TWO APPROACHES TO SPINOR FIELDS ON MANIFOLDS

What are they?

1. Weyl, Wigner and Fock (1929): refer spinor fields on curved manifolds to orthonormal frames (Weyl: *Achsenkreuze*; later: *Vierbeine*), use constant Dirac matrices; (much) later formalized by mathematicians as spin structures (principal bundles)

2. Tetrode (1928), Schrödinger (1932), Infeld and van der Waerden (1933), Bergmann (1957): assume Dirac matrices to be point-dependent and covariantly constant (vector bundles)

Plan of talk

Spinors according to the Ancient Greeks

Words of the masters and the controversies

Two approaches to vector and spinor fields

Double valuedness of spinor fields on $S^2$ and the Schrödinger solution

Dirac operator on hypersurfaces

Example: the sphere $S_m \subset \mathbb{R}^{m+1}$

SPINORS ACCORDING TO THE ANCIENT GREEKS

Spinors are implicit in solution of the Pythagorean equation

$$x^2 + y^2 = z^2$$

which is equivalent to

$$(G) \quad \begin{pmatrix} z + x & y \\ y & z - x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix}$$

and gives (Euclid) $x = p^2 - q^2$, $y = 2pq$, $z = p^2 + q^2$.

In $\mathbb{R}^3$ with $g$ of signature $(2, 1)$: null vector = (spinor)$^2$.

Lessons from Euclid

1. Generalize to higher dim

   totally null multivector of max dim = (pure spinor)$^2$.

   (Veblen, Cartan; pure essential in dim $\geq 7$)

2. Spin groups

   Multiply $(G)$ on the left by real unimodular matrix $A$, on the right by its transpose, take det to get

   $$1 \to \mathbb{Z}_2 \to SL_2(\mathbb{R}) = \text{Spin}_{2,1}^0 \to SO_{2,1}^0 \to 1$$

   NB. If $A \in GL_2(\mathbb{R})$, then vectors transform by rotations and dilations, but not under general linear transformations.

3. Idea of Clifford algebras
Multiply \((G)\) by \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
on the left, take square
\[
\begin{pmatrix}
y & z-x \\
-z-x & -y
\end{pmatrix}^2 = (x^2 + y^2 - z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

“Quadratic form linearized”

Replace \(z\) by \(iz\) to get complex spinors, Pauli matrices, Spin\(_3\) = SU\(_2\), etc.

4. **Non-trivial topology involved**

Rotation by \(\alpha\) in \((p, q)\) plane:

\[
p' = p \cos \alpha + q \sin \alpha, \quad q' = -p \sin \alpha + q \cos \alpha
\]

induces rotation by \(2\alpha\) in \((x, y)\) plane:

\[
x' = x \cos 2\alpha + y \sin 2\alpha, \quad y' = -x \sin 2\alpha + y \cos 2\alpha
\]

Take \(0 \leq \alpha \leq \pi\) to conclude: **Spinors change sign when rotated by \(2\pi\)**

**WORDS OF THE MASTERS**

The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors.


The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures.


**THE CONTROVERSY**

E. Schrödinger mildly criticized the approach to spinor fields based on Weyl’s *Achsenkreuze*:

Bei diesem Verfahren ist es ein bißchen schwer, zu erkennen, ob die Einsteinsche Idee des Fernparallelismus, auf die teilweise direkt Bezug genommen wird, wirklich hereinspielt oder ob man davon unabhängig ist.


NB. Remarkable paper; contains the first derivation of formula for the square of the Dirac operator. Criticized by Cartan.


Footnote on p. 150:

Certain physicists regard spinors as entities which are, in a sense, unaffected by the rotations which classical geometric entities (vectors, etc.) can undergo, and of which the components in a given reference system are susceptible of undergoing linear transformations which are in a sense autonomous. See for example L. Infeld and B. L. van der Waerden, “Die Wellengleichungen des Elektrons in der allgemeinen Relativitätstheorie”, *S. B. preuss. Akad. Wiss.*, 1933, 380.

Note that Cartan dismisses van der Waerden as a ‘certain physicist’
From p. 151:

Theorem. With the geometric sense we have given to the word “spinor” it is impossible to introduce fields of spinors into the classical Riemannian technique; that is, having chosen an arbitrary system of co-ordinates $x^i$ for the space, it is impossible to represent a spinor by any finite number $N$ whatsoever, of components $u_\alpha$ such that the $u_\alpha$ have covariant derivatives of the form

$$u_{\alpha,i} = \partial u_\alpha / \partial x^i + \Lambda^\beta_{\alpha i} u_\beta$$

where the $\Lambda^\beta_{\alpha i}$ are determinate functions of $x^h$.

Cartan defines spinor fields by reference to orthonormal frames and emphasizes that mere curvilinear coordinates are not enough.

**TWO APPROACHES TO VECTOR AND SPINOR FIELDS**

Recall two ways of describing vector fields on $m$-dim manifold:

1. start from principal bundle $P$ of linear frames

   $$\text{GL}_m(\mathbb{R}) \xrightarrow{\quad} P \xrightarrow{X} \mathbb{R}^m$$

   vector field $X$ is equivariant map: $X(ea) = a^{-1}X(e)$, $a \in \text{GL}_m(\mathbb{R})$, $e \in P$. If $s : M \to P$ is a section, then $X \circ s : M \to \mathbb{R}^m$ gives the components of $X$ with respect to the field of frames $s$.

2. tangent bundle

   $$\xymatrix@C=20pt{ TM \ar[d]_{\pi} \ar[r]^<>(0.5){X'} & M \ar[u]_{\quad}\ar@/_1.5pc/[ll]_{X} }$$

   vector field $X'$ is section of the tangent bundle

**Connection between the two:**

$1 \Rightarrow 2$ by forming associated bundle:

$$(P \times \mathbb{R}^m) / \text{GL}_m(\mathbb{R}) = TM$$

$1 \Leftarrow 2$ define

$$P = \{ e : \mathbb{R}^m \to T_x M \text{ linear isom.} \mid x \in M \}$$

Consider analogous definitions for spinors. For simplicity of notation assume $(M, g)$ to be proper Riemannian, orientable manifold. Recall that given a quadratic space $(V, h)$, one has the Clifford algebra $\text{Cl}(V, h)$ and the spin group $\text{Spin}(V, h) \subset \text{Cl}(V, h)$; if $V = \mathbb{R}^m$ and $h$ definite, then write $\text{Cl}_m$, $\text{Spin}_m$, $\text{SO}_m$, etc. A representation $\gamma : \text{Cl}_m \to \text{End} S$ defines by restriction a representation

$$\gamma : \text{Spin}_m \to \text{GL}(S)$$

in the vector space $S$ of spinors.
start from principal spin bundle $Q$, a reduction of the bundle of orthonormal frames $P$ on $M$,

$$\begin{align*}
\text{Spin}_m & \to Q \overset{\psi}{\to} S \\
\downarrow & \downarrow \\
\text{SO}_m & \to P \\
\downarrow & \\
M &
\end{align*}$$

spinor field $\Psi$ of type $\gamma$ is equivariant map:

$$\Psi(qa) = \gamma(a^{-1})\Psi(q), \quad a \in \text{Spin}_m.$$ 

The Levi-Civita connection lifts from $P$ to $Q$ and defines Dirac operator $\gamma^\mu \nabla_\mu$ acting on spinor fields $\Psi : Q \to S$. No teleparallelism required.

2. To define bundle of spinors $\Sigma \to M$, analogous to $TM \to M$, consider the Clifford bundle

$$\text{Cl}(g) = \bigcup_{x \in M} \text{Cl}(T_xM, g_x) \to M$$

and a map of bundles

$$\gamma : \text{Cl}(g) \to \text{End} \Sigma$$

such that for $x \in M$, the restriction

$$\gamma_x : \text{Cl}(T_xM, g_x) \to \text{End} \Sigma_x$$

is a representation of the Clifford algebra $\text{Cl}(T_xM, g_x)$ in $\Sigma_x$. Sections $\Psi : M \to \Sigma$ are spinor fields.

NB. Schrödinger assumed $\gamma_x$ to be the Dirac repr., whereas Bergmann considered 2-component Weyl spinors and split $\Sigma = \Sigma_+ \oplus \Sigma_-$ so that $\gamma = \begin{pmatrix} 0 & \tau \\ \sigma & 0 \end{pmatrix}$.

If $X : M \to TM \to \text{Cl}(g)$ is a vector field, then $\gamma(X) : M \to \text{End} \Sigma$ is a field of matrices acting on spinor fields. Covariant differentiation is introduced by requiring

$$\nabla_X(\gamma(Y)\Psi) = \gamma(\nabla_X^L Y)\Psi + \gamma(Y)\nabla_X \Psi$$


**DOUBLE VALUEDNESS OF SPINOR FIELDS ON $S_2$ AND THE SCHRODINGER TRICK**

As example, construct spin structure on $M = S_2$; note first $\text{Spin}_2 = U_1$. The spin bundle is

$$Q = \text{Spin}_3 = \text{SU}_2 = S_3 = \{q = (q_1, q_2) \in \mathbb{C}^2 : |q_1|^2 + |q_2|^2 = 1\},$$

the action of $\text{Spin}_2$ on $Q$ is $(q, a) \mapsto qa = (q_1a, q_2a), a \in U_1$, and the bundle of orth. frames $P = \text{SO}_3 = \text{SU}_2 / \mathbb{Z}_2$ so that

$$Q \to P \text{ is } (q_1, q_2) \mapsto \pm(q_1, q_2) \quad \text{and}$$
Indeed, the spin fields become double-valued on spinor fields $\Psi(\pm)$ coming from the stereographic projection; there is then the section $\chi$ of spin spherical harmonics. This and similar problems were noticed, considered at length, and solved by the masters: E. Schrödinger, Eigenschwingungen des sphärischen Raums, Comment. Pontificia Acad. Scientiarum 2 (1938) 321–364. W. Pauli, Über ein Kriterium für Ein- oder Zweiwertigkeit der Eigenfunktionen in der Wellenmechanik, Helv. Phys. Acta 12 (1939) 147–168.

Note: double valuedness does not arise if one starts from a field of orth. frames on $S^2$, coming from the stereographic projection; there is then the section $\Psi: Q \to S_3 \to S = \mathbb{C}^2$ satisfies $\Psi(qa) = \gamma(a^{-1})\Psi(q)$ so that $\Psi(-q) = -\Psi(q)$. Take section $s: S^2 \setminus \{\varphi = 0\} \to S_3$

$$s(\theta, \varphi) = (e^{\frac{1}{2}i\varphi} \cos \frac{1}{2}\theta, e^{-\frac{1}{2}i\varphi} \sin \frac{1}{2}\theta)$$

then $\lim_{\varphi \to 0} s = -\lim_{\varphi \to 2\pi} s$. Look at the ‘usual’ description of spinor field: $\Psi \circ s$.


Schrödinger’s solution of the problem:

Note: double valuedness does not arise if one starts from a field of orth. frames on $S^2$, coming from the stereographic projection; there is then the section $s': S^2 \setminus \{\theta = 0\} \to S_3$, $s'(\theta, \varphi) = (e^{i\varphi} \cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta)$ and $\Psi \circ s'$ is single-valued. This approach taken in theory of spin spherical harmonics.

Consider the spinor field (equivariant map) $\Psi: \text{Spin}_3 \to \mathbb{C}^2$. Form $\Phi(q) = \sigma(q)\Psi(q)$, then for $a \in \text{Spin}_2$ one has $\Phi(qa) = \Phi(q)$ so that $\Phi$ descends to a globally defined, single-valued map $\psi: S_2 \to \mathbb{C}^2$. This shows: bundle $\Sigma$ of spinors on $S_2$ is trivial. But the tangent bundle is not.
Theorem. Every hypersurface $M$ immersed in $\mathbb{R}^{m+1}$ has a (s)pin structure and the bundle of Dirac ($m$ even) or Pauli ($m$ odd) spinors, associated with it, is trivial. If $m$ even and $M$ orientable, then there is the bundle of Weyl (chiral) spinors over $M$; it need not be trivial.

The Dirac operator, originally defined to act on the equivariant map $\Psi : Q \to S$, for a hypersurface $M$ descends to act on the corresponding field $\psi : M \to S$.

**DIRAC OPERATOR ON HYPERSURFACES**

Assume $M$ to be an even-dimensional orientable hypersurface in $\mathbb{R}^{m+1}$ and let $n^\mu, \mu = 1, \ldots, m+1$, be a field of unit normals on it. Let $\gamma_\mu$ be the Dirac matrices satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu} I$$

and put $N = \gamma_\mu n^\mu$,

$$\sigma_{\mu\nu} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \quad \text{and} \quad L_{\mu\nu} = n_\nu \partial_\mu - n_\mu \partial_\nu.$$ 

The vector fields $L_{\mu\nu}$ are tangent to $M$. One derives the following formula for the Dirac operator acting on $\psi : M \to S$:

$$D = N (\sigma_{\mu\nu} L_{\mu\nu} - \frac{1}{2} \text{div} n)$$

and

$$DN + ND = 0$$

so that $N$ plays the role of $\gamma_5$ in SRT. (Analogy with Polyakov’s hedgehog solution.)

There is more if there is a foliation of $\mathbb{R}^{m+1}$ by hypersurfaces; introducing $\frac{\partial}{\partial r} = n^\mu \partial_\mu$ one has the decomposition

$$\gamma^\mu \partial_\mu = D + N (\frac{\partial}{\partial r} + \frac{1}{2} \text{div} n)$$

**EXAMPLE: THE SPHERE $S_m \subset \mathbb{R}^{m+1}$**

Consider $\mathbb{R}^{m+1}$ foliated by spheres $r = \text{const.}$, then $n^\mu = x^\mu / r$ and $\text{div} n = m / r$; for $m = 3$ the vector with components $rL_{23}, rL_{31}, rL_{12}$ is the angular momentum operator.

Recall formula for the Laplacian:

$$\Delta_{\mathbb{R}^{m+1}} = r^{-2} \Delta_{S_m} + r^{-m} \frac{\partial}{\partial r} r^m \frac{\partial}{\partial r}$$

Evaluating on a harmonic polynomial homogeneous of degree $l$ gives the eigenvalues $-l(l+m-1)$ of $\Delta_{S_m}$.

An analogous formula for the Dirac operator is

$$\gamma^\mu \partial_\mu = r^{-1} D_{S_m} + N r^{-m/2} \frac{\partial}{\partial r} r^{m/2}$$

Take now for $\phi : \mathbb{R}^{m+1} \to S$ a harmonic, homogeneous of degree $l+1$, $S$-valued polynomial, then $\psi = \gamma^\mu \partial_\mu \phi$ is of degree $l$ and satisfies $\gamma^\mu \partial_\mu \psi = 0$, therefore

$$D_{S_m} \psi = -(l + \frac{1}{2} m) \psi \quad \text{and} \quad D_{S_m} N \psi = (l + \frac{1}{2} m) N \psi$$

so that the spectrum of the Dirac operator on $S_m$ is the set $\pm (l + \frac{1}{2} m), l = 0, 1, \ldots$.