A theorem on kernel in the theory of operator-valued distributions

by

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1. Introduction. Let \( S(\mathbb{R}^n) \) denote the topological vector space of test functions for tempered distributions introduced by L. Schwartz [3]. For any two functions \( \varphi \in S(\mathbb{R}^n) \), \( \psi \in S(\mathbb{R}^m) \) we put

\[
(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y),
\]

where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( (x, y) \in \mathbb{R}^{n+m} \). It is known that this formula defines a continuous bilinear mapping

\[
\otimes : S(\mathbb{R}^n) \times S(\mathbb{R}^m) \to S(\mathbb{R}^{n+m}).
\]

Let \( L \) be a topological vector space. Any continuous linear mapping

\[
A : S(\mathbb{R}^n) \to L
\]

is called a \( L \)-valued distribution defined on \( \mathbb{R}^n \). For the special case \( L = C^0 \) this definition coincides with the definition of tempered distributions given by L. Schwartz. The second special case \( L = L(D) \), where \( D \) is a dense linear subset of a Hilbert space \( H \) and \( L(D) \) denotes the \( * \)-algebra of operators acting in \( D \) (the strict definition of \( L(D) \) is given below), is of great importance in the quantum field theory [4]. \( L(D) \)-valued distributions are often called operator-valued distributions.

We say that the topological vector space \( L \) satisfies the theorem on kernel if for any separately continuous bilinear mapping

\[
B : S(\mathbb{R}^n) \times S(\mathbb{R}^m) \to L
\]

there exists a continuous linear mapping

\[
\hat{B} : S(\mathbb{R}^{n+m}) \to L
\]

such that \( B(\varphi, \psi) = \hat{B}(\varphi \otimes \psi) \) for any \( \varphi \in S(\mathbb{R}^n) \) and \( \psi \in S(\mathbb{R}^m) \).
It is known that $C^4$ satisfies the theorem on kernel. This fact discovered by L. Schwartz is of great importance for the theory of tempered distributions. Similarly interesting problems concerning $L$-valued distributions can be solved if it is known that $L$ satisfies the theorem on kernel. For example assuming that $L(D)$ satisfies the theorem on kernel one can easily show that for any two $L(D)$-valued distributions $A_1, A_2$ defined on $R^n$ and $R^m$ respectively there exists a $L(D)$-valued distribution $B$ defined on $R^{n+m}$ such that $B(\varphi \otimes \psi) = A_1(\varphi)A_2(\psi)$ for any $\varphi \in S(R^n)$ and $\psi \in S(R^m)$.

One can check that in general case $L(D)$ does not satisfy the theorem on kernel. However, it appears that there exist sufficiently many dense linear subsets $\bar{D} \subset H$ such that $L(\bar{D})$ satisfies the theorem on kernel. More exactly, for any dense linear subset $D \subset H$ we shall construct a linear subset $\bar{D} \supset D$ such that:

1. $L(\bar{D})$ satisfies the theorem on kernel.
2. For any $L(D)$-valued distribution $A$ there exists one and only one $L(\bar{D})$-valued distribution $\bar{A}$ such that $A(\varphi) = \bar{A}(\varphi)$ for all test functions $\varphi$. (Relation $A \subset \bar{A}$, where $A, \bar{A}$ are operators acting in the Hilbert space $H$ means that $D_A \subset D_{\bar{A}}$ and $A_{\bar{A}} = B_{\bar{A}}$ for any $\varphi \in D_{\bar{A}}$.)

2.2. Sequentially complete spaces. Assume that $L$ is a locally convex topological vector space i.e. that the topology of $L$ is given by system of seminorms $(q_\alpha : \alpha \in A)$. Let us remind that a sequence $(A_\alpha)_{\alpha \in A}$ of elements of $L$ is called a Cauchy sequence if for any $\alpha \in A$ and any positive number $\varepsilon$ one can find an integer $N$ such that $q_\alpha(A_k - A_N) < \varepsilon$ for any $k, N \geq N$. The space $L$ is sequentially complete if any Cauchy sequence is convergent.

Let $(A_\alpha)_{\alpha \in A}$ be a sequence of elements of the space $L$. Assume that $L$ is sequentially complete and that

$$\sum_{\alpha \in A} q_\alpha(A_\alpha) < \infty$$

for any $\alpha \in A$. Then the series $\sum_{\alpha \in A} A_\alpha$ is convergent and

$$q_\alpha\left(\sum_{\alpha \in A} A_\alpha\right) \leq \sum_{\alpha \in A} q_\alpha(A_\alpha) \quad (1)$$

for any $\alpha \in A$.

**Theorem 1.** Any locally convex and sequentially complete topological vector space satisfies the theorem on kernel.

**Proof.** Let

$$S(R^n) \times S(R^m) \ni (\varphi, \psi) \mapsto B(\varphi, \psi) \in L$$

be a partially continuous linear mapping. We have to find a continuous linear mapping

$$S(R^{n+m}) \ni \varphi \mapsto B(\varphi) \in L$$

such that $B(\varphi \otimes \psi) = B(\varphi, \psi)$ for any $\varphi \in S(R^n)$ and $\psi \in S(R^m)$.

Assume for simplicity that $n = m = 1$. (the proof in the general case is slightly more complicated). Let

$$h_k(\varphi) = \frac{1}{\sqrt{\pi^{2k} k!}} e^{-\varphi^2/2} \int_{-\infty}^{\infty} e^{-x^2} d\varphi$$

be the $k$-th normalized Hermite function. It is known that $h_k \in S(R^1)$.

Moreover any element $\varphi \in S(R^1)$ is a linear combination

$$\varphi = \sum_{k \geq 0} C_k(\varphi)h_k,$$

where complex coefficients $C_k(\varphi)$ decrease rapidly when $k \to \infty$. It means that

$$P_N(\varphi) = \sup_k |C_k(\varphi)(1+k)^N| < \infty$$

for any natural $N$. One can check that (3) introduces the system of norms $(p_N : N \text{-integer})$ on the space $S(R^1)$ and that the topology of $S(R^1)$ given by this system coincides with the well known topology introduced by L. Schwartz.

Similarly any element $\chi \in S(R^1)$ can be written as a linear combination

$$\chi = \sum_{k \geq 0} C_k(\chi)h_k,$$

where $C_k(\chi)$ decrease rapidly when $k \to \infty$, i.e.

$$P_N(\chi) = \sup_k |C_k(\chi)(1+k)^N| < \infty$$

for any natural $N$ and $N'$. The topology of $S(R^1)$ is defined by the system of norms $(p_{N,N'} : N, N' \text{-integer})$ introduced by (4). $S(R^1)$ is a Fréchet space and therefore according to the theorem of Mazur, Orlicz and Bourbaki (see [2]) we can conclude that (3) is a continuous mapping. It means that for any $\alpha \in A$ there exists integers $N(a)$, $N'(a)$ and a positive number $K_a$ such that

$$\|q_\alpha(B(\varphi, \psi))\| \leq K_a P_{N(a)}(\varphi) P_{N'(a)}(\psi)$$

for any $\varphi, \psi \in S(R^1)$. Setting $h_k$ and $h_{k'}$ instead of $\varphi$ and $\psi$ we obtain:

$$\|q_\alpha(B(h_k, h_{k'}))\| \leq K_a (1+k)^{N(a)} (1+k')^{N'(a)}.$$
On the other hand the formula (4) says that

\[ |\varrho_{\alpha}(\chi)| \leq \frac{P_{\alpha}(\lambda_{\theta}^{N_{\theta}^{(a_{\theta}+1)}})}{(1 + k)^{N_{\theta}(a_{\theta}+1)}}. \]

Combining these two formulas one can check that

\[ \sum_{x \in \mathbb{X}} |\varrho_{\alpha}(\chi)| g_{\alpha}(x) = \left( \frac{e^2}{6} \right) K_{\alpha} \varrho_{\alpha}(\lambda_{\theta}^{N_{\theta}^{(a_{\theta}+1)}})(x), \]

where \( \left( \frac{e^2}{6} \right) = \frac{1}{(1 + k)^{N_{\theta}^{(a_{\theta}+1)}}}. \) This inequality holds for any \( \alpha \in A. \)

Therefore \( (L) \) is assumed to be sequentially complete the series

\[ \sum_{\alpha \in \mathbb{X}} g_{\alpha}(x) = \sum_{\alpha \in \mathbb{X}} \varrho_{\alpha}(\chi) \]

is convergent. It is seen that this formula defines a linear mapping

\[ \tilde{B} : S(\mathbb{R}^n) \rightarrow L. \]

This mapping is continuous. Indeed by virtue of (5) and (1) we have

\[ g_{\alpha}(\chi) \leq \left( \frac{e^2}{6} \right) K_{\alpha} \varrho_{\alpha}(\lambda_{\theta}^{N_{\theta}^{(a_{\theta}+1)}})(x). \]

Let \( \varphi, \psi \in S(\mathbb{R}^n). \) Then \( \varrho_{\alpha}(\varphi \otimes \psi) = \varrho_{\alpha}(\varphi) \varrho_{\alpha}(\psi) \) and equation (6) shows that \( B(\varphi \otimes \psi) = B(\varphi, \psi). \) This completes the proof.}

### 3. \( LD \)-valued distributions

Let \( D \) be a dense linear subset of a Hilbert space \( H. \) Symbol \( LD(D) \) will denote the set of all (in general unbounded) operators acting in \( H \) and such that:

- \( D_\alpha \subseteq D, \)
- \( A \subseteq D, \)
- \( D_\alpha \subseteq D_\beta \) and \( A \alpha \subseteq A \beta \)

where \( D_\alpha, D_\beta \) is the domain of an operator \( A. \) For any \( A \epsilon LD(D) \) the domain \( D_\alpha \) is dense in \( H \) and therefore the operator \( A \) is preclosed.

It is seen that \( LD(D) \) is a \( \ast \)-algebra. It means that \( A + B, AB \) and \( A^\ast = A^\ast |_D \) belong to \( LD(D) \) for any \( A, B \in LD(D), \lambda \epsilon C^n. \)

Let \( u \epsilon D. \) Then the mapping

\[ LD(D) \rightarrow A \rightarrow (u | Au) \epsilon C^n \]

is a linear functional on \( LD(D) \). The topology of \( LD(D) \) induced by the family of all functionals of the form (7) is called weak topology. It can be proved that \( LD(D) \) provided with the weak topology is a locally convex topological vector space and the mappings

\[ LD(D) \rightarrow A \rightarrow A + eLD(D), \]

\[ LD(D) \rightarrow A \rightarrow AB \epsilon LD(D), \]

\[ LD(D) \rightarrow A \rightarrow BA \epsilon LD(D), \]

(where \( B \) is a fixed element of \( LD(D) \)) are continuous.

A sequence \( (u_n) \) of elements of \( H \) is called \( (D) \)-fundamental if \( u_n \rightarrow D \) (where \( A \) is the closure of an operator \( A \)) and the sequence \( (\tilde{A}u_n) \) is convergent for any \( A \epsilon LD(D) \). Any \( (D) \)-fundamental sequence \( (u_n) \) is convergent (since \( I \subseteq LD(D) \)). Moreover \( u_n \epsilon D_\alpha \)

is convergent and \( \lim \tilde{A}u_n = \tilde{A} \lim u_n \)

for any \( A \epsilon LD(D). \)

Let \( \tilde{D} \) be the smallest linear subset of the Hilbert space \( H \) such that:

1. \( D \subseteq \tilde{D} \subseteq D_\alpha \) for any \( A \epsilon LD(D), \)
2. \( \lim u_n \epsilon \tilde{D} \) for any \( (D) \)-fundamental sequence \( (u_n) \) of elements of \( \tilde{D} \).

For any \( B \epsilon LD(D) \) we put

\[ \tilde{B} \subseteq B |_{D_\alpha}. \]

It is seen that \( B \subseteq \tilde{D} \subseteq D. \) We are going to prove

**Theorem 2.**

1. \( \tilde{B} \subseteq LD(D) \) for any \( B \epsilon LD(D) \)
2. The mapping

\[ LD(D) \rightarrow B \subseteq LD(D) \]

is a homomorphism of the \( \ast \)-algebras.

3. \( \tilde{D} \subseteq \tilde{D} \).

**Proof.** Let \( D_1 \) be the set consisting of all elements \( u \epsilon \tilde{D} \) such that

\[ \tilde{B}u \epsilon D, \]

\[ \tilde{A}Bu = \tilde{A} \tilde{B}u, \]

\[ \tilde{A}Bu = \tilde{A}u + \tilde{B}u, \]

\[ u \epsilon D, \]

\[ \tilde{B}u = \tilde{B}u. \]
for any $A, B \in L(D)$. It is seen that $D \subset D_1 = \tilde{D}$. Let $(u_n)$ be a $(D)$-fundamental sequence of elements of $D_1$. Then

a) $u_n = \lim_{n \to \infty} u_n \in \tilde{D}$ (since $(u_n)$ is a $(D)$-fundamental sequence of elements of $\tilde{D}$).

b) The sequence $(\tilde{B}u_n)$ is convergent for any $A, B \in L(D)$ and equation (11) shows that $(\tilde{B}u_n)$ is a $(D)$-fundamental sequence. By using (8) one can see that $\tilde{B}u_n = \lim_{n \to \infty} B u_n \in \tilde{D}$ (as a limit of a $(D)$-fundamental sequence of elements $\tilde{D}$).

c) Taking $\lim_{n \to \infty}$ in the both sides of equations

\[
\frac{\tilde{A} \tilde{B} u_n}{\tilde{A}+\tilde{B}} = \tilde{A} \tilde{B} u_n,
\]

we have (see (8)):

\[
\frac{\tilde{A} \tilde{B} u_n}{\tilde{A}+\tilde{B}} = \tilde{A} \tilde{B} u_n = \tilde{A} u_n + \tilde{B} u_n.
\]

we have (see (8)):

\[
\frac{\tilde{A} \tilde{B} u_n}{\tilde{A}+\tilde{B}} = \tilde{A} \tilde{B} u_n,
\]

\[
\frac{\tilde{A} \tilde{B} u_n}{\tilde{A}+\tilde{B}} = \tilde{A} u_n + \tilde{B} u_n.
\]

\[
\frac{\tilde{A} \tilde{B} u_n}{\tilde{A}+\tilde{B}} = \tilde{A} u_n + \tilde{B} u_n.
\]

The obtained results imply that $u_n \in D_1$. We have proved that any $(D)$-fundamental sequence of elements of $D_1$ is convergent to an element of $D_1$. Now the definition of $\tilde{D}$ says that $D_1 = \tilde{D}$. Therefore the equations (10)-(14) hold for any $u \in \tilde{D}$. Let us notice that $B^* = B^*$. Now the statement 1° of the theorem follows immediately from (10), (13) and (14).

The statement 2° is a simple conclusion of (11), (12) and (14).

We are going to prove the statement 3°. Let $(u_n)$ be a $(\tilde{D})$-fundamental sequence of elements of $\tilde{D}$. It is sufficient to prove that $\lim_{n \to \infty} u_n \in \tilde{D}$. The sequence $(\tilde{A}u_n)$ is convergent for any $A \in L(D)$ (since $\tilde{A} \in L(D)$). But $\tilde{A}u_n = \tilde{A}u_n$. It means that $(u_n)$ is a $(D)$-fundamental sequence and therefore $\lim_{n \to \infty} u_n \in \tilde{D}$.

Briefly speaking the Theorem 2 says that all operators from $L(D)$ can be extended to the larger domain $\tilde{D}$ and that this extension preserves all algebraic relations. However, one can easily prove that (excluding the trivial case $\tilde{D} = D$) mapping $B \to \tilde{B}$ is not continuous. Therefore the following theorem needs a proof.

**Theorem 3.** Let

\[
A : S(R^n) \to L(D)
\]

be a $L(D)$-valued distribution. Then the mapping

\[
\tilde{A} : S(R^n) \to L(\tilde{D}),
\]

where $\tilde{A}(\varphi) = A(\varphi)^* \varphi$ for any $\varphi \in S(R^n)$, is continuous, i.e. $\tilde{A}$ is a $L(\tilde{D})$-valued distribution.

**Proof.** For any $u \in \tilde{D}$ we put

\[
f_u(\varphi) = \langle \varphi, A(\varphi)^* \varphi \rangle, \quad \varphi \in S(R^n).
\]

$f_u$ is a linear functional on $S(R^n)$. Let $D_1$ be the subset of $H$ consisting of all vectors $u \in \tilde{D}$ such that the functional $f_u$ is continuous. The continuity of (15) means that $D \subset D_1$.

Let $(u_n)$ be a $(D)$-fundamental sequence of elements of $D_1$. By virtue of (8) one can see that for any $A \in L(D)$:

\[
\lim (u_n \langle A \varphi, u \rangle) = (u \langle A \varphi, u \rangle),
\]

where $u_n = \lim u_n$. Setting $A = A(\varphi)$ we get:

\[
\lim f_u(\varphi) = f_u(\varphi)
\]

for any $\varphi \in S(R^n)$. The functionals $f_u$ are continuous since $u \in D$. Therefore we have the sequence of continuous functionals on $S(R^n)$, which is convergent at any point $\varphi \in S(R^n)$. In this situation the limit functional $f_u$ has to be continuous (cf. [1] Chapter II § 1, Theorem 17). It means that $u_n \to D_1$. Taking into account the definition of $\tilde{D}$ one can see that $D_1 = \tilde{D}$ i.e. for any $u \in \tilde{D}$ the functional

\[
S(R^n) \times \varphi \to \langle \varphi, A(\varphi)^* \varphi \rangle \in C^2
\]

is continuous. In order to complete the proof let us remind that the mapping (16) is continuous if and only if the functionals (17) are continuous for all $u \in \tilde{D}$. □

4. **Strong topology in $L(D)$.** Let $D$ be a dense linear subset of a Hilbert space $H$. The topology of $L(D)$ introduced by the family of mappings of the form

\[
L(D)(A) \to BAu \in H, \quad L(D)(A) \to BA^* u \in H,
\]

where $u \in D$ and $B \in L(D)$ is called a strong topology. This topology is stronger than the weak topology introduced before. It means that any strongly continuous mapping into $L(D)$ is the more weakly continuous.
The following theorem shows that these two topologies are equivalent from the point of view of the operator-valued distribution theory.

**Theorem 4.** Let

\[ A: S(R^n) \to L(D) \]

be a \( L(D) \)-valued distribution. Then \( A \) is a strongly continuous mapping.

**Proof.** It is sufficient to show that for any \( u \in D \) and \( B \in L(D) \) the mappings

\[ S(R^n) \ni \varphi \mapsto BA(\varphi)u \in H, \]

\[ S(R^n) \ni \varphi \mapsto BA(\varphi)^*u \in H \]

are continuous.

Assume that \( \lim \varphi_n = \varphi \) and \( \lim \text{BA}(\varphi_n)u = v \) for a sequence \( (\varphi_n) \) of test functions. Then for any \( u \in D \):

\[ \lim \{u, \text{BA}(\varphi_n)u\} = (u, v). \]

On the other hand the mapping \( A \) is weakly continuous and

\[ \lim \{u, \text{BA}(\varphi_n)u\} = \lim \{B^*u, \text{A}(\varphi_n)u\} = \{B^*u, A(\varphi)u\} = (u, \text{BA}(\varphi)u). \]

It implies that \( \text{BA}(\varphi)u = v \). This way we have proved that the mapping \( (18) \) has a closed graph. By using the closed graph theorem (see for example [1]) Chapter II § 3 theorem 4) we conclude that (18) is a continuous mapping. Similarly one can prove the continuity of (19).

It can be easily seen that (excluding the trivial case \( D = H \)) the space \( L(D) \) (provided with a weak topology) is not sequentially complete. The situation becomes better if one considers the strong topology.

**Theorem 5.** Let \( D = \tilde{D} \). Then the space \( L(D) \) provided with the strong topology is sequentially complete.

**Proof.** Assume that \( (A_u)_{u \in D} \) (where \( A_u \in L(D) \)) is a Cauchy sequence with respect to the strong topology. Then \( (B A_u)_{u \in D} \) and \( (B A_u^*)_{u \in D} \) are convergent in \( H \) for any \( B \in L(D) \) and \( u \in D \) (the Hilbert space \( H \) is complete and any Cauchy sequence of elements of \( H \) is convergent). It means that \( (A_u)_{u \in D} \) and \( (A_u^*)_{u \in D} \) are \( (D) \)-fundamental sequences. For any \( u \in D \) we put:

\[ A_u = \lim A_u u. \]

Then

\[ A_u \in D \]

as a limit of a \((D)\)-fundamental sequence of elements of \( D = \tilde{D} \). Let \( v \in D \). Then

\[ (v, \lim A_u^* u) = (v, A_u^* u) = (A_v, u) \]

Therefore \( u \in D \) and

\[ A_u = \lim A_u^* u \in D \]

because \( (A_u^* u)_{u \in D} \) is also \((D)\)-fundamental sequence of elements of \( D = \tilde{D} \). Relations (21) and (22) show now that \( A \in L(D) \).

By using (8) we get

\[ \lim B A u_u = B u_u \quad \text{and} \quad \lim B A_u^* u = B A^* u \]

for any \( B \in L(D) \) and \( u \in D \). It means that the sequence \( (A_u) \) is strongly convergent to the element \( A \in L(D) \).

5. **Final results.** By virtue of Theorems 1, 4 and 5 we immediately get:

**Theorem 6.** Let \( D \) be a dense linear subset of a Hilbert space \( H \) such that \( \tilde{D} = D \). Then \( L(D) \) satisfies the theorem on kernel.

For the general case (without assumption that \( \tilde{D} = D \)) we have:

**Theorem 7.** Let \( D \) be a dense linear subset of a Hilbert space \( H \) and let

\[ B: S(R^n) \times S(R^n) \to L(D) \]

be a separately continuous bilinear mapping. Then there exists a \( L(D) \)-valued distribution

\[ B: S(R^n) \to L(D) \]

such that

\[ B(\varphi, \psi) = B(\varphi \otimes \psi) \]

for any \( \varphi \in S(R^n) \) and \( \psi \in S(R^n) \).

This theorem follows immediately from the Theorems 3 and 6.

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