Geometry of null hypersurfaces

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February 5, 2004

Abstract

We review some basic natural geometric objects on null hypersurfaces. Gauss-Codazzi constraints are given in terms of the analog of canonical ADM momentum which is a well defined tensor density on the null surface. Bondi cones are analyzed with the help of this object.

1 Introduction

In Synge’s festshrift volume [10] Roger Penrose distinguished three basic structures which a null hypersurface $N$ in four-dimensional spacetime $M$ acquires from the ambient Lorentzian geometry:

• the degenerate metric $g|_N$ (see [9] for Cartan’s classification of them and the solution of the local equivalence problem)
• the concept of an affine parameter along each of the null geodesics from the two-parameter family ruling $N$
• the concept of parallel transport for tangent vectors along each of the null geodesics

Using all three concepts on $N$ one can define several natural geometric objects which we shall review in this article.

In Section 2 we remind the structures which are presented in [1]. In the next section we give solutions, which are mostly based on [2], to the following questions:

• What is the analog of canonical ADM momentum for the null surface?
• What are the "initial value constraints"?
• Are they intrinsic objects?

*supported by the Polish Research Council grant KBN 2 P03B 073 24
More precisely, we remind the construction of external geometry in terms of tensor density which is a well defined intrinsic object on a null surface. We already developed some applications of these object to the following subjects:

- Dynamics of the light-like matter shell from matter Lagrangian which is an invariant scalar density on $N$ [3]
- Dynamics of gravitational field in a finite volume with null boundary and its application to black holes thermodynamics [6] (see also in this volume)
- Geometry of crossing null shells [4].

In the last section we apply our construction to Bondi cones.

2 Natural geometric structures on $TN/K$

We remind some standard constructions on null hypersurfaces (see [1]):

- time-oriented Lorentzian manifold $M$ with signature $(-,+,+,+)$.
- null hypersurface $N$ – submanifold with codim=1 and degenerate induced metric $g|_N$ $(0,+,+,+)$, $K$ – time-oriented non-vanishing null vector field such that $K_p^+ = T_pN$ at each point $p \in N$
  1. $K$ is null and tangent to $N$, $g(X,K)=0$ iff $X$ is a vector field tangent to $N$
  2. integral curves of $K$ are null geodesic generators of $N$
  3. $K$ is determined by $N$ up to scaling factor being any positive function.
- $T_pN/K := \{ \overline{X} : X \in T_pN \}$ where $\overline{X} = [X]_{mod \ K}$ is an equivalence class of the relation mod $K$ defined as follows:
  $X \equiv Y (mod \ K) \iff X - Y$ is parallel to $K$.
- $TN/K := \cup_{p \in N} T_pN/K$ vector bundle over $N$ with 2-dimensional fibers (equipped with Riemannian metric $h$), the structure does not depend on the choice of $K$ (scaling factor)
  $h : T_pN/K \times T_pN/K \longrightarrow \mathbb{R}, \quad h(\overline{X},\overline{Y}) = g(X,Y)$.
- null Weingarten map $b_K$ (depending on the choice of scaling factor, in non-degenerate case one can always take unit normal to the hypersurface but in null case the vectorfield $K$ is no longer transversal to $N$ and has always scaling factor freedom because its length vanishes)
  \[ b_K : T_pN/K \longrightarrow T_pN/K, \quad b_K(\overline{X}) = \nabla_X K; \]
  \[ b_{fK} = fb_K, \quad f \in C^\infty(N), \quad f > 0. \]
null second fundamental form $B_K$ (bilinear form associated to $b_K$ via $h$)

$$B_K : T_pN/K \times T_pN/K \rightarrow \mathbb{R}$$

Moreover, $b_K$ is self-adjoint with respect to $h$ and $B_K$ is symmetric.

• $N$ is totally geodesic (i.e. restriction to $N$ of the Levi-Civita connection of $M$ is an affine connection on $N$, any geodesic in $M$ starting tangent to $N$ stays in $N$) $\iff B = 0$ (non-expanding horizon is a typical example).

• null mean curvature of $N$ with respect to $K$

$$\theta := \text{tr} b = \sum_{i=1}^{2} B_K(\tau_i, \tau_i) = \sum_{i=1}^{2} g(\nabla_{e_i} K, e_i)$$

where $\tau_i$ is an orthonormal basis for $T_pN/K$, $e_i$ is an orthonormal basis for $T_pS$ in the induced metric on $S$ which is a two-dimensional submanifold of $N$ transverse to $K$.

We assume now that $K$ is a geodesic vector field i.e. $\nabla_K K = 0$. Let us denote by prime covariant differentiation in the null direction:

$$\mathbf{Y}' := \nabla_{\mathbf{K}} \mathbf{Y}, \quad b'(\mathbf{Y}) := b(\mathbf{Y})' - b(\mathbf{Y}')$$

From Riemann tensor we build the following curvature endomorphism

$$R : T_pN/K \rightarrow T_pN/K, \quad R(\mathbf{X}) = R_{\text{Riemann}}(\mathbf{X}, K)K$$

and we get a Ricatti equation

$$b' + b^2 + R = 0 \quad (1)$$

Taking the trace of (1) we obtain well-known Raychaudhuri equation:

$$\theta' = -\text{Ricci}(K, K) - B^2, \quad B^2 = \sigma^2 + \frac{1}{2} \theta^2 \quad (2)$$

where $\sigma$ is a shear scalar corresponding to the trace free part of $B$. A standard application of the Raychaudhuri equation gives the following

**Proposition 1.** Let $M$ be a spacetime which obeys the null energy condition, i.e. $\text{Ricci}(X, X) \geq 0$ for all null vectors $X$, and let $N$ be a smooth null hypersurface in $M$. If the null generators of $N$ are future geodesically complete then $N$ has nonnegative null mean curvature i.e. $\theta \geq 0$.

### 3 Canonical momentum on null surface

For non-degenerate hypersurface we define the canonical ADM momentum:

$$p^{\hat{k}l} := \sqrt{\det g_{mn}} (g^{\hat{k}k} g^{\hat{l}l} K_{ij} - K^{k \hat{l}i} ), \quad (3)$$

where $K^{k \hat{l}i}$ is the second fundamental form (external curvature) of the imbedding of the hypersurface into the spacetime $M$. 


Gauss-Codazzi equations for non-degenerate hypersurface are the following:

\[ P_i^l|l = \sqrt{\det g_{mn}} G_{\mu\nu} n^\mu \quad (= 8\pi \sqrt{\det g_{mn}} T_{\mu\nu} n^\nu), \quad (4) \]

\[ (\det g_{mn}) R - P_{kl} P_{kl} + \frac{1}{2} (P_{kl} g_{kl})^2 = 2(\det g_{mn}) G_{\mu\nu} n^\mu n^\nu \quad (= 16\pi (\det g_{mn}) T_{\mu\nu} n^\mu n^\nu), \quad (5) \]

where \( R \) is the (three–dimensional) scalar curvature of \( g_{kl} \), \( n^\nu \) is a four–vector normal to the hypersurface, \( T_{\mu\nu} \) is an energy–momentum tensor of the matter field, and the calculations have been made with respect to the non-degenerate induced three–metric \( g_{kl} \) (“\( | \)” denotes covariant derivative, indices are raised and lowered with respect to that metric etc.).

A null hypersurface in a Lorentzian spacetime \( M \) is a three-dimensional submanifold \( N \subset M \) such that the restriction \( g_{ab} \) of the spacetime metric \( g_{\mu\nu} \) to \( N \) is degenerate.

We shall often use adapted coordinates, where coordinate \( x^3 \) is constant on \( N \). Space coordinates will be labeled by \( k, l = 1, 2, 3 \); coordinates on \( N \) will be labeled by \( a, b = 0, 1, 2 \); finally, coordinates on \( S \) will be labeled by \( A, B = 1, 2 \). Spacetime coordinates will be labeled by Greek characters \( \alpha, \beta, \mu, \nu \).

We will show in the sequel that null-like counterpart of initial data \((g_{kl}, P^k)\) consists of the metric \( g_{ab} \) and tensor density \( Q^a_b \) which is a mixed (contravariant-covariant) tensor density.

The non-degeneracy of the spacetime metric implies that the metric \( g_{ab} \) induced on \( N \) from the spacetime metric \( g_{\mu\nu} \) has signature \((0,+,+)\). This means that there is a non-vanishing null-like vector field \( K^a \) on \( N \), such that its four-dimensional embedding \( K^\mu \) to \( M \) (in adapted coordinates \( K^3 = 0 \)) is orthogonal to \( N \). Hence, the covector \( K_\nu = K^\mu g_{\mu\nu} = K^a g_{a\nu} \) vanishes on vectors tangent to \( N \) and, therefore, the following identity holds:

\[ K^a g_{ab} \equiv 0. \quad (6) \]

It is easy to prove that integral curves of \( K^a \), after a suitable reparameterization, are geodesic curves of the spacetime metric \( g_{\mu\nu} \). Moreover, any null hypersurface \( N \) may always be embedded in a one-parameter congruence of null hypersurfaces.

We assume that topologically we have \( N = \mathbb{R}^1 \times S^2 \). Since our considerations are purely local, we fix the orientation of the \( \mathbb{R}^1 \) component and assume that null-like vectors \( K \) describing degeneracy of the metric \( g_{ab} \) of \( N \) will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate \( x^0 \) increases in the direction of \( K \), i.e. inequality \( K(x^0) = K^0 > 0 \) holds. In these coordinates degeneracy fields are of the form \( K = f(\partial_0 - n^A \partial_A) \), where \( f > 0 \), \( n_A = g_{0A} \) and we rise indices with the help of the two-dimensional matrix \( \tilde{g}^{AB} \), inverse to \( g_{AB} \).

If by \( \lambda \) we denote the two-dimensional volume form on each surface \( \{x^0 = \text{const.}\} \):

\[ \lambda := \sqrt{\det g_{AB}}, \quad (7) \]
then for any degeneracy field $K$ of $g_{ab}$ the following object

$$v_K := \frac{\lambda}{K(x^a)}$$

is a well defined scalar density on $N$. This means that

$$v_K := v_K dx^0 \wedge dx^1 \wedge dx^2$$

is a coordinate-independent differential three-form on $N$. However, $v_K$ depends upon the choice of the field $K$.

It follows immediately from the above definition that the following object:

$$\Lambda = v_K K$$

is a well defined (i.e. coordinate-independent) vector density on $N$.

Obviously, it does not depend upon any choice of the field $K$:

$$\Lambda = \lambda (\partial_0 - n^A \partial_A) \quad (8)$$

and it is an intrinsic property of the internal geometry $g_{ab}$ of $N$. The same is true for the divergence $\partial_a \Lambda^a$ which is, therefore, an invariant, $K$-independent, scalar density on $N$. Mathematically (in terms of differential forms) the quantity $\Lambda$ represents the two-form:

$$L := \Lambda^a (\partial_a \lrcorner dx^0 \wedge dx^1 \wedge dx^2)$$

whereas the divergence represents its exterior derivative (a three-from):

$$dL := (\partial_a \Lambda^a) dx^0 \wedge dx^1 \wedge dx^2.$$ 

In particular, a null surface with vanishing $dL$ is the non-expanding horizon.

Both objects $L$ and $v_K$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $p \in N$ the tangent space $T_p N$ may be quotiented with respect to the degeneracy subspace spanned by $K$. The quotient space $T_p N/K$ carries a non-degenerate Riemannian metric $h$ and, therefore, is equipped with a volume form $\omega$ (its coordinate expression would be: $\omega = \lambda dx^1 \wedge dx^2$).

The two-form $L$ is equal to the pull-back of $\omega$ from the quotient space $T_p N/K$ to $T_p N$:

$$\pi : T_p N \longrightarrow T_p N/K, \quad L := \pi^* \omega.$$ 

The three-form $v_K$ may be defined as a product:

$$v_K = \alpha \wedge L,$$

where $\alpha$ is any one-form on $N$, such that $< K, \alpha > \equiv 1$.

We have

$$dL = \theta v_K$$

where $\theta$ is a null mean curvature of $N$.

The degenerate metric $g_{ab}$ on $N$ does not allow to define via the compatibility condition $\nabla g = 0$, any natural connection, which could be applied to generic tensor fields on $N$. Nevertheless, there is one exception: the degenerate metric defines uniquely a certain covariant, first
order differential operator. The operator may be applied only to mixed (contravariant-covariant) tensor density fields \( H^a_{\ b} \), satisfying the following algebraic identities:

\[
H^a_{\ b} K^b = 0, \quad H_{a b} = H_{b a},
\]

where \( H_{a b} \) is the extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor densities of this type.

The operator, which we denote by \( \nabla_a \), is defined by means of the four-dimensional metric connection in the ambient spacetime \( M \) in the following way:

Given \( H^a_{\ b} \), take any its extension \( H^\mu_{\nu} \) to a four-dimensional, symmetric tensor density, “orthogonal” to \( N \), i.e. satisfying \( H^\perp_{\nu} = 0 \) (“\( \perp \)” denotes the component transversal to \( N \)). Define \( \nabla_a H^a_{\ b} \) as the restriction to \( N \) of the four-dimensional covariant divergence \( \nabla_\mu H^\mu_{\nu} \). The ambiguities, which arise when extending three-dimensional object \( H^a_{\ b} \) living on \( N \) to the four-dimensional one, cancel finally and the result is unambiguously defined as a covector density on \( N \). It turns out, however, that this result does not depend upon the spacetime geometry and may be defined intrinsically on \( N \) as follows:

\[
\nabla_a H^a_{\ b} = \partial_a H^a_{\ b} - \frac{1}{2} H^{ac} g_{ac, b},
\]

where \( g_{ac, b} := \partial_b g_{ac} \), a tensor density \( H^a_{\ b} \) satisfies identities (9), and moreover, \( H^{ac} \) is any symmetric tensor density, which reproduces \( H^a_{\ b} \) when lowering an index:

\[
H^a_{\ b} = H^{ac} g_{cb}.
\]

It is easily seen, that such a tensor density always exists due to identities (9), but the reconstruction of \( H^{ac} \) from \( H^a_{\ b} \) is not unique because \( H^{ac} + CK^a K^c \) also satisfies (11) if \( H^{ac} \) does. Conversely, two such symmetric tensors \( H^{ac} \) satisfying (11) may differ only by \( CK^a K^c \). Fortunately, this non-uniqueness does not influence the value of (10).

Hence, the following definition makes sense:

\[
\nabla_a H^a_{\ b} := \partial_a H^a_{\ b} - \frac{1}{2} H^{ac} g_{ac, b}.
\]

The right-hand-side does not depend upon any choice of coordinates (i.e. it transforms like a genuine covector density under change of coordinates).

To express directly the result in terms of the original tensor density \( H^a_{\ b} \), we observe that it has five independent components and may be uniquely reconstructed from \( H^0_{\ A} \) (2 independent components) and the symmetric two-dimensional matrix \( H_{AB} \) (3 independent components). Indeed, identities (9) may be rewritten as follows:

\[
H^A_{\ B} = \hat{g}^{AC} H_{CB} - n^A H^0_{\ B},
\]

\[
H^0_{\ B} = H^{0 A} n_A,
\]

\[
H^0_0 = \left( \hat{g}^{BC} H_{CA} - n^B H^0_{\ A} \right) n_A.
\]
The correspondence between $H^{ab}$ and $(H^{0A}, H_{A\mu})$ is one-to-one.

To reconstruct $H^{ab}$ from $H^a_b$ up to an arbitrary additive term $CK^\mu K^\mu$, take the following (coordinate dependent) symmetric quantity:

$$ F^{AB} := C^{AC} g^{CD} C^{DA} - n^A H_0^C g^{CB} - n^B H_0^C g^{CA}, $$

$$ F^{0A} := -H_0^b g^{CA}, $$

$$ F^{00} := 0. $$

(16) (17) (18)

It is easy to observe that any $H^{ab}$ satisfying (11) must be of the form:

$$ H^{ab} = F^{ab} + H^{00} K^a K^b. $$

(19)

The non-uniqueness in the reconstruction of $H^{ab}$ is, therefore, completely described by the arbitrariness in the choice of the value of $H^{00}$. Using these results, we finally obtain:

$$ \nabla_a H^a_b = \frac{1}{2} H^{ac} g_{ac,b} = \frac{1}{2} H^{ac} g_{ac,b} - \frac{1}{2} F^{ac} g_{ac,b}, $$

(20)

The operator on the right-hand-side of (20) is called the (three-dimensional) covariant derivative of $H^a_b$ on $N$ with respect to its degenerate metric $g_{ab}$. It is well defined (i.e. coordinate-independent) for a tensor density $H^a_b$ fulfilling conditions (9). One can also show that the above definition coincides with the one given in terms of the four-dimensional metric connection and, due to (10), it equals:

$$ \nabla_a H^a_b = \frac{1}{2} H^{ac} g_{ac,b} = \frac{1}{2} H^{ac} g_{ac,b} - \frac{1}{2} F^{ac} g_{ac,b}, $$

(21)

hence, it coincides with $\nabla_a H^a_b$ defined intrinsically on $N$.

To describe exterior geometry of $N$ we begin with covariant derivatives along $N$ of the “orthogonal vector $K$“. Consider the tensor $\nabla_a K^\mu$. Unlike in the non-degenerate case, there is no unique “normalization” of $K$ and, therefore, such an object does depend upon a choice of the field $K$. The length of $K$ vanishes. Hence, the tensor is again orthogonal to $N$, i.e. the components corresponding to $\mu = 3$ vanish identically in adapted coordinates. This means that $\nabla_a K^a$ is a purely three-dimensional tensor living on $N$. For our purposes it is useful to use the “ADM-momentum” version of this object, defined in the following way:

$$ Q^a_b (K) := -s \left\{ v_K (\nabla_b K^a - \delta^a_b \nabla_c K^c) + \delta^a_b \partial_c \Lambda^c \right\}, $$

(22)

where $s := \text{sgn} g^{03} = \pm 1$. Due to the above convention, the object $Q^a_b (K)$ feels only external orientation of $N$ and does not feel any internal orientation of the field $K$.

Remark: If $N$ is a non-expanding horizon, the last term in the above definition vanishes.

The last term in (22) is $K$-independent. It has been introduced in order to correct algebraic properties of the quantity

$$ v_K (\nabla_b K^a - \delta^a_b \nabla_c K^c). $$
One can show that $Q^a_b$ satisfies identities (9) and, therefore, its covariant divergence with respect to the degenerate metric $g_{ab}$ on $N$ is uniquely defined. This divergence enters into the Gauss–Codazzi equations, which relate the divergence of $Q$ with the transversal component $\mathcal{G}^b_\perp$ of the Einstein tensor density $G^\mu_\nu = \sqrt{\text{det} g} \left( R^\mu_\nu - \delta^\mu_\nu \frac{1}{2} R \right)$. The transversal component of such a tensor density is a well defined three-dimensional object living on $N$. In coordinate system adapted to $N$, i.e. such that the coordinate $x^3$ is constant on $N$, we have $\mathcal{G}^b_\perp = G^3_b$. Due to the fact that $G$ is a tensor density, components $G^3_b$ do not change with changes of the coordinate $x^3$, provided it remains constant on $N$. These components describe, therefore, an intrinsic covector density living on $N$.

**Proposition 2.** The following null-like-surface version of the Gauss–Codazzi equation is true:

$$\nabla_a Q^a_b (K) + s v_K \partial_b \left( \frac{\partial_c \Lambda^c}{v_K} \right) \equiv -G^\perp_b . \quad (23)$$

The proof is given in [3]. We remind the reader that the ratio between two scalar densities: $\partial_c \Lambda^c$ and $v_K$, is a scalar function $\theta$. Its gradient is a covector field. Finally, multiplied by the density $v_K$, it produces an intrinsic covector density on $N$. This proves that also the left-hand-side is a well defined geometric object living on $N$.

The component $K^b G^\perp_b$ of the equation (23) is nothing but a densitized form of Raychaudhuri equation (2) for the congruence of null geodesics generated by the vector field $K$.

### 4 Initial data on asymptotic Bondi cones

Recall (see [7]) that in Bondi-Sachs coordinates $(u, x, x^A)$ the space-time metric takes the form:

$$^4g = -x V e^{2\beta} du^2 + 2 e^{2\beta} x^{-2} dx dx + x^{-2} h_{AB} (dx^A - U^A du) (dx^B - U^B du) . \quad (24)$$

Let us derive explicitly canonical data $(g_{ab}, Q^a_b)$ on null surfaces $N := \{u = \text{const.}\}$ which we call Bondi cones. The intrinsic coordinates on null surface $N$ are $x^a = (x, x^A)$. We choose null field

$$K := e^{-2\beta} x^2 \partial_3 . \quad (25)$$

The components of the degenerate metric $g_{ab}$ are as follows:

$$g_{AB} = x^{-2} h_{AB} , \quad g_{xA} = 0 = g_{xx} .$$

From (24), (25) and (22) we obtain the following formulæ:

$$sQ^x_x (K) = 0 \quad (26)$$

$$sQ^A_B (K) = -\frac{1}{2} \sin \theta h^{AC} (x^{-2} h_{CB})_x . \quad (27)$$

$$sQ^x_A (K) = x^{-2} \sin \theta \left( \beta_A + \frac{1}{2} e^{-2\beta} h_{AB} U^B_x \right) . \quad (28)$$
If we assume that Bondi cone data is polyhomogeneous and conformally $C^1 \times C^0$-compactifiable, it follows that (cf. [8])

\[ h_{AB} = \hat{h}_{AB}(1 + \frac{x^2}{4} \chi^{CD}_{\chi_{CD}} + x^2 \zeta_{AB} + x^3 \xi_{AB} + O_{ln^*x}(x^4)) , \]

where $\zeta_{AB}$ and $\xi_{AB}$ are polynomials in $\ln x$ with coefficients which smoothly depend upon the $x^A$'s. By definition of the Bondi coordinates we have $\det h = \det \hat{h} = \sin \theta$, which implies $\hat{h}_{AB} \chi_{AB} = \hat{h}_{AB} \zeta_{AB} = 0$. Further,

\[ \beta = -\frac{1}{32} \chi^{CD}_{\chi_{CD}} x^2 + B x^3 + O_{ln^*x}(x^4) , \]

\[ h_{AB} U^B = -\frac{1}{2} \chi_{AB}^{\|B} x^2 + W_A x^3 + O_{ln^*x}(x^4) , \]

where $B$ and $W_A$ are again polynomials in $\ln x$ with smooth coefficients depending upon the $x^A$'s, while $\|$ denotes covariant differentiation with respect to the unit sphere metric $\hat{h}$. This leads to the following approximate formulae:

\[ sQ_{A B}(K) = x^{-2} \sin \theta \left( x^{-1} \delta_{A B} - \chi_{A B} + O(x^2) \right) \]

\[ sQ_{A A}(K) = x^{-2} \sin \theta \left( -\frac{1}{2} x \chi_{A}^{\|B} + O(x^2) \right) \]

\[ g_{AB} = x^{-2} \left( \hat{h}_{AB} + x \chi_{AB} + O(x^2) \right) \]

It is easy to verify that the asymptotic behaviour of canonical data $(g_{ab}, Q^a_b)$ is determined by “free data” $\chi_{AB}$ which agrees with standard Bondi-Sachs approach to the null initial value formulation.

We hope that the variational formula on a truncated cone, which is space-like inside and light-like near $\text{Scri}$, (proposed in [5]) can be formulated with the help of the object $Q^a_b$ for arbitrary hypersurfaces, i.e. without assumption that the null part of the initial surface is a Bondi cone.

References


