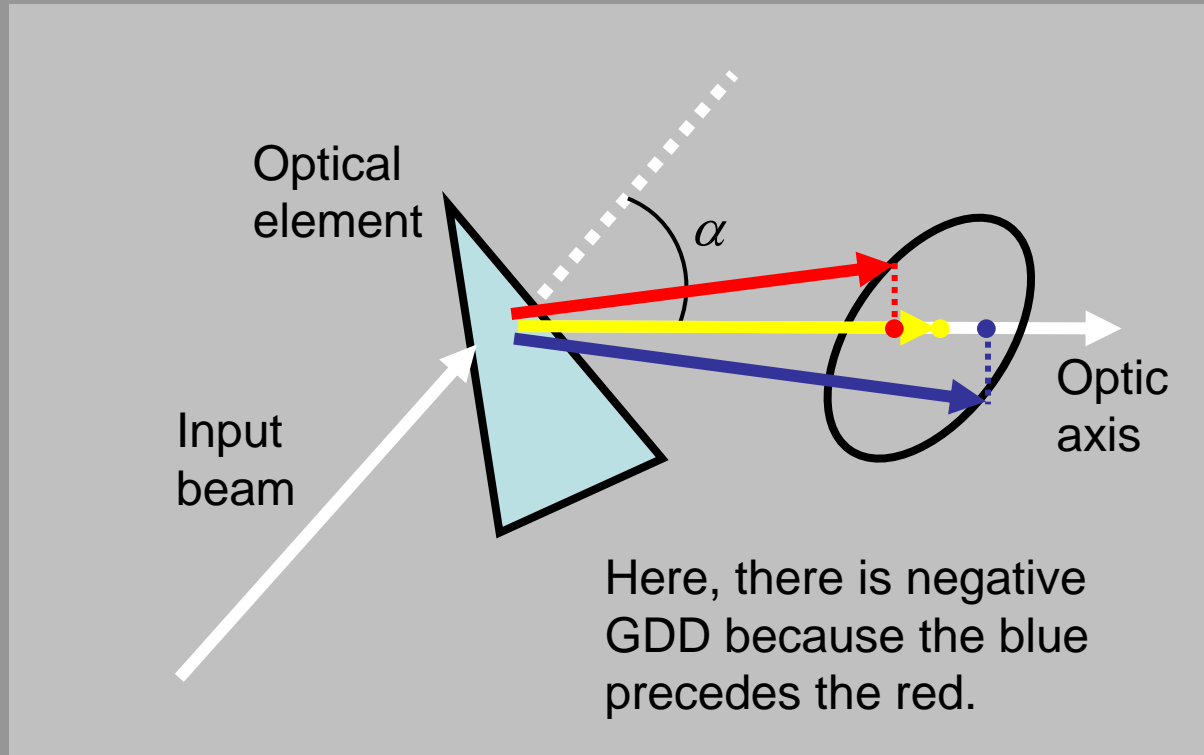
# So how can we generate negative GDD?

This is a big issue because pulses spread further and further as they propagate through materials.

We need a way of generating negative GDD to compensate.

# Angular dispersion yields negative GDD.

Suppose that an optical element introduces angular dispersion.

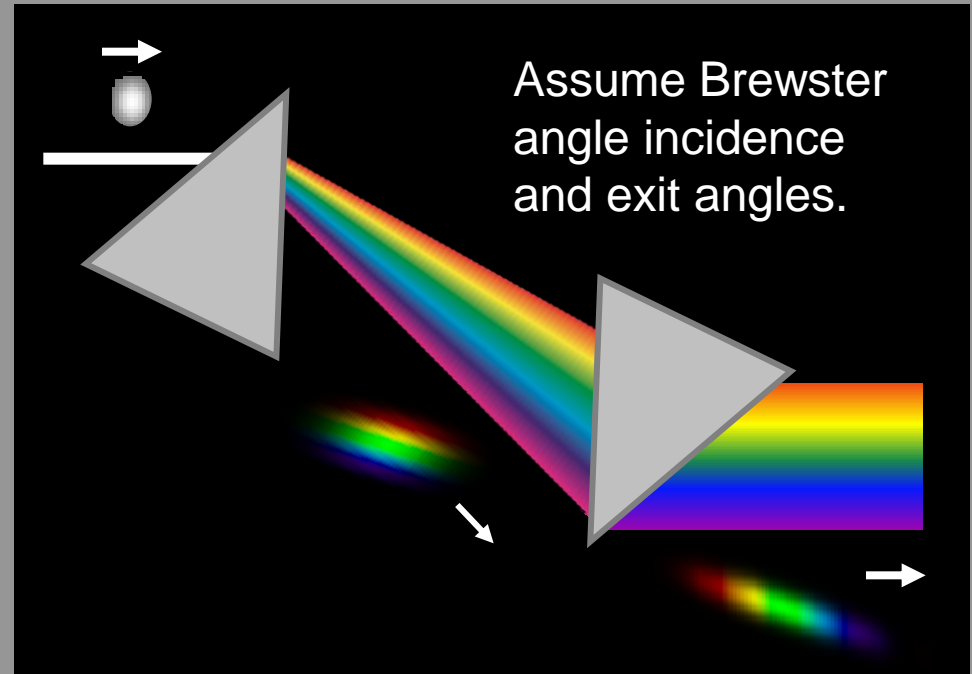


We'll need to compute the projection onto the optic axis (the propagation direction of the center frequency of the pulse).

# A prism pair has negative GDD.

How can we use dispersion to introduce negative chirp conveniently?

Let  $L_{prism}$  be the path through each prism and  $L_{sep}$  be the prism separation.



$$\left. \frac{d^2 \varphi}{d\omega^2} \right|_{\omega_0} \approx -4L_{sep} \frac{\lambda_0^3}{2\pi c^2} \left( \left. \frac{dn}{d\lambda} \right|_{\lambda_0} \right)^2 + L_{prism} \frac{\lambda_0^3}{2\pi c^2} \left. \frac{d^2 n}{d\lambda^2} \right|_{\lambda_0}$$

Always negative!

This term assumes that the beam grazes the tip of each prism

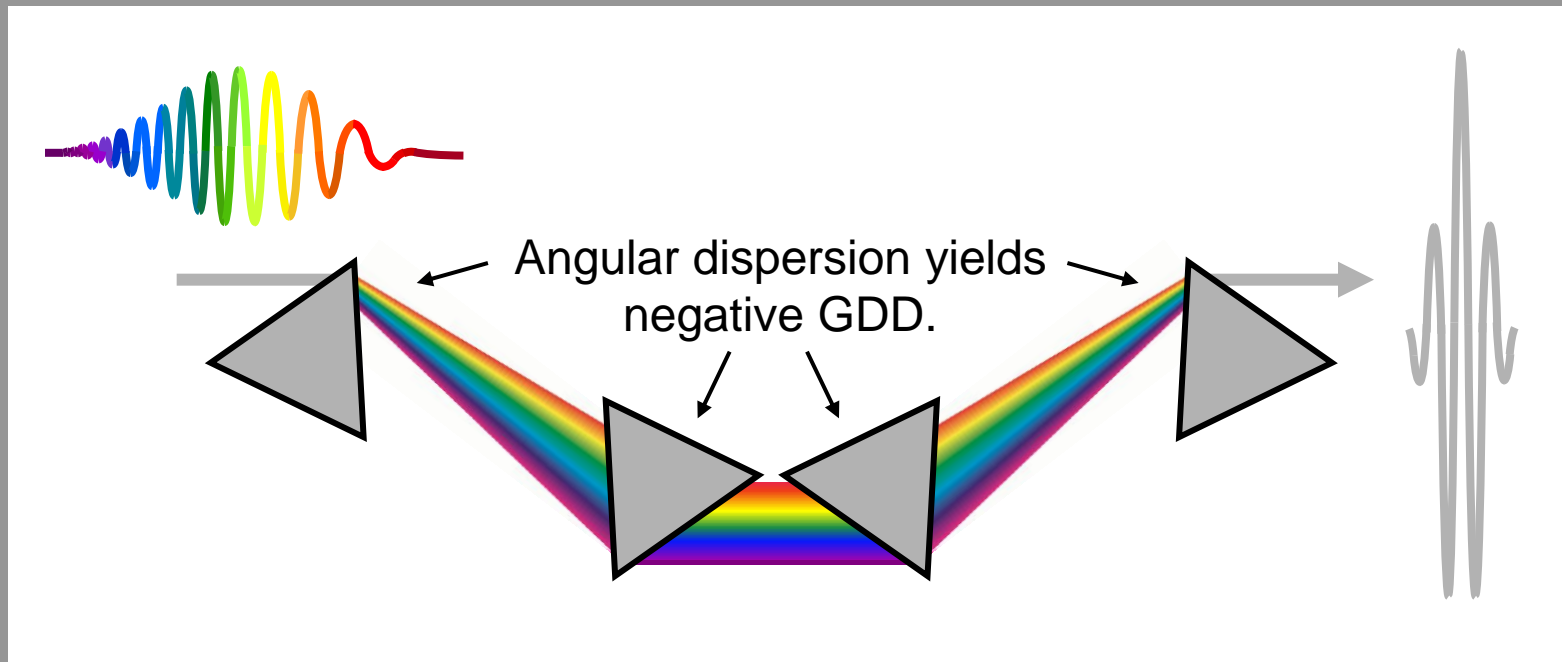
This term allows the beam to pass through an additional length,  $L_{prism}$ , of prism material.

Always positive (in visible and near-IR)

Vary  $L_{sep}$  or  $L_{prism}$  to tune the GDD!

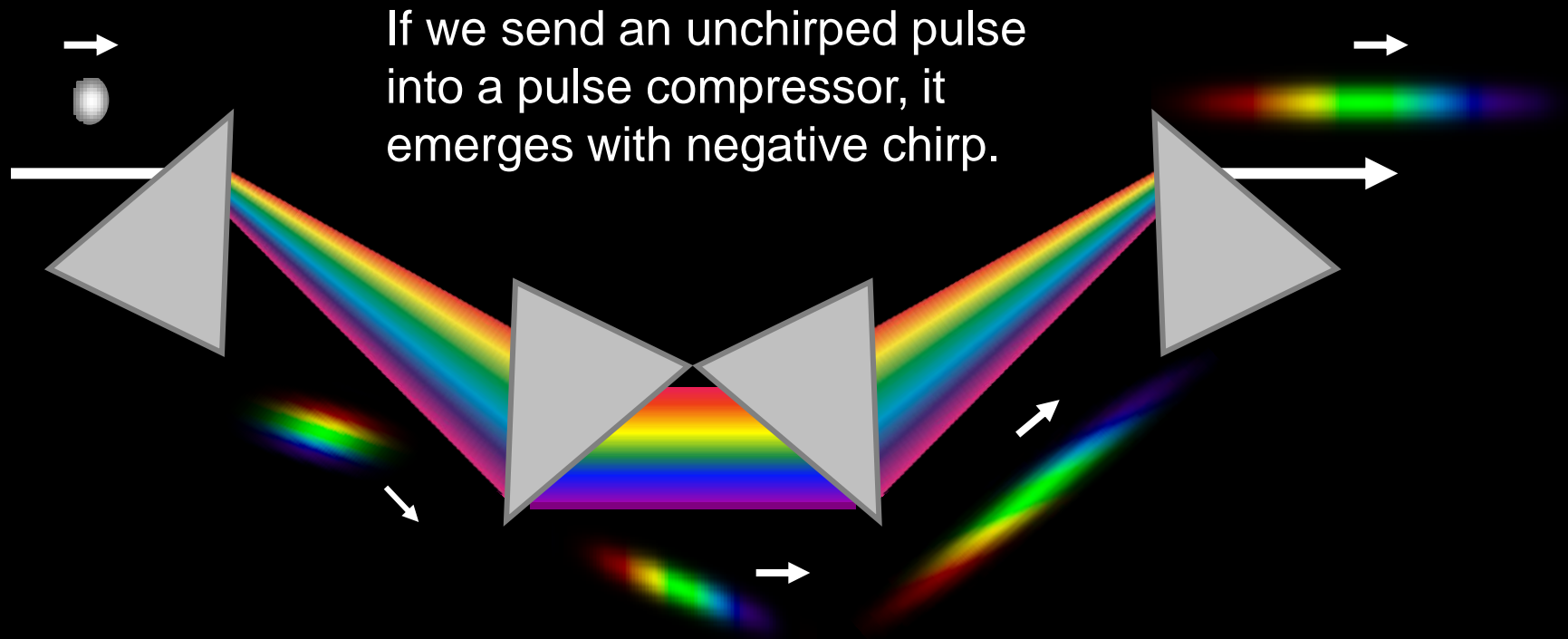
# Pulse Compressor

This device, which also puts the pulse back together, has **negative** group-delay dispersion and hence can compensate for propagation through materials (i.e., for positive chirp).



It's routine to stretch and then compress ultrashort pulses by factors of  $>1000$ .

# What does the pulse look like inside a pulse compressor?



Note the unintuitive color variation of the pulse after the first prism. To see the effect on a positively chirped pulse, read right to left!

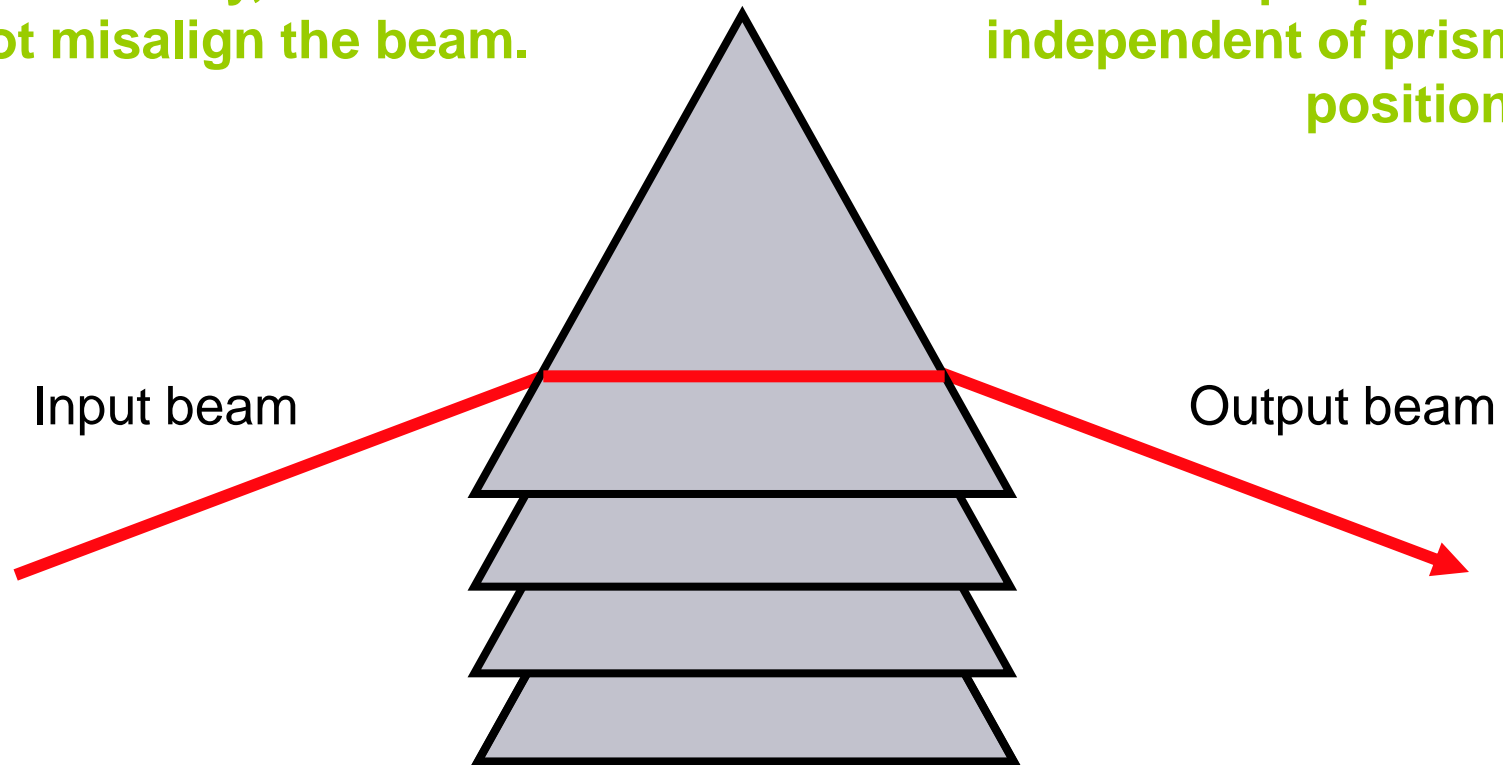


# Adjusting the GDD maintains alignment.

Any prism in the compressor can be translated perpendicular to the beam path to add glass and reduce the magnitude of negative GDD.

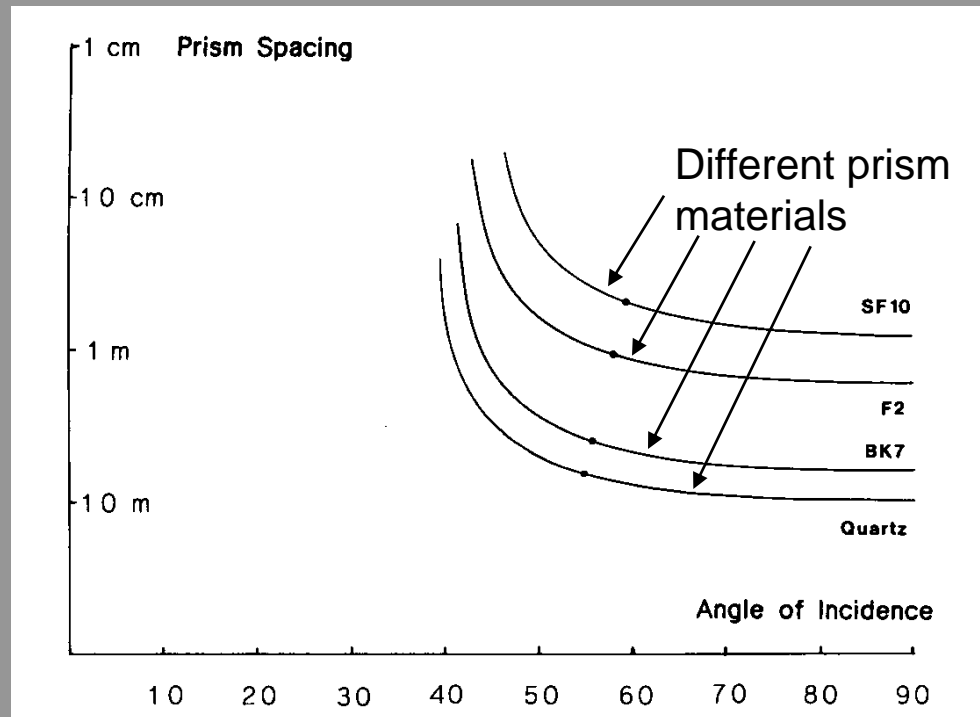
Remarkably, this does not misalign the beam.

The output path is independent of prism position.



# The required separation between prisms in a pulse compressor can be large.

The GDD  $\propto$  the prism separation and the square of the dispersion.



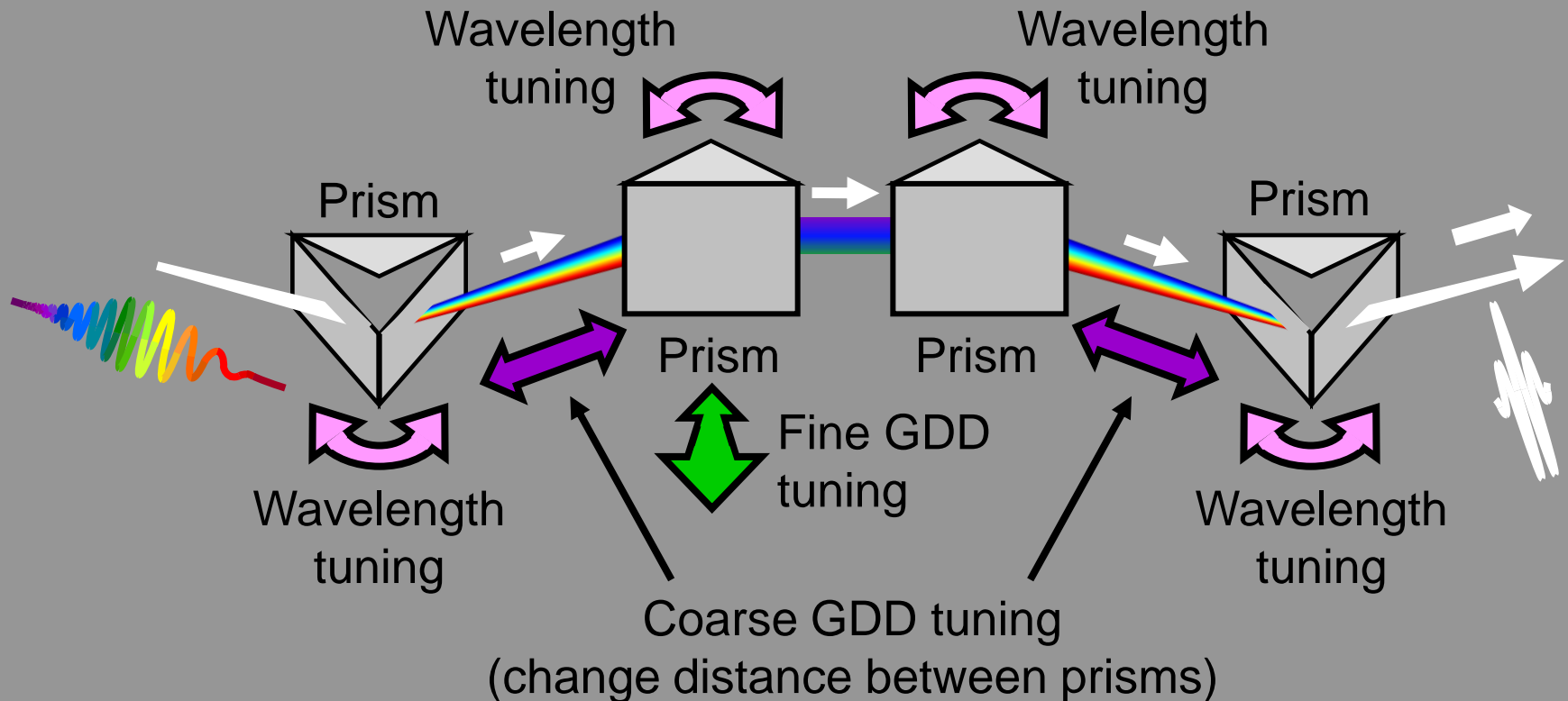
Compression of a 1-ps, 600-nm pulse with 10 nm of bandwidth (to about 50 fs).

Kafka and Baer,  
Opt. Lett., 12,  
401 (1987)

It's best to use highly dispersive glass, like SF10, or gratings. But compressors can still be  $> 1$  m long.

# Four-prism pulse compressor

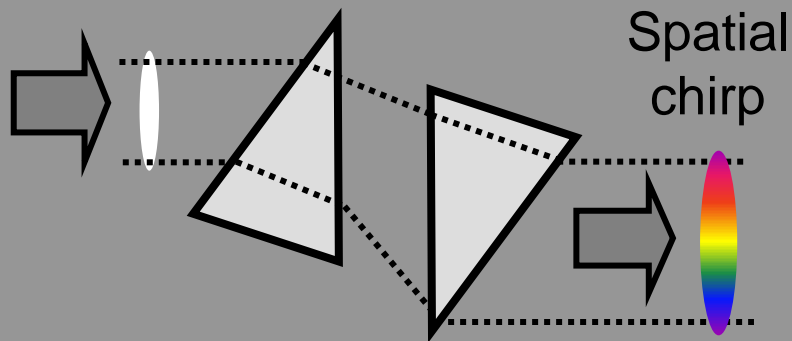
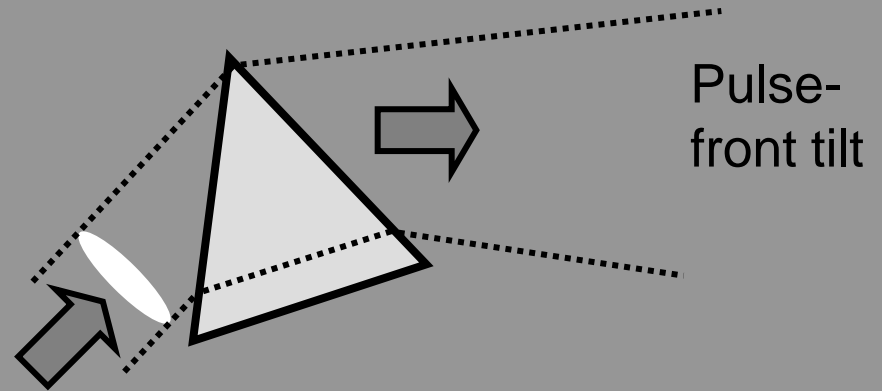
Also, alignment is critical, and many knobs must be tuned.



All prisms and their incidence angles must be identical.

# Pulse-compressors have alignment issues.

Pulse compressors are notorious for their large size, alignment complexity, and spatio-temporal distortions.

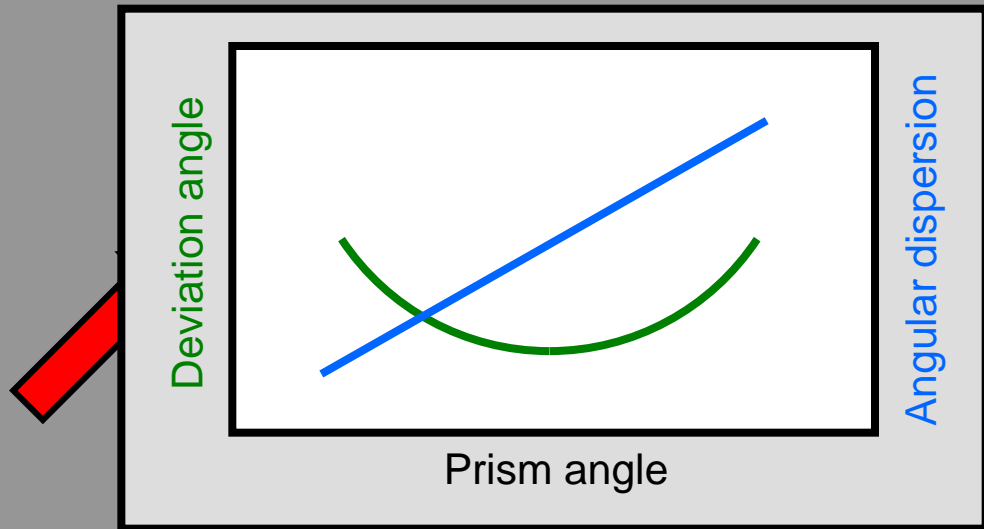


Unless the compressor is aligned perfectly, the output pulse has significant:

1. 1D beam magnification
2. Angular dispersion
3. Spatial chirp
4. Pulse-front tilt

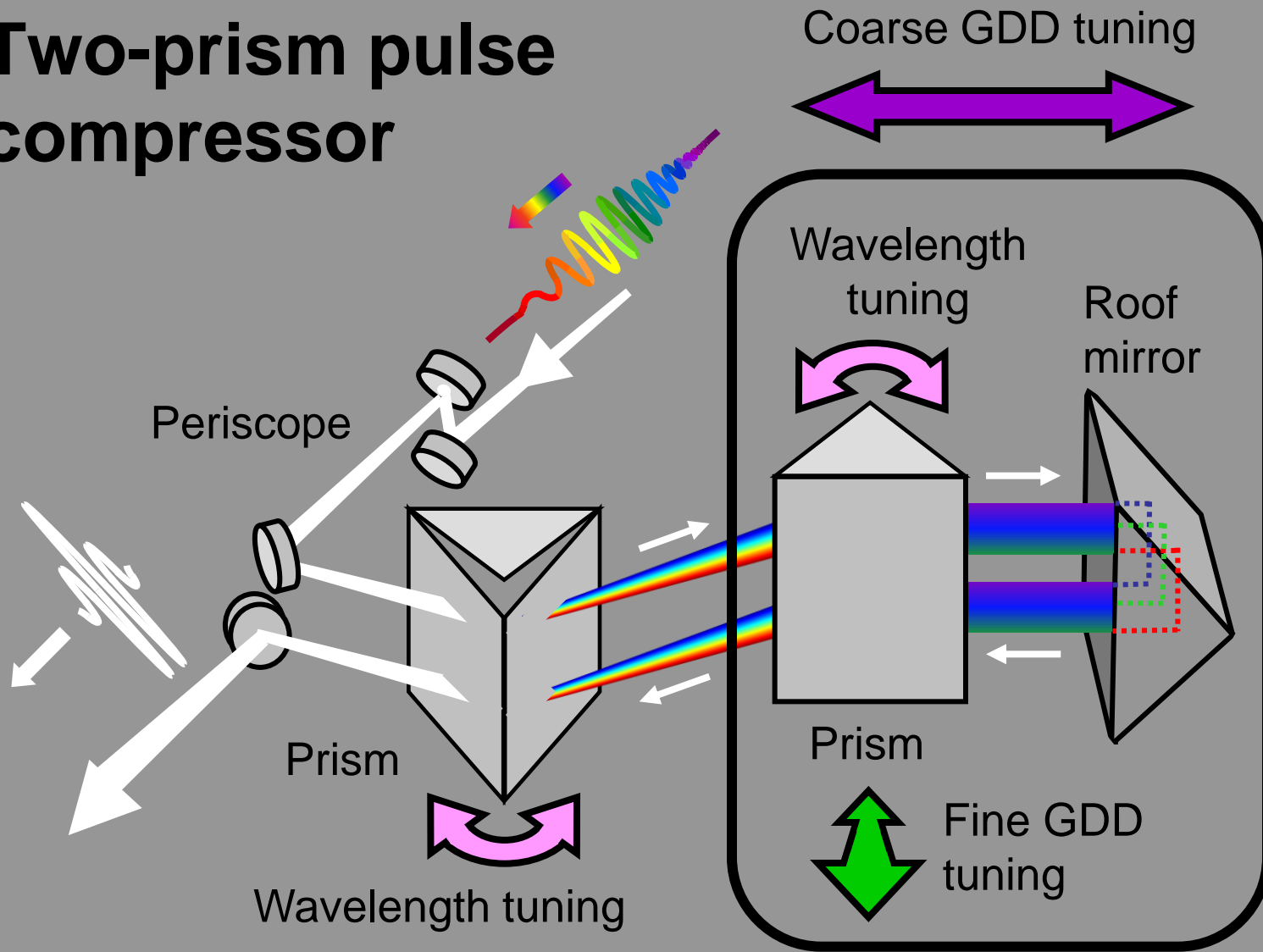
# Why is it difficult to align a pulse compressor?

The prisms are usually aligned using the **minimum deviation** condition.



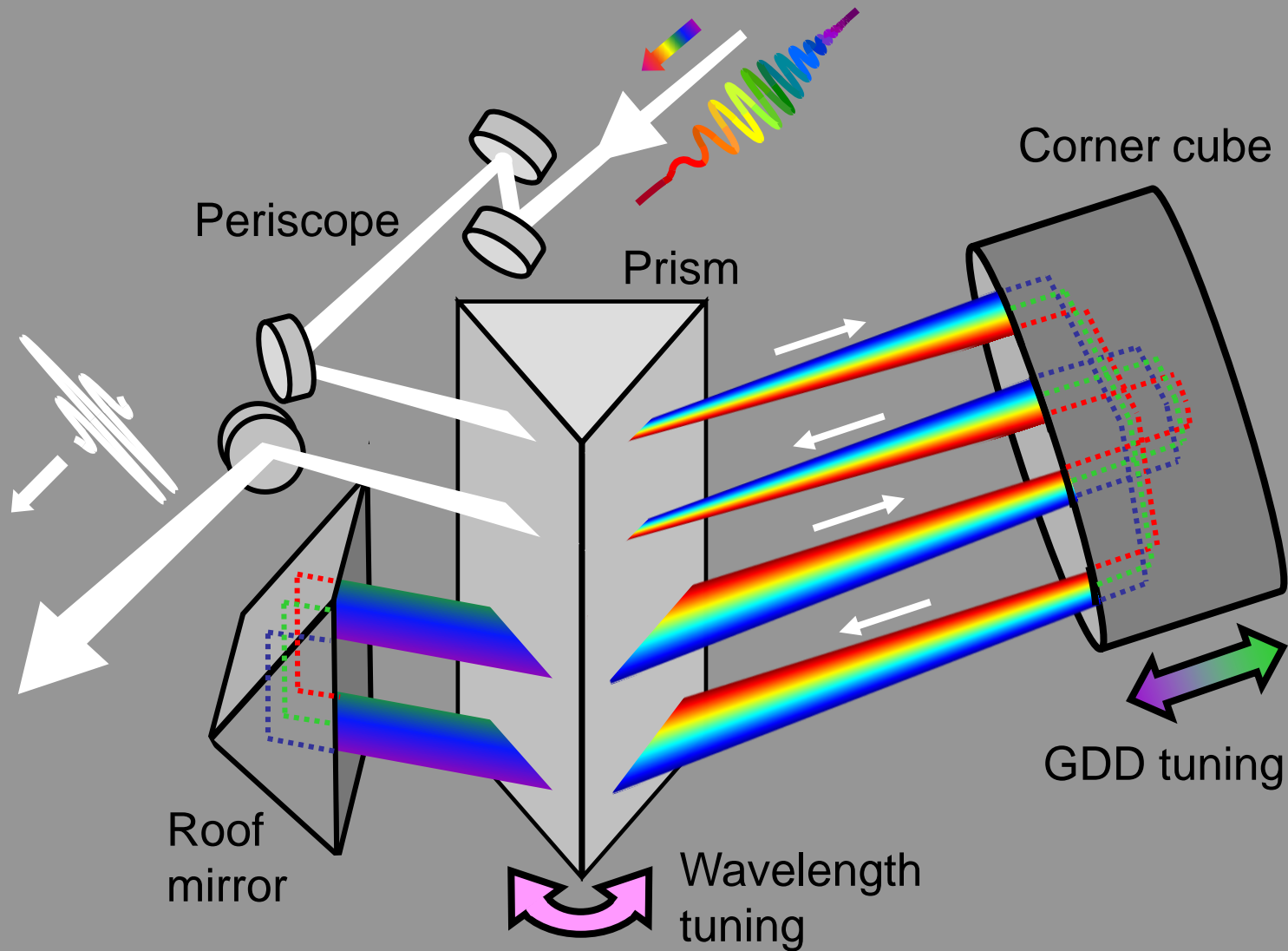
The variation of the **deviation angle** is **2<sup>nd</sup> order** in the prism angle. But what matters is the prism **angular dispersion**, which is **1<sup>st</sup> order**! Using a 2<sup>nd</sup>-order effect to align a 1<sup>st</sup>-order effect is a bad idea.

# Two-prism pulse compressor



This design cuts the size and alignment issues in half.

# Single-prism pulse compressor



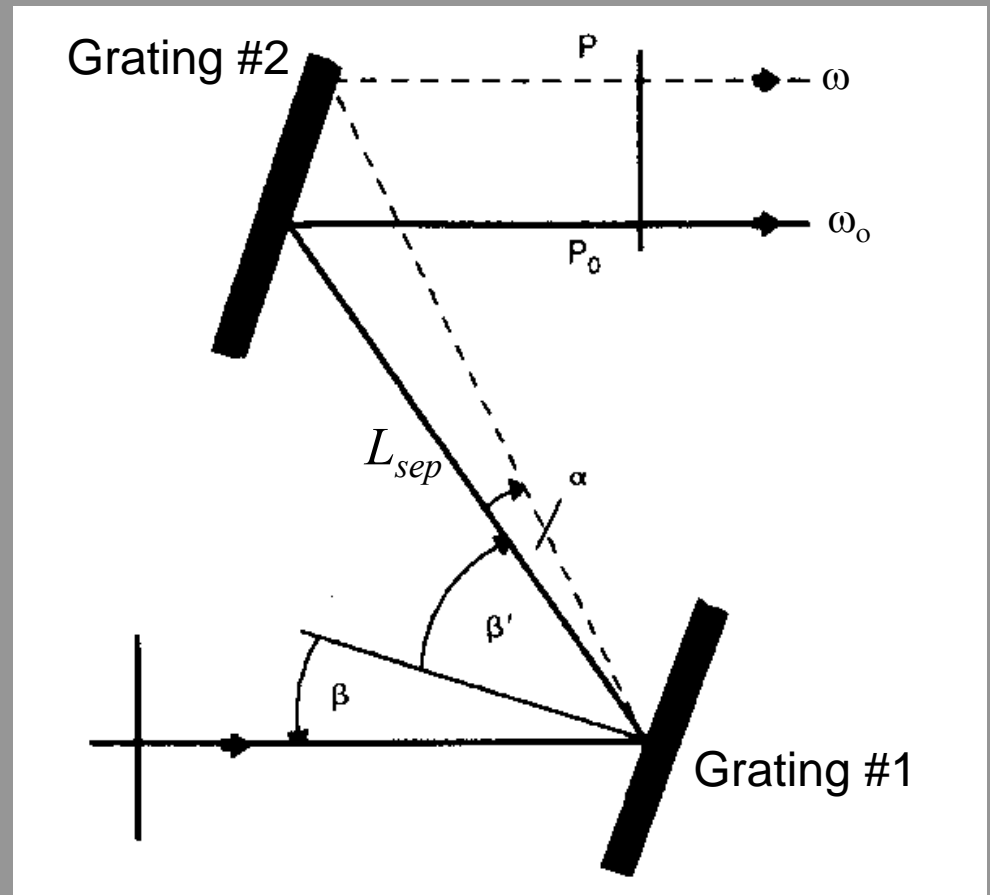
# Diffraction-grating pulse compressor

The grating pulse compressor also has negative GDD.

$$\left. \frac{d^2 \varphi}{d\omega^2} \right|_{\omega_0} \approx - \frac{\lambda_0^3}{2\pi c^2 d^2} \frac{L_{sep}}{\cos^2(\beta')}$$

where  $d$  = grating spacing  
(same for both gratings)

Note that, as in the prism pulse compressor, the larger  $L_{sep}$ , the larger the negative GDD.





# 2nd- and 3rd-order phase terms for prism and grating pulse compressors

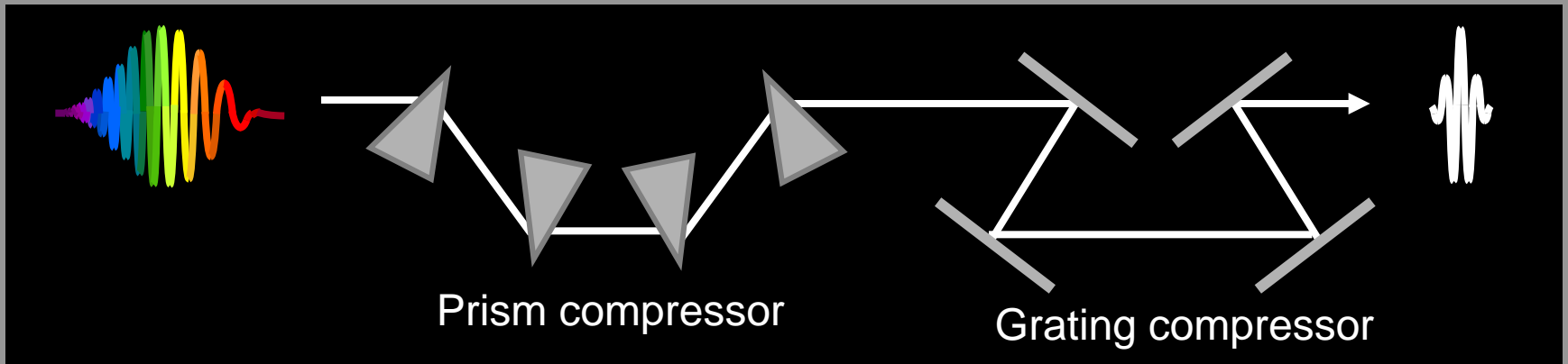
Grating compressors offer more compression than prism compressors.

Device	$\lambda_\ell$ [nm]	$\omega_\ell$ [fs <sup>-1</sup> ]	$\phi''$ [fs <sup>-2</sup> ]	$\phi'''$ [fs <sup>-3</sup> ]
SQ1 ( $L = 1$ cm)	620	3.04	550	240
Piece of glass	800	2.36	362	280
Brewster prism pair, SQ1	620	3.04	-760	-1300
$\ell = 50$ cm	800	2.36	-523	-612
grating pair	620	3.04	$-8.2 \cdot 10^4$	$1.1 \cdot 10^5$
$b = 20$ cm; $\beta = 0^\circ$ $d = 1.2 \mu\text{m}$	800	2.36	$-3 \cdot 10^6$	$6.8 \cdot 10^6$

Note that the relative signs of the 2nd and 3rd-order terms are opposite for prism compressors and grating compressors.

# Compensating 2nd and 3rd-order spectral phase

Use both a prism and a grating compressor. Since they have 3rd-order terms with opposite signs, they can be used to achieve almost arbitrary amounts of both second- and third-order phase.



Given the 2nd- and 3rd-order phases of the input pulse,  $\varphi_{input2}$  and  $\varphi_{input3}$ , solve simultaneous equations:

$$\varphi_{input2} + \varphi_{prism2} + \varphi_{grating2} = 0$$

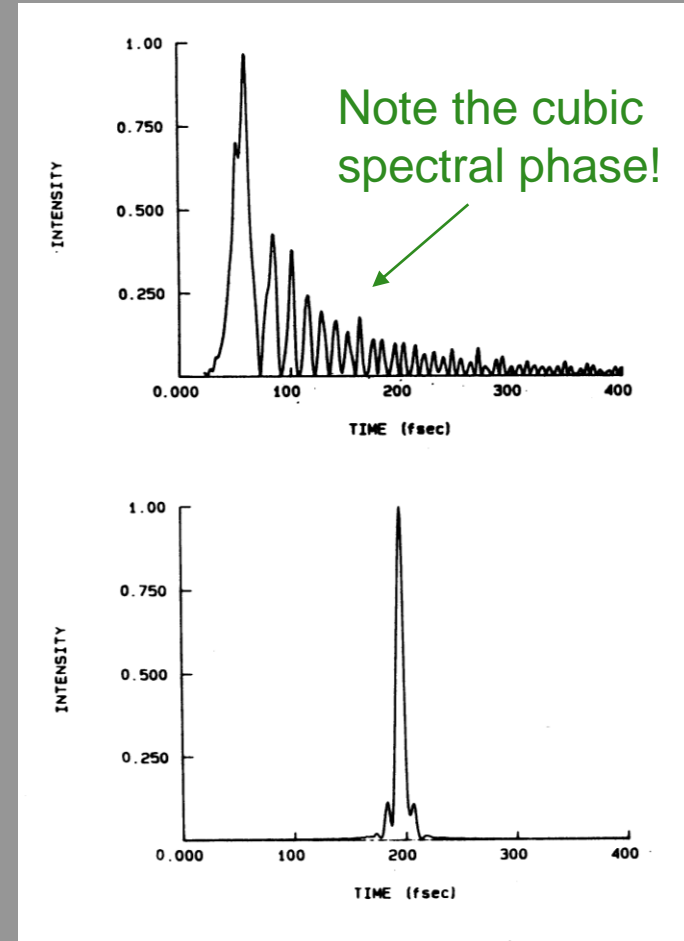
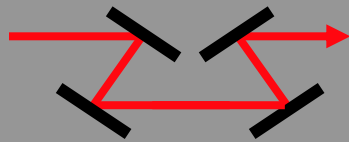
$$\varphi_{input3} + \varphi_{prism3} + \varphi_{grating3} = 0$$

This design was used by Fork and Shank at Bell Labs in the mid 1980's to achieve a 6-fs pulse, a record that stood for over a decade.

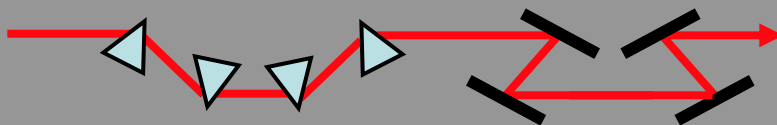
# Pulse Compression Simulation

Using prism and grating pulse compressors vs. only a grating compressor

Resulting intensity vs. time  
with only a grating compressor:

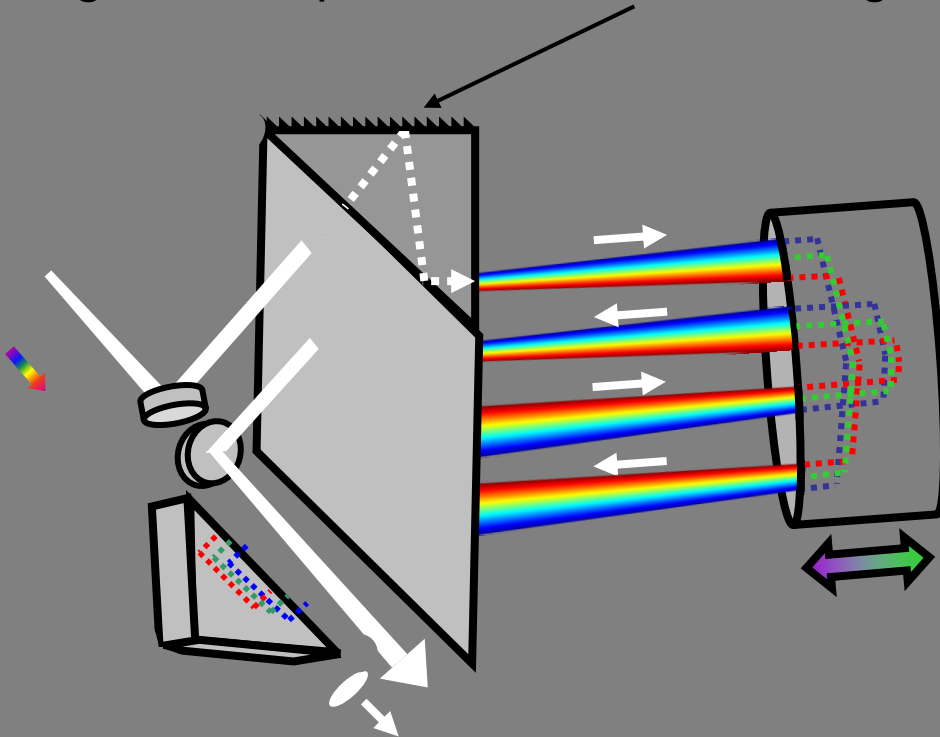


Resulting intensity vs. time  
with a grating compressor  
**and** a prism compressor:



# The **grism** pulse compressor has tunable third-order dispersion.

A grism is a prism with a diffraction grating etched onto it.



The (transmission) grism equation is:

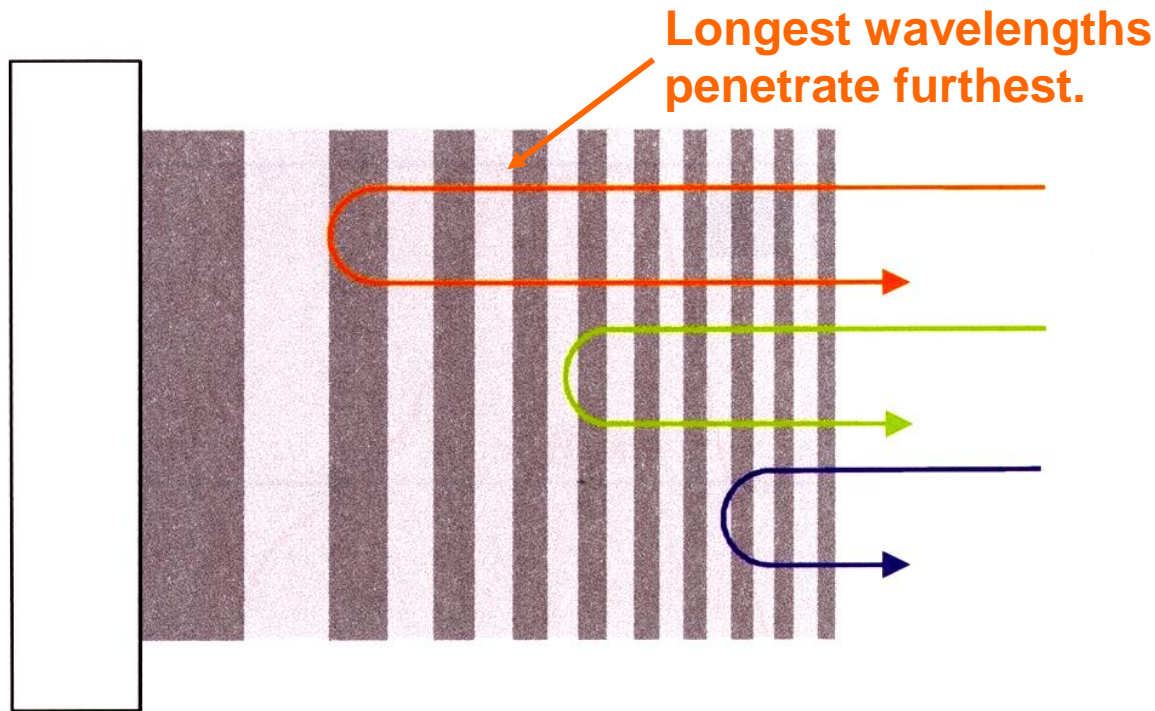
$$a [\sin(\theta_m) - n \sin(\theta_i)] = m\lambda$$

Note the factor of  $n$ , which does not occur for a diffraction grating.

A grism compressor can compensate for both 2<sup>nd</sup> and 3<sup>rd</sup>-order dispersion due even to many meters of fiber.

# Chirped mirror coatings also yield dispersion compensation.

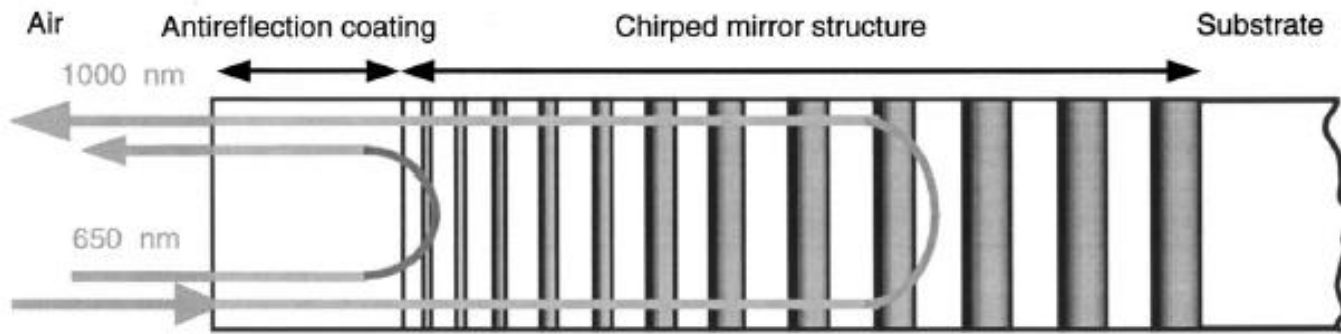
## Ultrabroad-Band Chirped Multilayer Mirrors



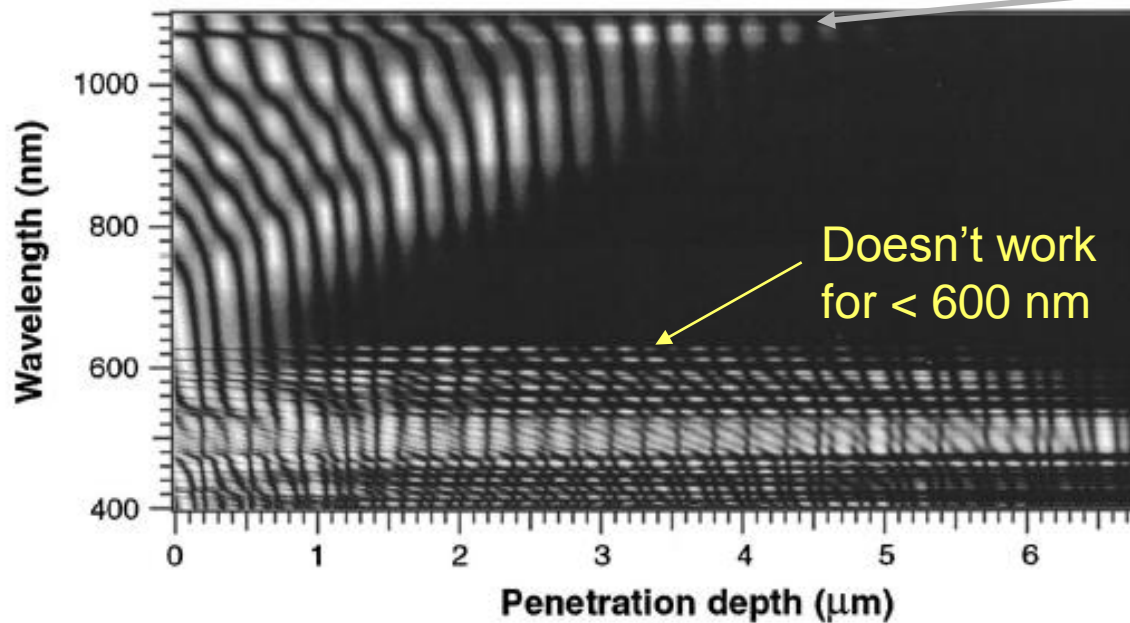
Group delay is nearly linear with the wavelength

Such mirrors avoid spatio-temporal effects, but they have limited GDD.

# Chirped mirror coatings

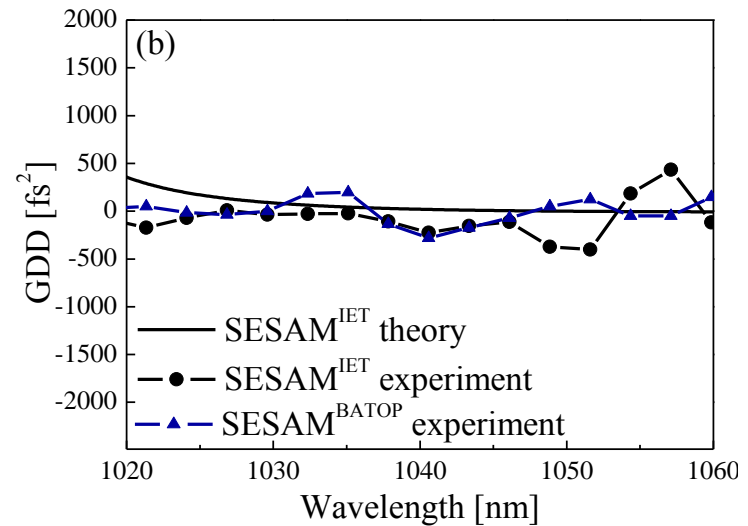
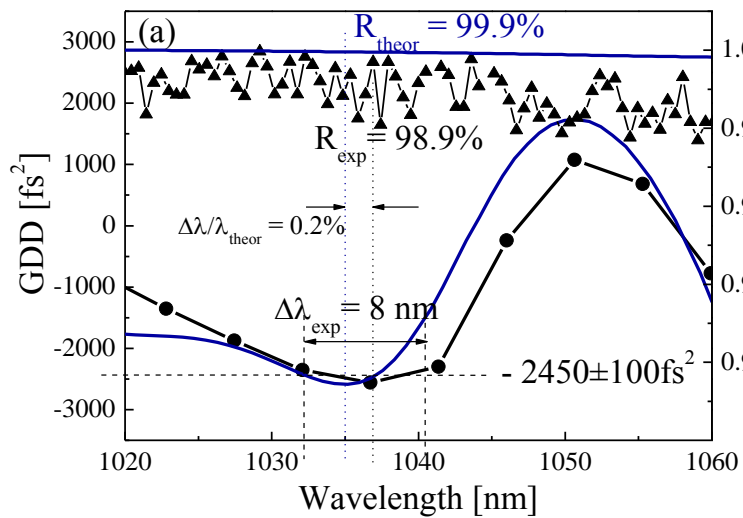
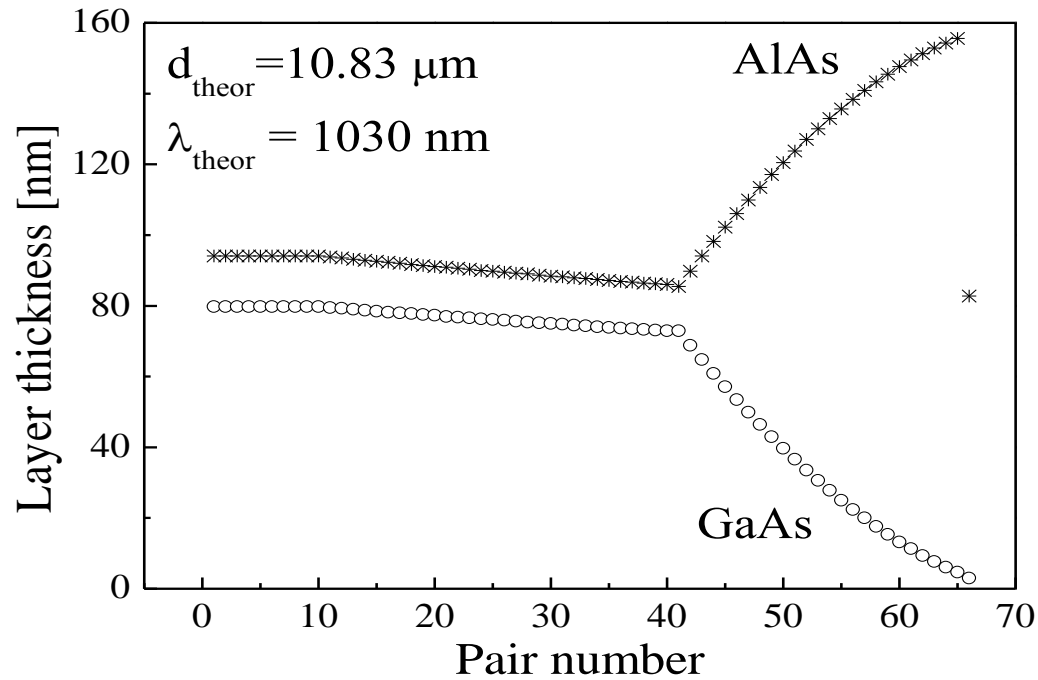


(a)



(b)

# Real-life chirped mirror coating (from Agata Jasik group at IET Warsaw)



# Tunable chirped mirror

