CLIFFORD ALGEBRAS AND THEIR REPRESENTATIONS

Andrzej Trautman, Uniwersytet Warszawski, Warszawa, Poland

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Introduction

Introductory and historical remarks

Clifford (1878) introduced his 'geometric algebras' as a generalization of Grassmann algebras, complex numbers and quaternions. Lipschitz (1886) was the first to define groups constructed from 'Clifford numbers' and use them to represent rotations in a Euclidean space. É. Cartan discovered representations of the Lie algebras $\mathrm{so}_n(\mathbb{C})$ and $\mathrm{so}_n(\mathbb{R})$, $n > 2$, that do not lift to representations of the orthogonal groups. In physics, Clifford algebras and spinors appear for the first time in Pauli's nonrelativistic theory of the 'magnetic electron'. Dirac (1928), in his work on the relativistic wave equation of the electron, introduced matrices that provide a representation of the Clifford algebra of Minkowski space. Brauer and Weyl (1935) connected the Clifford and Dirac ideas with Cartan's spinorial representations of Lie algebras; they found, in any number of dimensions, the spinorial, projective representations of the orthogonal groups.

Clifford algebras and spinors are implicit in Euclid's solution of the Pythagorean equation $x^2 - y^2 + z^2 = 0$ which is equivalent to

$$\begin{pmatrix} y-x & z \\ z & y+x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix} \tag{1}$$

so that $x = q^2 - p^2,\ y = p^2 + q^2,\ z = 2pq$. If the numbers appearing in (1) are real, then this equation can be interpreted as providing a representation of a vector $(x, y, z) \in \mathbb{R}^3$, null with respect to a quadratic form of signature $(1, 2)$, as the 'square' of a spinor $(p, q) \in \mathbb{R}^2$. The pure spinors of Cartan (1938) provide a generalization of this observation to higher dimensions.

Multiplying the square matrix in (1) on the left by a real, 2×2 unimodular matrix, on the right by its transpose, and taking the determinant, one arrives at the exact sequence of group homomorphisms

$$1 \to \mathbb{Z}_2 \to \mathrm{SL}_2(\mathbb{R}) = \mathrm{Spin}^0_{1,2} \to \mathrm{SO}^0_{1,2} \to 1.$$

Multiplying the same matrix by

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

on the left and computing the square of the product, one obtains

$$\begin{pmatrix} z & x+y \\ x-y & -z \end{pmatrix}^2 = (x^2 - y^2 + z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This equation is an illustration of the idea of representing a quadratic form as the square of a linear form in a Clifford algebra. Replacing y by $\mathrm{i}y$ one arrives at complex spinors, the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \mathrm{i}\varepsilon, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathrm{Spin}_3 = \mathrm{SU}_2$, etc.

This article reviews Clifford algebras, the associated groups and their representations, for quadratic spaces over complex or real numbers. These notions have been generalized by Chevalley (1954) to quadratic spaces over arbitrary number fields.

Notation

If S is a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, then S^* denotes its dual, i.e. the vector space over K of all K-linear maps from S to K. The value of $\omega \in S^*$ on $s \in S$ is sometimes written as $\langle s, \omega \rangle$. The transpose of a linear map $f : S_1 \to S_2$ is the map $f^* : S_2^* \to S_1^*$ defined by $\langle s, f^*(\omega) \rangle = \langle f(s), \omega \rangle$ for every $s \in S_1$ and $\omega \in S_2^*$. If S_1 and S_2 are complex vector spaces, then a map $f : S_1 \to S_2$ is said to be semi-linear if it is $\mathbb{R}$-linear and $f(\mathrm{i}\, s) = -\mathrm{i}\, f(s)$. The complex conjugate of a finite-dimensional complex vector space S is the complex vector space $\bar{S}$ of all semi-linear maps from S^* to $\mathbb{C}$. There is a natural semi-linear isomorphism (complex conjugation) $S \to \bar{S}$, $s \mapsto \bar{s}$ such that $\langle \omega, \bar{s} \rangle = \overline{\langle s, \omega \rangle}$ for every $\omega \in S^*$. The space $\bar{\bar{S}}$ can be identified with S and then $\bar{\bar{s}} = s$. The spaces $(\bar{S})^*$ and $\overline{S^*}$ are identified. If $f : S_1 \to S_2$ is a complex-linear map, then there is the complex conjugate map $\bar{f} : \bar{S}_1 \to \bar{S}_2$ given by $\bar{f}(\bar{s}) = \overline{f(s)}$ and the Hermitian conjugate map $f^\dagger \stackrel{\text{def}}{=} \bar{f}^* : \bar{S}_2^* \to \bar{S}_1^*$. A linear map $A : S \to \bar{S}^*$ such that $A^\dagger = A$ is said to be Hermitian. $K(N)$ denotes, for $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the set of all N by N matrices with elements in K.

Real, complex and quaternionic structures

A *real structure* on a complex vector space S is a complex-linear map $C : S \to \bar{S}$ such that $\bar{C}C = \mathrm{id}_S$. A vector $s \in S$ is said to be real if $\bar{s} = C(s)$. The set $\mathrm{Re}\, S$ of all real vectors is a real vector space; its real dimension is the same as the complex dimension of S.

A complex-linear map $C : S \to \bar{S}$ such that $\bar{C}C = -\,\mathrm{id}_S$ defines on S a *quaternionic structure*; a necessary condition for such a structure to exist is that the complex dimension m of S be even, $m = 2n$, $n \in \mathbb{N}$. The space S with a quaternionic structure can be made into a right vector space over the field $\mathbb{H}$ of quaternions. In the context of quaternions, it is convenient to represent the imaginary unit of $\mathbb{C}$ as $\sqrt{-1}$. Multiplication on the right by the quaternion unit i is realized as the multiplication (on the left) by $\sqrt{-1}$. If j and $\mathrm{k} = \mathrm{ij}$ are the other two quaternion units and $s \in S$, then one puts $s\mathrm{j} = \bar{C}(\bar{s})$ and $s\mathrm{k} = (s\mathrm{i})\mathrm{j}$.

A real vector space S can be *complexified* by forming the tensor product $\mathbb{C} \otimes_{\mathbb{R}} S = S \oplus \mathrm{i}\, S$.

The *realification* of a complex vector space S is defined here as the real vector space having S as its set of vectors so that $\dim_{\mathbb{R}} S = 2 \dim_{\mathbb{C}} S$. The complexification of a realification of S is the 'double' $S \oplus S$ of the original space.

Inner product spaces and their groups

Definitions: quadratic and symplectic spaces

A bilinear map $B : S \times S \to K$ on a vector space S over K is said to make S into an *inner product space*. To save on notation, one writes also $B : S \to S^*$ so that $\langle s, B(t)\rangle = B(s,t)$ for all $s, t \in S$. The group of automorphisms of an inner product space,

$$\mathrm{Aut}(S,B) = \{R \in \mathrm{GL}(S) | R^* \circ B \circ R = B\},$$

is a Lie subgroup of the general linear group $\mathrm{GL}(S)$. An inner product space (S,B) is said here to be *quadratic* (resp., *symplectic*) if B is symmetric (resp., anti-symmetric and non-singular). A quadratic space is characterized by its *quadratic form* $s \mapsto B(s,s)$. For $K = \mathbb{C}$, a Hermitian map $A : S \to \bar{S}^*$ defines a Hermitian scalar product $A(s,t) = \langle \bar{s}, A(t)\rangle$.

An *orthogonal space* is defined here as a quadratic space (S,B) such that $B : S \to S^*$ is an isomorphism. The group of automorphisms of an orthogonal space is the orthogonal group $\mathrm{O}(S,B)$. The group of automorphisms of a symplectic space is the symplectic group $\mathrm{Sp}(S,B)$. The dimension of a symplectic space is even. If $S = K^{2n}$ is a symplectic space over $K = \mathbb{R}$ or $\mathbb{C}$, then its symplectic group is denoted by $\mathrm{Sp}_{2n}(K)$. Two quaternionic symplectic groups appear in the list of spin groups of low-dimensional spaces:

$$\mathrm{Sp}_2(\mathbb{H}) = \{a \in \mathbb{H}(2) | a^\dagger a = I\}$$

and

$$\mathrm{Sp}_{1,1}(\mathbb{H}) = \{a \in \mathbb{H}(2) | a^\dagger \sigma_z a = \sigma_z\}.$$

Here $a^\dagger$ denotes the matrix obtained from a by transposition and quaternionic conjugation.

Contractions, frames and orthogonality

From now on, unless otherwise specified, (V,g) is a quadratic space of dimension m. Let $\wedge V = \bigoplus_{p=0}^m \wedge^p V$ be its exterior (Grassmann) algebra. For every $v \in V$ and $w \in \wedge V$ there is the *contraction* $g(v) \lrcorner w$ characterized as follows. The map $V \times \wedge V \to \wedge V$, $(v,w) \mapsto g(v) \lrcorner w$, is bilinear; if $x \in \wedge^p V$, then $g(v) \lrcorner (x \wedge w) = (g(v) \lrcorner x) \wedge w + (-1)^p x \wedge (g(v) \lrcorner w)$ and $g(v) \lrcorner v = g(v,v)$.

A frame (e_μ) in a quadratic space (V,g) is said to be a *quadratic frame* if $\mu \neq \nu$ implies $g(e_\mu, e_\nu) = 0$.

For every subset W of V there is the orthogonal subspace $W^\perp$ containing all vectors that are orthogonal to every element of W.

If (V,g) is a real orthogonal space, then there is an *orthonormal frame* (e_μ), $\mu = 1,\dots,m$, in V such that k frame vectors have squares equal to -1, l frame vectors have squares equal to 1 and $k+l=m$. The pair (k,l) is the *signature* of g. The quadratic form g is said to be *neutral* if the orthogonal space (V,g) admits two maximal totally null subspaces W and W' such that $V = W \oplus W'$. Such a space V is $2n$-dimensional, either complex or real with g of signature (n,n). A *Lorentzian* space has maximal totally null subspaces of dimension 1 and a *Euclidean* space, characterized by a definite quadratic form, has no null subspaces of positive dimension. The *Minkowski* space is a Lorentzian space of dimension 4.

If (V,g) is a complex orthogonal space, then an *orthonormal frame* (e_μ), $\mu = 1,\dots,m$, can be chosen in V so that, defining $g_{\mu\nu} = g(e_\mu, e_\nu)$, one has $g_{\mu\mu} = (-1)^{\mu+1}$ and, if $\mu \neq \nu$, then $g_{\mu\nu} = 0$.

If $A : S \to \bar{S}^*$ is a Hermitian isomorphism, then there is a (pseudo)unitary frame (e_α) in S such that the matrix $A_{\bar\alpha\beta} = A(e_\alpha, e_\beta)$ is diagonal, has p ones and q minus ones on the diagonal, $p + q = \dim S$. If $p = q$, then A is said to be neutral. A is definite if either p or $q = 0$.

Algebras

Definitions

An algebra over K is a vector space $\mathcal{A}$ over K with a bilinear map $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$, $(a,b) \mapsto ab$, which is distributive with respect to addition. The algebra is *associative* if $(ab)c = a(bc)$ holds for all $a,b,c \in \mathcal{A}$. It is *commutative* if $ab = ba$ for all $a,b \in \mathcal{A}$. An element $1_\mathcal{A}$ is the *unit* of $\mathcal{A}$ if $1_\mathcal{A} a = a 1_\mathcal{A} = a$ holds for every $a \in \mathcal{A}$.

From now on, unless otherwise specified, the bare word *algebra* denotes a finite-dimensional, associative algebra over $K = \mathbb{R}$ or $\mathbb{C}$, with a unit element. If S is an N-dimensional vector space over K, then the set $\operatorname{End} S$ of all endomorphisms of S is an N^2-dimensional algebra over K, the product being defined by composition; if $f, g \in \operatorname{End} S$, then one writes fg instead of $f \circ g$; the unit of $\operatorname{End} S$ is the identity map I. By definition, homomorphisms of algebras map units into units. The map $K \to \mathcal{A}$, $a \mapsto a1_\mathcal{A}$ is injective and one identifies K with its image in $\mathcal{A}$ by this map so that the unit can be represented by $1 \in K \subset \mathcal{A}$. A set $\mathcal{B} \subset \mathcal{A}$ is said to *generate* $\mathcal{A}$ if every element of $\mathcal{A}$ can be represented as a linear combination of products of elements of $\mathcal{B}$. For example, if V is a vector space over K, then its *tensor algebra*

$$\mathcal{T}(V) = \oplus_{p=0}^{\infty} \otimes^p V$$

is an (infinite-dimensional) algebra over K generated by $K \oplus V$. The algebra of all N by N matrices with entries in an algebra $\mathcal{A}$ is denoted by $\mathcal{A}(N)$. Its unit element is the unit matrix I. In particular, $\mathbb{R}(N)$, $\mathbb{C}(N)$ and $\mathbb{H}(N)$ are algebras over $\mathbb{R}$. The algebra $\mathbb{R}(2)$ is generated by the set $\{\sigma_x, \sigma_z\}$. As a vector space, the algebra $\mathbb{R}(2)$ is *spanned* by the set $\{I, \sigma_x, \varepsilon,\ \sigma_z\}$.

The direct sum $\mathcal{A} \oplus \mathcal{B}$ of the algebras $\mathcal{A}$ and $\mathcal{B}$ over K is an algebra over K such that its underlying vector space is $\mathcal{A} \times \mathcal{B}$ and the product is defined by $(a, b) \cdot (a', b') = (aa', bb')$ for every $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Similarly, the product in the tensor product algebra $\mathcal{A} \otimes_K \mathcal{B}$ is defined by

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'. \tag{3}$$

For example, if $\mathcal{A}$ is an algebra over $\mathbb{R}$, then the tensor product algebra $\mathbb{R}(N) \otimes_{\mathbb{R}} \mathcal{A}$ is isomorphic to $\mathcal{A}(N)$ and

$$K(N) \otimes_K K(N') = K(NN') \tag{4}$$

for $K = \mathbb{R}$ or $\mathbb{C}$ and $N, N' \in \mathbb{N}$. There are isomorphisms of algebras over $\mathbb{R}$:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &= \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &= \mathbb{C}(2) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &= \mathbb{R}(4). \end{aligned} \tag{5}$$

An algebra over $\mathbb{R}$ can be complexified by complexifying its underlying vector space; it follows from (5) that $\mathbb{C}(2)$ is the complex algebra obtained by complexification of the real algebra $\mathbb{H}$.

The *center* of an algebra $\mathcal{A}$ is the set

$$\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} \mid ab = ba \ \forall\ b \in \mathcal{A}\}.$$

The center is a commutative subalgebra containing K. An algebra over K is said to be *central* if its center coincides with K. The algebras $\mathbb{R}(N)$ and $\mathbb{H}(N)$ are central over $\mathbb{R}$. The algebra $\mathbb{C}(N)$ is central over $\mathbb{C}$, but not over $\mathbb{R}$.

Simplicity and representations

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subsets of the algebra $\mathcal{A}$. Define $\mathcal{B}_1\mathcal{B}_2 = \{b_1 b_2 \mid b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$. A vector subspace $\mathcal{B}$ of $\mathcal{A}$ is said to be a *left* (resp., *right*) *ideal* of $\mathcal{A}$ if $\mathcal{A}\mathcal{B} \subset \mathcal{B}$ (resp, $\mathcal{B}\mathcal{A} \subset \mathcal{B}$). A two-sided ideal—or simply an ideal—is a left and right ideal. An algebra $\mathcal{A} \neq \{0\}$ is said to be *simple* if its only two-sided ideals are $\{0\}$ and $\mathcal{A}$.

For example, the algebras $\mathbb{R}(N)$ and $\mathbb{H}(N)$ are simple over $\mathbb{R}$; the algebra $\mathbb{C}(N)$ is simple when considered as an algebra over both $\mathbb{R}$ and $\mathbb{C}$; every associative, finite-dimensional simple algebra over $\mathbb{R}$ or $\mathbb{C}$ is isomorphic to one of them.

A *representation* of an algebra $\mathcal{A}$ over K in a vector space S over K is a homomorphism of algebras $\rho : \mathcal{A} \to \operatorname{End} S$. If ρ is injective, then the representation is said to be *faithful*. For example, the *regular representation* $\rho : \mathcal{A} \to \operatorname{End} \mathcal{A}$ of an algebra $\mathcal{A}$, defined by $\rho(a)b = ab$ for all $a, b \in \mathcal{A}$, is faithful. A vector subspace T of the vector space S carrying a representation ρ of $\mathcal{A}$ is said to be *invariant* for ρ if $\rho(a)T \subset T$ for every $a \in \mathcal{A}$; it is *proper* if distinct from both $\{0\}$ and S. For example, a left ideal of $\mathcal{A}$ is invariant for the regular representation. Given an invariant subspace T of ρ one can *reduce* ρ to T by forming the representation $\rho_T : \mathcal{A} \to \operatorname{End} T$, where $\rho_T(a)s = \rho(a)s$ for every $a \in \mathcal{A}$ and $s \in T$. A representation is *irreducible* if it has no proper invariant subspaces.

A linear map $F : S_1 \to S_2$ is said to *intertwine* the representations $\rho_1 : \mathcal{A} \to \operatorname{End} S_1$ and $\rho_2 : \mathcal{A} \to \operatorname{End} S_2$ if $F\rho_1(a) = \rho_2(a)F$ holds for every $a \in \mathcal{A}$. If F is an isomorphism, then the representations ρ_1 and ρ_2 are said to be *equivalent*, $\rho_1 \sim \rho_2$. The following two propositions are classical:

(A) (i) *An algebra over K is simple if, and only if, it admits a faithful irreducible representation in a vector space over K. Such a representation is unique, up to equivalence.*
(ii) *The complexification of a central simple algebra over $\mathbb{R}$ is a central simple algebra over $\mathbb{C}$.*

For *real* algebras, one often considers *complex* representations, i.e. representations in complex vector spaces. Two such representations $\rho_1 : \mathcal{A} \to \operatorname{End} S_1$ and $\rho_2 : \mathcal{A} \to \operatorname{End} S_2$ are said to be *complex-equivalent* if there is a complex isomorphism $F : S_1 \to S_2$ intertwining the representations; they are *real-equivalent* if there is an isomorphism among the realifications of S_1 and S_2, intertwining the representations. For example, $\mathbb{C}$, considered as an algebra over $\mathbb{R}$, has two complex-inequivalent representations in $\mathbb{C}$: the identity representation and its complex conjugate. The realifications of these representations, given by $\mathrm{i} \mapsto \varepsilon$ and $\mathrm{i} \mapsto -\varepsilon$, respectively, are real-equivalent: they are intertwined by σ_z. The real algebra $\mathbb{H}$, being central simple, has only one, up to complex equivalence, representation in $\mathbb{C}^2$: every such representation is equivalent to the one given by

$$\mathrm{i} \mapsto \sigma_x/\sqrt{-1}, \;\; \mathrm{j} \mapsto \sigma_y/\sqrt{-1}, \;\; \mathrm{k} \mapsto \sigma_z/\sqrt{-1}.$$

This representation extends to an injective homomorphism of algebras $i : \mathbb{H}(N) \to \mathbb{C}(2N)$ which is used to define the quaternionic determinant of a matrix $a \in \mathbb{H}(N)$ as $\det_{\mathbb{H}}(a) = \det i(a)$ so that $\det_{\mathbb{H}}(a) \geqslant 0$ and $\det_{\mathbb{H}}(ab) = \det_{\mathbb{H}}(a)\det_{\mathbb{H}}(b)$ for every $a, b \in \mathbb{H}(N)$. In particular, if $q \in \mathbb{H}$ and $\lambda, \mu \in \mathbb{R}$, then $\det_{\mathbb{H}}(q) = \bar{q}q$ and

$$\det_{\mathbb{H}} \begin{pmatrix} \lambda & q \\ -\bar{q} & \mu \end{pmatrix} = (\lambda\mu + \bar{q}q)^2. \tag{6}$$

There are quaternionic unimodular groups $\mathrm{SL}_N(\mathbb{H}) = \{a \in \mathbb{H}(N) | \det_{\mathbb{H}}(a) = 1\}$. For example, the group $\mathrm{SL}_1(\mathbb{H})$ is isomorphic to SU_2 and $\mathrm{SL}_2(\mathbb{H})$ is a non-compact, 15-dimensional Lie group, one of the spin groups in 6 dimensions.

Antiautomorphisms and inner products

An *automorphism* of an algebra $\mathcal{A}$ is a linear isomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(ab) = \alpha(a)\alpha(b)$. An invertible element $c \in \mathcal{A}$ defines an *inner* automorphism $\mathrm{Ad}(c) \in \mathrm{GL}(\mathcal{A})$, $\mathrm{Ad}(c)a = cac^{-1}$. Complex conjugation in $\mathbb{C}$, considered as an algebra over $\mathbb{R}$, is an automorphism that is not inner. An *antiautomorphism* of an algebra $\mathcal{A}$ is a linear isomorphism $\beta : \mathcal{A} \to \mathcal{A}$ such that $\beta(ab) = \beta(b)\beta(a)$ for all $a, b \in \mathcal{A}$. An (anti)automorphism β is *involutive* if $\beta^2 = \mathrm{id}$. For example, conjugation of quaternions defines an involutive antiautomorphism of $\mathbb{H}$.

Let $\rho : \mathcal{A} \to \mathrm{End}\, S$ be a representation of an algebra with an involutive antiautomorphism β. There is then the *contragredient* representation $\check{\rho} : \mathcal{A} \to \mathrm{End}\, S^*$ given by $\check{\rho}(a) = (\rho(\beta(a)))^*$. If, moreover, $\mathcal{A}$ is central simple and ρ is faithful irreducible, then there is an isomorphism $B : S \to S^*$ intertwining ρ and $\check{\rho}$ which is either symmetric, $B^* = B$, or antisymmetric, $B^* = -B$. It defines on S the structure of an inner product space. This structure extends to $\mathrm{End}\, S$: there is a symmetric isomorphism $B \otimes B^{-1} : \mathrm{End}\, S \to (\mathrm{End}\, S)^* = \mathrm{End}\, S^*$ given, for every $f \in \mathrm{End}\, S$, by $(B \otimes B^{-1})(f) = BfB^{-1}$.

Let $K^\times = K \smallsetminus \{0\}$ be the multiplicative group of the field K. Given a simple algebra $\mathcal{A}$ with an involutive antiautomorphism β, one defines $N(a) = \beta(a)a$ and the group

$$\mathcal{G}(\beta) = \{a \in \mathcal{A} \mid N(a) \in K^\times\}.$$

Let $\rho : \mathcal{A} \to \mathrm{End}\, S$ be the faithful irreducible representation as above, then, for $a \in \mathcal{A}$ and $s, t \in S$, one has

$$B(\rho(a)s, \rho(a)t) = N(a)B(s, t).$$

If $a \in \mathcal{G}(\beta)$ and $\lambda \in K^\times$, then $\lambda a \in \mathcal{G}(\beta)$ and the *norm* N satisfies $N(\lambda a) = \lambda^2 N(a)$. The inner product B is invariant with respect to the action of the group

$$\mathcal{G}_1(\beta) = \{a \in \mathcal{G}(\beta) \mid N(a) = 1.\}$$

(B) *Let $\mathcal{A}$ be a central simple algebra over K with an involutive antiautomorphism β and a faithful irreducible representation ρ so that*

$$\check{\rho}(a) = B\rho(a)B^{-1}.$$

The map $h : \mathcal{A} \times \mathcal{A} \to K$ defined by

$$h(a, b) = \mathrm{Tr}\, \rho(\beta(a)b)$$

is bilinear, symmetric and non-degenerate. The map ρ is an isometry of the quadratic space $(\mathcal{A}, h)$ on its image in the quadratic space $(\mathrm{End}\, S, B \otimes B^{-1})$.

Graded algebras

Definitions

An algebra $\mathcal{A}$ is said to be $\mathbb{Z}$-graded (resp., $\mathbb{Z}_2$-graded) if there is a decomposition of the underlying vector space $\mathcal{A} = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}^p$ (resp. $\mathcal{A} = A^0 \oplus A^1$) such that $\mathcal{A}^p\mathcal{A}^q \subset \mathcal{A}^{p+q}$. In a $\mathbb{Z}_2$-graded algebra it is understood that $p+q$ is reduced mod 2. If $a \in \mathcal{A}^p$, then a is said to be homogeneous of degree p. The exterior algebra $\bigwedge V$ of a vector space V is $\mathbb{Z}$-graded. Every $\mathbb{Z}$-graded algebra becomes $\mathbb{Z}_2$-graded when one reduces the degree of every element mod 2. A *graded isomorphism* of graded algebras is an isomorphism that preserves the grading.

A $\mathbb{Z}_2$-grading of $\mathcal{A}$ is characterized by the involutive automorphism α such that, if $a \in \mathcal{A}^p$, then $\alpha(a) = (-1)^p a$. From now on, grading means $\mathbb{Z}_2$-grading unless otherwise specified. The elements of $\mathcal{A}^0$ (resp., $\mathcal{A}^1$) are said to be even (resp., odd). It is often convenient to denote the graded algebra as

$$\mathcal{A}^0 \to \mathcal{A}. \tag{7}$$

Given such an algebra over K and $N \in \mathbb{N}$, one constructs the graded algebra $\mathcal{A}^0(N) \to \mathcal{A}(N)$. Two graded algebras over K, $\mathcal{A}^0 \to \mathcal{A}$ and $\mathcal{A}'^0 \to \mathcal{A}'$ are said to be of the same *type* if there are integers N and N' such that the algebras $\mathcal{A}^0(N) \to \mathcal{A}(N)$ and $\mathcal{A}'^0(N') \to \mathcal{A}'(\mathcal{N}')$ are graded isomorphic. The property of being of the same type is an equivalence relation in the set of all graded algebras over K.

Given an algebra $\mathcal{A}$, one constructs two 'canonical' graded algebras as follows:
(i) the *double* algebra

$$\mathcal{A} \to \mathcal{A} \oplus \mathcal{A}$$

graded by the 'swap' automorphism, $\alpha(a_1, a_2) = (a_2, a_1)$ for $a_1, a_2 \in \mathcal{A}$;
(ii) the algebra

$$\mathcal{A} \oplus \mathcal{A} \to \mathcal{A}(2)$$

is defined by declaring the diagonal (resp., antidiagonal) elements of $\mathcal{A}(2)$ to be even (resp., odd).

The real algebra $\mathbb{R}(2)$ has also another grading, given by the involutive automorphism α such that $\alpha(a) = \varepsilon a \varepsilon^{-1}$, where $a \in \mathbb{R}(2)$ and ε is as in (2). In this case, (7) reads

$$\mathbb{C} \to \mathbb{R}(2).$$

There are also graded algebras over $\mathbb{R}$

$$\mathbb{R} \to \mathbb{C}, \quad \mathbb{C} \to \mathbb{H} \quad \text{and} \quad \mathbb{H} \to \mathbb{C}(2).$$

The grading of the last algebra can be defined by declaring the Pauli matrices and $\mathrm{i}I$ to be odd.

Super Lie algebras

A *super Lie algebra* is a graded algebra $\mathcal{A}$ such that the product $(a,b) \mapsto [a,b]$ is super anticommutative, $[a,b] = -(-1)^{pq}[b,a]$, and satisfies the super Jacobi identity,

$$[a,[b,c]] = [[a,b],c] + (-1)^{pq}[b,[a,c]]$$

for every $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$ and $c \in \mathcal{A}$. To every graded associative algebra $\mathcal{A}$ there corresponds a super Lie algebra $\mathcal{GLA}$: its underlying vector space and grading are as in $\mathcal{A}$ and the product, for $a \in \mathcal{A}^p$ and $b \in \mathcal{A}^q$, is given as the *supercommutator* $[a,b] = ab - (-1)^{pq}ba$.

Super centrality and graded simplicity

A graded algebra $\mathcal{A}$ over K is *supercentral* if $\mathcal{Z}(\mathcal{A}) \cap \mathcal{A}^0 = K$. The algebra $\mathbb{R} \to \mathbb{C}$ is supercentral, but the real ungraded algebra $\mathbb{C}$ is not central.

A subalgebra $\mathcal{B}$ of a graded algebra $\mathcal{A}$ is said to be a graded subalgebra if $\mathcal{B} = \mathcal{B} \cap \mathcal{A}^0 \oplus \mathcal{B} \cap \mathcal{A}^1$. A graded ideal of $\mathcal{A}$ is an ideal that is a graded subalgebra. A graded algebra $\mathcal{A} \neq \{0\}$ is said to be *graded simple* if it has no graded ideals other than $\{0\}$ and $\mathcal{A}$. The double algebra of a simple algebra is graded simple, but not simple.

The graded tensor product

Let $\mathcal{A}$ and $\mathcal{B}$ be graded algebras; the tensor product of their underlying vector spaces admits a natural grading, $(\mathcal{A} \otimes \mathcal{B})^p = \bigoplus_q \mathcal{A}^q \otimes \mathcal{B}^{p-q}$. The product defined in (3) makes $\mathcal{A} \otimes \mathcal{B}$ into a graded algebra. There is another 'super' product in the same graded vector space given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{pq} aa' \otimes bb'$$

for $a' \in \mathcal{A}^p$ and $b \in \mathcal{B}^q$. The resulting graded algebra is referred to as the *graded tensor product* and denoted by $\mathcal{A} \hat{\otimes} \mathcal{B}$. For example, if V and W are vector spaces, then the Grassmann algebra $\wedge(V \oplus W)$ is isomorphic to $\wedge V \hat{\otimes} \wedge W$.

Clifford algebras

Definitions; the universal property and grading

The *Clifford algebra* associated with a quadratic space (V,g) is the quotient algebra

$$\mathcal{C}\ell(V,g) = \mathcal{T}(V)/\mathcal{J}(V,g), \tag{8}$$

where $\mathcal{J}(V,g)$ is the ideal in the tensor algebra $\mathcal{T}(V)$ generated by all elements of the form $v \otimes v - g(v,v)1_{\mathcal{T}(V)}$, $v \in V$.

The Clifford algebra is associative with a unit element denoted by 1. One denotes by κ the canonical map of $\mathcal{T}(V)$ onto $\mathcal{C}\ell(V,g)$ and by ab the product of two elements $a,b \in \mathcal{C}\ell(V,g)$ so that $\kappa(P \otimes Q) = \kappa(P)\kappa(Q)$ for $P,Q \in \mathcal{T}(V)$. The map κ is injective on $K \oplus V$ and one identifies this subspace of $\mathcal{T}(V)$ with its image under κ. With this identification, for all $u, v \in V$, one has

$$uv + vu = 2g(u, v).$$

Clifford algebras are characterized by their universal property described in proposition

(C) *Let $\mathcal{A}$ be an algebra with a unit $1_{\mathcal{A}}$ and let $f : V \to \mathcal{A}$ be a* Clifford map, *i.e. a linear map such that $f(v)^2 = g(v, v)1_{\mathcal{A}}$ for every $v \in V$. There then exists a homomorphism $\hat{f} : \mathcal{C}\ell(V, g) \to \mathcal{A}$ of algebras with units, an* extension *of f, so that $f(v) = \hat{f}(v)$ for every $v \in V$.*

As a corollary, one obtains

(D) *If f is an isometry of (V, g) into (W, h), then there is a homomorphism of algebras $\mathcal{C}\ell(f) : \mathcal{C}\ell(V, g) \to \mathcal{C}\ell(W, h)$ extending f so that there is the commutative diagram*

$$\begin{array}{ccc}
\mathcal{C}\ell(V, g) & \xrightarrow{\mathcal{C}\ell(f)} & \mathcal{C}\ell(W, h) \\
\uparrow & & \uparrow \\
V & \xrightarrow[f]{} & W
\end{array}$$

For example, the isometry $v \mapsto -v$ extends to the involutive *main automorphism* α of $\mathcal{C}\ell(V, g)$, defining its $\mathbb{Z}_2$-grading:

$$\mathcal{C}\ell(V, g) = \mathcal{C}\ell^0(V, g) \oplus \mathcal{C}\ell^1(V, g).$$

The algebra $\mathcal{C}\ell(V, g)$ admits also an involutive *canonical antiautomorphism* β characterized by $\beta(1) = 1$ and $\beta(v) = v$ for every $v \in V$.

The vector space structure of Clifford algebras

Referring to proposition (D), let $\mathcal{A} = \mathrm{End}(\bigwedge V)$ and, for every $v \in V$ and $w \in \bigwedge V$, put $f(v)w = v \wedge w + g(v) \lrcorner\, w$, then $f : V \to \mathrm{End}(\bigwedge V)$ is a Clifford map and the map

$$i : \mathcal{C}\ell(V, g) \to \bigwedge V \tag{9}$$

given by $i(a) = \hat{f}(a)1_{\wedge V}$ is an isomorphism of vector spaces. This proves

(E) *As a* vector space*, the algebra $\mathcal{C}\ell(V, g)$ is isomorphic to the exterior algebra $\bigwedge V$.*

If V is m-dimensional, then $\mathcal{C}\ell(V,g)$ is 2^m-dimensional. The linear isomorphism (9) defines a $\mathbb{Z}$-grading of the *vector space* underlying the Clifford algebra: if $i(a_k) \in \bigwedge^k V$, then a_k is said to be of Grassmann degree k. Every element $a \in \mathcal{C}\ell(V,g)$ decomposes into its Grassmann components, $a = \sum_{k\in\mathbb{Z}} a_k$. The Clifford product of two elements of Grassmann degrees k and l decomposes as follows: $a_k b_l = \sum_{p\in\mathbb{Z}} (a_k b_l)_p$, and $(a_k b_l)_p = 0$ if $p < |k-l|$ or $p \equiv k - l + 1 \bmod 2$ or $p > m - |m-k-l|$.

One often uses (9) to *identify* the vector spaces $\bigwedge V$ and $\mathcal{C}\ell(V,g)$; this having been done, one can write, for every $v \in V$ and $a \in \mathcal{C}\ell(V,g)$,

$$va = v \wedge a + g(v) \lrcorner a, \tag{10}$$

so that $[v,a] = 2g(v)\lrcorner a$, where $[\,,\,]$ is the supercommutator. It defines a super Lie algebra structure in the vector space $K \oplus V$. The quadratic form defined by g need not be non-degenerate; for example, if it is the zero form, then (10) shows that the Clifford and exterior multiplications coincide and $\mathcal{C}\ell(V,0)$ is isomorphic, as an algebra, to the Grassmann algebra.

Complexification of real Clifford algebras

(F) *If (V,g) is a real quadratic space, then the algebras $\mathbb{C} \otimes \mathcal{C}\ell(V,g)$ and $\mathcal{C}\ell(\mathbb{C}\otimes V, \mathbb{C}\otimes g)$ are isomorphic, as graded algebras over $\mathbb{C}$.*

From now on, through the end of the article, one assumes that (V,g) is an orthogonal space over $K = \mathbb{R}$ or $\mathbb{C}$.

The Clifford algebra associated with the orthogonal space $\mathbb{C}^m$ is denoted by $\mathcal{C}\ell_m$. The Clifford algebra associated with the orthogonal space $(\mathbb{R}^{k+l}, g)$, where g is of signature (k,l), is denoted by $\mathcal{C}\ell_{k,l}$, so that $\mathbb{C}\otimes \mathcal{C}\ell_{k,l} = \mathcal{C}\ell_{k+l}$. The algebra $\mathcal{C}\ell_{k,l}$ is a *real form* of the complex algebra $\mathcal{C}\ell_{k+l}$.

Relations between Clifford algebras in spaces of adjacent dimensions

Consider an orthogonal space (V,g) over K and the one-dimensional orthogonal space (K,h_1), having a unit vector $w \in K$, $h_1(w,w) = \epsilon$, where $\epsilon = 1$ or -1. The map $V \ni v \mapsto vw \in \mathcal{C}\ell^0(V\oplus K, g\oplus h_1)$ satisfies $(vw)^2 = -\epsilon g(v,v)$ and extends to the isomorphism of algebras $\mathcal{C}\ell(V,-\epsilon g) \to \mathcal{C}\ell^0(V\oplus K, g\oplus h_1)$. This proves

(G) *There are isomorphisms of algebras: $\mathcal{C}\ell_m \to \mathcal{C}\ell^0_{m+1}$ and $\mathcal{C}\ell_{k,l} \to \mathcal{C}\ell^0_{k+1,l}$.*

Consider the orthogonal space (K^2, h) with a neutral h such that, for $\lambda,\mu \in K$, one has $\langle(\lambda,\mu), h(\lambda,\mu)\rangle = \lambda\mu$. The map

$$K^2 \to K(2), \quad (\lambda,\mu) \mapsto \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$$

has the Clifford property and establishes the isomorphisms represented by the horizontal arrows in the diagram

$$\begin{array}{ccc} \mathcal{C}\ell(K^2,h) & \longrightarrow & K(2) \\ \uparrow & & \uparrow \\ \mathcal{C}\ell^0(K^2,h) & \longrightarrow & K\oplus K. \end{array} \tag{11}$$

(H) *If* (K^2, h) *is neutral and* (V, g) *is over* K*, then the algebra* $\mathcal{C}\ell(V \oplus K^2, g \oplus h)$ *is isomorphic to the algebra* $\mathcal{C}\ell(V, g) \otimes K(2)$*. Specifically, there are isomorphisms*

$$\begin{aligned} \mathcal{C}\ell_{k+1,l+1} &= \mathcal{C}\ell_{k,l} \otimes \mathbb{R}(2) \\ \mathcal{C}\ell_{m+2} &= \mathcal{C}\ell_m \otimes \mathbb{C}(2). \end{aligned} \tag{12}$$

The Chevalley theorem and the Brauer–Wall group

If (V, g) and (W, h) are quadratic spaces over K, then their sum is the quadratic space $(V \oplus W, g \oplus h)$ characterized by $g \oplus h : V \oplus W \to V^* \oplus W^*$ so that $(g \oplus h)(v, w) = (g(v), h(w))$. By noting that the map $V \oplus W \ni (v, w) \mapsto v \otimes 1 + 1 \otimes w \in \mathcal{C}\ell(V, g) \hat{\otimes} \mathcal{C}\ell(W, h)$ has the Clifford property, Chevalley proved

(I) *The algebra* $\mathcal{C}\ell(V \oplus W, g \oplus h)$ *is isomorphic to the algebra* $\mathcal{C}\ell(V, g) \hat{\otimes} \mathcal{C}\ell(W, h)$*.*

The *type* of the (graded) algebra $\mathcal{C}\ell(V \oplus W, g \oplus h)$ depends only on the types of $\mathcal{C}\ell(V, g)$ and $\mathcal{C}\ell(W, h)$. The Chevalley theorem (I) shows that the set of types of Clifford algebras over K forms an Abelian group for a multiplication induced by the graded tensor product. The unit of this *Brauer–Wall group* of K is the type of the algebra $\mathcal{C}\ell(K^2, h)$ described in (11); for a full account with proofs see Wall (1963).

The volume element and the centers

Let $e = (e_\mu)$ be an orthonormal frame in (V, g). The *volume element* associated with e is

$$\eta = e_1 e_2 \dots e_m.$$

If η' is the volume element associated with another orthonormal frame e' in the same orthogonal space, then either $\eta' = \eta$ (e and e' are of the same orientation) or $\eta' = -\eta$ (e and e' are of opposite orientation). For $K = \mathbb{C}$ one has $\eta^2 = 1$; for $K = \mathbb{R}$ and g of signature (k, l) one has

$$\eta^2 = (-1)^{\frac{1}{2}(k-l)(k-l+1)}. \tag{13}$$

It is convenient to define $\iota \in \{1, \mathrm{i}\}$ so that $\eta^2 = \iota^2$. For every $v \in V$ one has $v\eta = (-1)^{m+1}\eta v$. The structure of the centers of Clifford algebras is as follows:

(J) *If m is even, then $\mathcal{Z}(\mathcal{C}\ell(V,g)) = K$ and $\mathcal{Z}(\mathcal{C}\ell^0(V,g)) = K \oplus K\eta$.*
If m is odd, then $\mathcal{Z}(\mathcal{C}\ell(V,g)) = K \oplus K\eta$ and $\mathcal{Z}(\mathcal{C}\ell^0(V,g)) = K$.
The graded algebra $\mathcal{C}\ell(V,g)$ is supercentral for every m.

The structure of Clifford algebras

The complex case

Using (4) one obtains from (11) and (12) the isomorphisms of algebras

$$\mathcal{C}\ell^0_{2n+1} = \mathcal{C}\ell_{2n} = \mathbb{C}(2^n), \tag{14}$$

$$\mathcal{C}\ell_{2n+1} = \mathcal{C}\ell^0_{2n+2} = \mathbb{C}(2^n) \oplus \mathbb{C}(2^n), \tag{15}$$

for $n = 0, 1, 2, \ldots$. Therefore, there are only two types of complex Clifford algebras, represented by $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C}(2)$: the Brauer–Wall group of $\mathbb{C}$ is $\mathbb{Z}_2$.

The real case

In view of proposition (I) and $\mathcal{C}\ell_{1,1} = \mathbb{R}(2)$, the algebra $\mathcal{C}\ell_{k,l}$ is of the same type as $\mathcal{C}\ell_{k-l,0}$ if $k > l$ and of the same type as $\mathcal{C}\ell_{0,l-k}$ if $k < l$. Since $\mathcal{C}\ell_{k,l} \hat{\otimes} \mathcal{C}\ell_{l,k} = \mathcal{C}\ell_{k+l,k+l}$, the type of $\mathcal{C}\ell_{l,k}$ is the inverse of the type of $\mathcal{C}\ell_{k,l}$. The algebra $\mathcal{C}\ell^0_{4,0} \to \mathcal{C}\ell_{4,0}$ is isomorphic to $\mathbb{H} \oplus \mathbb{H} \to \mathbb{H}(2)$. Indeed, if $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \subset \mathcal{C}\ell_{4,0}$, and $q = \mathrm{i}x_1 + \mathrm{j}x_2 + \mathrm{k}x_3 + x_4 \in \mathbb{H}$, then an isomorphism is obtained from the Clifford map f,

$$f(x) = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}. \tag{16}$$

In view of (13), the volume element η satisfies $\eta^2 = 1$. By replacing $-\bar{q}$ with $\bar{q}$ in (16), one shows that $\mathcal{C}\ell_{0,4}$ is also isomorphic to $\mathbb{H}(2)$. The map $\mathbb{R}^4 \times \mathbb{R}^{k+l} \to \mathbb{H}(2) \otimes \mathcal{C}\ell_{k,l}$ given by $(x, y) \mapsto f(x) \otimes 1 + f(\eta) \otimes y$ has the Clifford property and establishes the isomorphism of algebras $\mathcal{C}\ell_{k+4,l} = \mathbb{H} \otimes \mathcal{C}\ell_{k,l}$. Since, similarly, $\mathcal{C}\ell_{k,l+4} = \mathbb{H} \otimes \mathcal{C}\ell_{k,l}$, one obtains the isomorphism

$$\mathcal{C}\ell_{k+4,l} = \mathcal{C}\ell_{k,l+4}\,.$$

Therefore,

$$\mathcal{C}\ell_{k+8,l} = \mathcal{C}\ell_{k+4,l+4} = \mathcal{C}\ell_{k,l+8} = \mathcal{C}\ell_{k,l} \otimes \mathbb{R}(16),$$

and the algebras $\mathcal{C}\ell_{k,l}$, $\mathcal{C}\ell_{k+8,l}$ and $\mathcal{C}\ell_{k,l+8}$ are all of the same type. This double periodicity of period 8 is subsumed by saying that real Clifford algebras can be arranged on a 'spinorial chessboard'. The type of $\mathcal{C}\ell^0_{k,l} \to \mathcal{C}\ell_{k,l}$ depends only on $k - l \mod 8$; the eight types have the following low-dimensional algebras as representatives: $\mathcal{C}\ell_{1,0}$, $\mathcal{C}\ell_{2,0}$, $\mathcal{C}\ell_{3,0}$, $\mathcal{C}\ell_{4,0} = \mathcal{C}\ell_{0,4}$, $\mathcal{C}\ell_{0,3}$, $\mathcal{C}\ell_{0,2}$ and $\mathcal{C}\ell_{0,1}$. The Brauer–Wall group of $\mathbb{R}$ is $\mathbb{Z}_8$, generated by the type of $\mathcal{C}\ell^0_{1,0} \to \mathcal{C}\ell_{1,0}$, i.e. by $\mathbb{R} \to \mathbb{C}$. Bearing in mind the isomorphism $\mathcal{C}\ell_{k,l} = \mathcal{C}\ell^0_{k+1,l}$ and abbreviating $\mathbb{C} \to \mathbb{R}(2)$

to $\mathbb{C} \to \mathbb{R}$, etc., one can arrange the types of real Clifford algebras in the form of a 'spinorial clock':

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "7"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "0"] & \mathbb{R} \arrow[d, "1"] \\
\mathbb{C} \arrow[u, "6"] & & \mathbb{C} \arrow[d, "2"] \\
\mathbb{H} \arrow[u, "5"] & \mathbb{H}\oplus\mathbb{H} \arrow[l, "4"'] & \mathbb{H} \arrow[l, "3"']
\end{tikzcd} \tag{17}$$

(K) *Recipe for determining* $\mathcal{C}\ell^0_{k,l} \to \mathcal{C}\ell_{k,l}$*:*
(i) *find the integers* μ *and* ν *such that* $k - l = 8\mu + \nu$ *and* $0 \leqslant \nu \leqslant 7$*;*
(ii) *from the spinorial clock, read off* $\mathcal{A}^0_\nu \xrightarrow{\nu} \mathcal{A}_\nu$ *and compute the real dimensions,* $\dim \mathcal{A}^0_\nu = 2^{\tau^0}$ *and* $\dim \mathcal{A}_\nu = 2^\tau$*;*
(iii) *form* $\mathcal{C}\ell^0_{k,l} = \mathcal{A}^0_\nu(2^{\frac{1}{2}(k+l-1-\tau^0)})$ *and* $\mathcal{C}\ell_{k,l} = \mathcal{A}_\nu(2^{\frac{1}{2}(k+l-\tau)})$.

The spinorial clock is symmetric with respect to the reflection in the vertical line through its center; this is a consequence of the isomorphism of algebras $\mathcal{C}\ell_{k,l+2} = \mathcal{C}\ell_{l,k} \otimes \mathbb{R}(2)$.

Note that the 'abstract' algebra $\mathcal{C}\ell_{k,l}$ carries, in general, less information than the Clifford algebra defined in (8), which contains V as a distinguished vector subspace with the quadratic form $v \mapsto v^2 = g(v,v)$. For example, the algebras $\mathcal{C}\ell_{8,0}$, $\mathcal{C}\ell_{4,4}$ and $\mathcal{C}\ell_{0,8}$ are all graded isomorphic.

Theorem on simplicity

From general theory (Chevalley 1954) or by inspection of (14), (15) and (17), one has

(L) *Let* m *be the dimension of the orthogonal space* (V,g) *over* K*.*

If m *is even (resp., odd), then the algebra* $\mathcal{C}\ell(V,g)$ *(resp.,* $\mathcal{C}\ell^0(V,g)$*) over* K *is central simple.*

If $K = \mathbb{C}$ *and* m *is odd (resp., even), then the algebra* $\mathcal{C}\ell(V,g)$ *(resp.,* $\mathcal{C}\ell^0(V,g)$*) is the direct sum of two isomorphic complex central simple algebras.*

If $K = \mathbb{R}$ *and* m *is odd (resp., even), then the algebra* $\mathcal{C}\ell(V,g)$ *(resp.,* $\mathcal{C}\ell^0(V,g)$*) when* $\eta^2 = 1$ *is the direct sum of two isomorphic central simple algebras and when* $\eta^2 = -1$ *is simple with a center isomorphic to* $\mathbb{C}$*.*

Representations

The Pauli, Cartan, Dirac and Weyl representations

Odd dimensions

Let (V, g) be of dimension $m = 2n + 1$ over K. From propositions (A) and (L) it follows that the central simple algebra $\mathcal{C}\ell^0(V, g)$ has a unique, up to equivalence, faithful and irreducible representation in the *complex* 2^n-dimensional vector space S of *Pauli spinors*. By putting $\sigma(\eta) = \iota I$ it is extended to a Pauli representation $\sigma : \mathcal{C}\ell(V, g) \to \operatorname{End} S$. Given an orthonormal frame (e_μ) in V, Pauli endomorphisms (matrices if S is identified with $\mathbb{C}^{2^n}$) are defined as $\sigma_\mu = \sigma(e_\mu) \in \operatorname{End} S$. The representations σ and $\sigma \circ \alpha$ are complex-inequivalent. For $K = \mathbb{C}$ none of them is faithful; their direct sum is the faithful *Cartan* representation of $\mathcal{C}\ell(V, g)$ in $S \oplus S$. For $K = \mathbb{R}$ and $\frac{1}{2}(k - l - 1)$ even, the representations σ and $\sigma \circ \alpha$ are real-equivalent and faithful. Computing $\beta(\eta)$ one finds that the contragredient representation $\check{\sigma}$ is equivalent to σ for n even and to $\sigma \circ \alpha$ for n odd.

Even dimensions

Similarly, for (V, g) of dimension $m = 2n$ over K, the central simple algebra $\mathcal{C}\ell(V, g)$ has a unique, up to equivalence, faithful and irreducible representation $\gamma : \mathcal{C}\ell(V, g) \to \operatorname{End} S$ in the 2^n-dimensional *complex* vector space S of *Dirac spinors*. The Dirac endomorphisms (matrices) are $\gamma_\mu = \gamma(e_\mu)$. Put $\mathit{\Gamma} = \iota\gamma(\eta)$ so that $\mathit{\Gamma}^2 = I$: the matrix $\mathit{\Gamma}$ generalizes the familiar γ_5. The Dirac representation γ restricted to $\mathcal{C}\ell^0(V, g)$ decomposes into the sum $\gamma_+ \oplus \gamma_-$ of two irreducible representations in the vector spaces

$$S_\pm = \{s \in S | \mathit{\Gamma} s = \pm s\}$$

of *Weyl (chiral) spinors*. The elements of S_+ are said to be of opposite *chirality* with respect to those of S_-. The transpose $\mathit{\Gamma}^*$ defines a similar split of S^*. The representations γ_+ and γ_- are never complex-equivalent, but they are real-equivalent and faithful for $K = \mathbb{R}$ and $\frac{1}{2}(k - l)$ odd.

The representations $\gamma \circ \alpha$ and $\check{\gamma}$ are both equivalent to γ. It is convenient to describe simultaneously the properties of the transpositions of the Pauli and Dirac matrices; let ρ_μ be either the Pauli matrices for V of dimension $2n+1$ or the Dirac matrices for V of dimension $2n$. There is a complex isomorphism $B : S \to S^*$ such that

$$\rho_\mu^* = (-1)^n B \rho_\mu B^{-1}. \tag{18}$$

In the case of the Dirac matrices, the factor $(-1)^n$ in (18) implies that this equation holds also for $\mathit{\Gamma}$ in place of ρ_μ. The isomorphism B preserves (resp., changes) the chirality of Weyl spinors for n even (resp., odd). Every matrix of the form $B\gamma_{\mu_1} \ldots \gamma_{\mu_p}$, where

$$1 \leqslant \mu_1 < \cdots < \mu_p \leqslant 2n, \tag{19}$$

is either symmetric or antisymmetric, depending on p and the symmetry of B. A simple argument, based on counting the number of such products of one symmetry, leads to the equation

$$B^* = (-1)^{\frac{1}{2}n(n+1)}B$$

valid in dimensions $2n$ and $2n+1$.

Inner products on spinor spaces

Let S be the complex vector space of Dirac or Pauli spinors associated with (V,g) over K. The isomorphism $B : S \to S^*$ defines on S an inner product $B(s,t) = \langle s, B(t)\rangle,\ s,t \in S$, which is orthogonal for $m \equiv 0,1,6$ or $7 \mod 8$ and symplectic for $m \equiv 2,3,4$ or $5 \mod 8$. For $m \equiv 0 \mod 4$, this product restricts to an inner product on the spaces of Weyl spinors that is orthogonal for $m \equiv 0 \mod 8$ and symplectic for $m \equiv 4 \mod 8$. For $m \equiv 2 \mod 4$, the map B defines the isomorphisms $B_\pm : S_\pm \to S^*_\mp$.

Example

One of the most often used representations $\gamma : \mathcal{C}\ell_{3,1} \to \mathbb{C}(4)$ is given by the Dirac matrices

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \\ \gamma_3 &= \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{20}$$

Charge conjugation and Majorana spinors

Throughout this section and next one assumes $K = \mathbb{R}$ so that, given a representation $\rho : \mathcal{C}\ell(V,g) \to \operatorname{End} S$, one can form the *complex* ("charge") *conjugate* representation $\bar\rho : \mathcal{C}\ell(V,g) \to \operatorname{End} \bar S$ defined by $\bar\rho(a) = \overline{\rho(a)}$ and the *Hermitian conjugate* representation $\rho^\dagger : \mathcal{C}\ell(V,g) \to \operatorname{End} \bar S^*$, where $\rho^\dagger(a) = \overline{\check\rho(a)}$.

Even dimensions

The representations $\bar\gamma$ and γ are equivalent: there is an isomorphism $C : S \to \bar S$ such that

$$\bar\gamma_\mu = C\gamma_\mu C^{-1}. \tag{21}$$

The automorphism $\bar C C$ is in the commutant of γ; it is, therefore, proportional to I and, by a change of scale, one can achieve $\bar C C = I$ for $k - l \equiv 0$ or $6 \mod 8$ and $\bar C C = -I$ for $k - l \equiv 2$ or $4 \mod 8$.

The spinor $s_c = C^{-1}\bar s \in S$ is the *charge conjugate* of $s \in S$. If $\psi : V \to S$ is a solution of the Dirac equation

$$(\gamma^\mu(\partial_\mu - \mathrm{i}qA_\mu) - \kappa)\psi = 0$$

for a particle of electric charge q, then ψ_c is a solution of the same equation with the opposite charge. Since

$$\bar{\Gamma} = \iota^2 C \Gamma C^{-1},$$

charge conjugation preserves (resp., changes) the chirality of Weyl spinors for $\frac{1}{2}(k-l)$ even (resp., odd).

If $\bar{C}C = I$, then

$$\mathrm{Re}\, S = \{s \in S | s_c = s\}$$

is a real vector space of dimension 2^n, the space of *Dirac–Majorana spinors*. The representation γ is real: restricted to $\mathrm{Re}\, S$ and expressed with respect to a frame in this space, it is given by real 2^n by 2^n matrices. For $k - l \equiv 0 \mod 8$ the representations γ_+ and γ_- are both real: in this case there are *Weyl–Majorana spinors.*

Odd dimensions

Computing $\overline{\sigma(\eta)}$ one finds that the conjugate representation $\bar{\sigma}$ is equivalent to σ (resp., $\sigma \circ \alpha$) if $\eta^2 = 1$ (resp., $\eta^2 = -1$). There is an isomorphism $C : S \to \bar{S}$ such that

$$\bar{\sigma}_\mu = (-1)^{\frac{1}{2}(k-l+1)} C \sigma_\mu C^{-1} \tag{22}$$

and $\bar{C}C = I$ (resp., $\bar{C}C = -I$) for $k - l \equiv 1$ or $7 \mod 8$ (resp., $k - l \equiv 3$ or $5 \mod 8$). For $k - l \equiv 1 \mod 8$, the restriction of the Pauli representation to $\mathcal{C}\ell^0_{k,l}$ is real and the Pauli matrices are pure imaginary; for $k - l \equiv 7 \mod 8$, the Pauli representations of $\mathcal{C}\ell_{k,l}$ are both real and so are the Pauli matrices, In both these cases there are *Pauli–Majorana spinors.*

Hermitian scalar products and multivectors

For $m = k + l$ odd and C as in (22), the map $A = \bar{B}C : S \to \bar{S}^*$ intertwines the representations $\sigma^\dagger$ and σ (resp., $\sigma \circ \alpha$) for k even (resp., odd),

$$\sigma_\mu^\dagger = (-1)^k A \sigma_\mu A^{-1}.$$

By rescaling of B, the map A can be made Hermitian. The corresponding Hermitian form $s \mapsto A(s, s)$ is definite if, and only if, k or $l = 0$; otherwise, it is neutral.

For $m = k + l$ even, the representations $\gamma^\dagger$ and γ are equivalent and one can define a Hermitian isomorphism $A : S \to \bar{S}^*$ so that

$$\gamma_\mu^\dagger = A \gamma_\mu A^{-1}. \tag{23}$$

The isomorphism $A' = A\Gamma$ intertwines the representations $\gamma^\dagger$ and $\gamma \circ \alpha$; it can also be made Hermitian by rescaling. The Hermitian form $A(s, s)$ is definite for $k = 0$ and $A'(s, s)$ is definite for $l = 0$; otherwise, these forms are neutral. For example, in the familiar representation (20), one has $A = \gamma_4$, a neutral form.

Given two spinors s and $t \in S$ and an integer p such that $0 \leqslant p \leqslant m = 2n$, one defines a p-vector with components

$$A_{\mu_1 \dots \mu_p}(s,t) = \langle \bar{s}, A\gamma_{\mu_1} \dots \gamma_{\mu_p} t \rangle, \tag{24}$$

where the indices are as in (19). The Hermiticity of A and (23) imply

$$\overline{A_{\mu_1 \dots \mu_p}(s,t)} = (-1)^{\frac{1}{2}p(p-1)} A_{\mu_1 \dots \mu_p}(t,s).$$

In view of $\Gamma^\dagger = (-1)^k A \Gamma A^{-1}$, the map A defines, for k even, a non-degenerate Hermitian scalar product on the spaces $S_\pm$ whereas $A(s,t) = 0$ if s and t are Weyl spinors of opposite chiralities. For k odd, A changes the chirality.

The Radon–Hurwitz numbers

(M) *For every integer $m > 0$, the algebra $\mathcal{C}\ell_{m,0}$ has an irreducible real representation ρ of dimension $2^{\chi(m)}$, where $\chi(m)$ is the mth Radon–Hurwitz number given by*

$$\begin{array}{rcccccccc} m = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \chi(m) = & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \end{array}$$

and $\chi(m+8) = \chi(m) + 4$. The matrices $\rho_\mu \in \mathbb{R}(2^{\chi(m)})$, $\mu = 1, \dots, m$, defining these representations satisfy

$$\rho_\mu \rho_\nu + \rho_\nu \rho_\mu = -2\delta_{\mu\nu} I$$

and can be chosen so as to be antisymmetric. In all dimensions other than $m \equiv 3 \mod 4$ the representations are faithful.

For $m \equiv 2$ and $4 \mod 8$ (resp., $m \equiv 1, 3$ and $5 \mod 8$) the representations ρ are the realifications of the corresponding Dirac (resp., Pauli) representations. In dimensions $m \equiv 0$ and $6 \mod 8$ (resp., $m \equiv 7 \mod 8$) the Dirac (resp., Pauli) representations themselves are real.

Inductive construction of representations

An inductive construction of the Pauli representations

$$\sigma : \mathcal{C}\ell_{n-1,n} \to \mathbb{R}(2^{n-1}), \quad n = 1, 2, \dots,$$

and of the Dirac representations

$$\gamma : \mathcal{C}\ell_{n,n} \to \mathbb{R}(2^n), \quad n = 1, 2, \dots,$$

is as follows.

(i) In dimension 1, put $\sigma_1 = 1$.

(ii) Given $\sigma_\mu \in \mathbb{R}(2^{n-1})$, $\mu = 1, \ldots, 2n-1$, define

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \text{ for } \mu = 1, \ldots, 2n-1,$$

and

$$\gamma_{2n} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

(iii) Given $\gamma_\mu \in \mathbb{R}(2^n)$, $\mu = 1, \ldots, 2n$, define $\sigma_\mu = \gamma_\mu$ for $\mu = 1, \ldots, 2n$, and $\sigma_{2n+1} = \gamma_1 \ldots \gamma_{2n}$ so that, for $n > 0$,

$$\sigma_{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

All entries of these matrices are either 0, 1 or -1; therefore, they can be used to construct representations of Clifford algebras of orthogonal spaces over any commutative field of characteristic $\neq 2$.

By induction, one has $\sigma_\mu^* = (-1)^{\mu+1}\sigma_\mu$. Therefore, the isomorphisms appearing in (18) are $B = \gamma_2\gamma_4 \ldots \gamma_{2n}$ for both $m = 2n$ and $2n+1$.

By multiplying some of the matrices σ_μ or γ_μ by the imaginary unit one obtains complex representations of the Clifford algebras associated with quadratic forms of other signatures. For example, in dimension 3, $(\sigma_1, \mathrm{i}\sigma_2, \sigma_3)$ are the Pauli matrices. In dimension 4, multiplying γ_2 by i one obtains the Dirac matrices for g of signature $(1,3)$, in the 'chiral representation':

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \\ \gamma_3 &= \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{25}$$

To obtain the real *Majorana representation* one uses the following fact:

(N) *If the matrix* $C \in \mathbb{R}(2^n)$ *is such that* $C^2 = I$ *and* (21) *holds, then the matrices* $(I + \mathrm{i}C)\gamma_\mu(I + \mathrm{i}C)^{-1}$, $\mu = 1, \ldots, 2n$, *are real.*

For the matrices (25) one can take $C = \gamma_1\gamma_3\gamma_4$ to obtain

$$\begin{aligned} \gamma_1' &= \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \gamma_2' = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ \gamma_3' &= \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad \gamma_4' = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \end{aligned}$$

The real representations described in proposition (M) can be obtained by the following direct inductive construction. Consider the following 7 real antisymmetric and anticommuting 8 by 8 matrices

$$\begin{aligned}
\rho_1 &= \sigma_z \otimes I \otimes \varepsilon \\
\rho_2 &= \sigma_z \otimes \varepsilon \otimes \sigma_x \\
\rho_3 &= \sigma_z \otimes \varepsilon \otimes \sigma_z \\
\rho_4 &= \sigma_x \otimes \varepsilon \otimes I \\
\rho_5 &= \sigma_x \otimes \sigma_x \otimes \varepsilon \\
\rho_6 &= \sigma_x \otimes \sigma_z \otimes \varepsilon \\
\rho_7 &= \varepsilon \otimes I \otimes I
\end{aligned} \tag{26}$$

For $m = 4, 5, 6$ and 7 the matrices $\rho_1, \dots, \rho_m$ generate the representations of $\mathcal{C}\ell_{m,0}$ in $\mathbb{R}^8$. The 8 matrices $\theta_\mu = \sigma_x \otimes \rho_\mu$, $\mu = 1, \dots, 7$, and $\theta_8 = \varepsilon \otimes I \otimes I \otimes I$ give the required representation of $\mathcal{C}\ell_{8,0}$ in $\mathbb{R}^{16}$. Dropping the first factor in ρ_1, ρ_2, ρ_3 one obtains the matrices generating a representation of $\mathcal{C}\ell_{3,0}$ in $\mathbb{R}^4$, etc. The symmetric matrix $\Theta = \theta_1 \dots \theta_8 = \sigma_z \otimes I \otimes I \otimes I$ anticommutes with all the θs and $\Theta^2 = I$. If the matrices $\rho_\mu \in \mathbb{R}(2^{\chi(m)})$ correspond to a representation of $\mathcal{C}\ell_{m,0}$, then the $m+8$ matrices $\Theta \otimes \rho_1, \dots, \Theta \otimes \rho_m, \theta_1 \otimes I, \dots, \theta_8 \otimes I$ generate the required representation of $\mathcal{C}\ell_{m+8,0}$.

Vector fields on spheres and division algebras

It is known that even-dimensional spheres have no nowhere vanishing tangent vector fields. All such fields on odd-dimensional spheres can be constructed with the help of the representation ρ described in proposition (M). Given a positive even integer N, let m be the largest integer such that $N = 2^{\chi(m)}p$, where p is an odd integer. Consider the unit sphere $\mathbb{S}_{N-1} = \{x \in \mathbb{R}^N | \|x\| = 1\}$ of dimension $N - 1$. For $v \in \mathbb{R}^m$, put $\rho'(v) = \rho(v) \otimes I$, where $I \in \mathbb{R}(p)$ is the unit matrix. Since $\rho(v)$ is antisymmetric, so is the matrix $\rho'(v) \in \mathbb{R}(N)$. Therefore, for every $x \in \mathbb{S}_{N-1}$, the vector $\rho'(v)x$ is orthogonal to x. The map $x \mapsto \rho'(v)x$ defines a vector field on $\mathbb{S}_{N-1}$ that vanishes nowhere unless $v = 0$: the $(N-1)$-sphere admits a set of m tangent vector fields which are linearly independent at every point. Using methods of algebraic topology, it has been shown that this method gives the maximum number of linearly independent tangent vector fields on spheres.

If $m = 1, 3$ or 7, then $m + 1 = 2^{\chi(m)}$ and, for these values of m, the sphere $\mathbb{S}_m$ is parallelizable. Moreover, one can then introduce in $\mathbb{R}^{m+1}$ the structure of an algebra $\mathcal{A}_m$ as follows. Put $\rho_0 = I$. If $e_0 \in \mathbb{R}^{m+1}$ is a unit vector and $e_\mu = \rho_\mu(e_0)$, then $(e_0, e_1, \dots, e_m)$ is an orthonormal frame in $\mathbb{R}^{m+1}$. The product of $x = \sum_{\mu=0}^m x_\mu e_\mu$ and $y = \sum_{\mu=0}^m y_\mu e_\mu$ is defined to be

$$x \cdot y = \textstyle\sum_{\mu,\nu=0}^m x_\mu y_\nu \rho_\mu(e_\nu)$$

so that e_0 is the unit element for this product. Defining $\operatorname{Re} x = x_0 e_0$, $\operatorname{Im} x = x - \operatorname{Re} x$, $\bar{x} = \operatorname{Re} x - \operatorname{Im} x$ one has $\bar{x} \cdot x = e_0 \|x\|^2$ and $\bar{x} \cdot (x \cdot y) = (\bar{x} \cdot x) \cdot y$ so that $x \cdot y = 0$ implies $x = 0$ or $y = 0$: $\mathcal{A}_m$ is a normed algebra without zero divisors. The algebras $\mathcal{A}_1$ and $\mathcal{A}_3$ are isomorphic to $\mathbb{C}$ and $\mathbb{H}$, respectively, and $\mathcal{A}_7$ is, by definition, the algebra $\mathbb{O}$ of *octonions* discovered by Graves and Cayley. The algebra $\mathbb{O}$ is non-associative; its multiplication table is obtained with the help of (26).

Spinor groups

Let (V, g) be a quadratic space over K. If $u \in V$ is not null, then it is invertible as an element of $\mathcal{C}\ell(V, g)$ and the map $v \mapsto -uvu^{-1}$ is a reflection in the hyperplane orthogonal to u. The orthogonal group $\mathrm{O}(V, g) = \mathrm{O}(V, -g) = \{R \in \mathrm{GL}(V) | R^* \circ g \circ R = g\}$ is generated by the set of all such reflections. A *spinor group* G is a subset of $\mathcal{C}\ell(V, g)$ that is a group with respect to multiplication induced by the product in the algebra, with a homomorphism $\rho : G \to \mathrm{GL}(V)$ whose image contains the connected component $\mathrm{SO}^0(V, g)$ of the group of rotations of (V, g). In the case of real quadratic spaces, one considers also spinor groups that are subsets of $\mathbb{C} \otimes \mathcal{C}\ell(V, g)$ with similar properties. By restriction, every representation of $\mathcal{C}\ell(V, g)$ or $\mathbb{C} \otimes \mathcal{C}\ell(V, g)$ gives *spinor representations* of the spinor groups it contains.

Pin groups

It is convenient to define a unit vector $v \in V \subset \mathcal{C}\ell(V, g)$ to be such that $v^2 = 1$ for V complex and $v^2 = 1$ or -1 for V real. The group $\mathrm{Pin}(V, g)$ is defined as consisting of products of all finite sequences of unit vectors. Defining now the *twisted adjoint representation* $\widetilde{\mathrm{Ad}}$ by $\widetilde{\mathrm{Ad}}(a)v = \alpha(a) v a^{-1}$, one obtains the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(V, g) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V, g) \to 1. \tag{27}$$

If $\dim V$ is even, then the *adjoint representation* $\mathrm{Ad}(a)v = ava^{-1}$ also yields an exact sequence like (27); if it is odd, then the image of Ad is $\mathrm{SO}(V, g)$ and the kernel is the four-element group $\{1, -1, \eta, -\eta\}$.

Given an orthonormal frame (e_μ) in (V, g) and $a \in \mathrm{Pin}(V, g)$, one defines the orthogonal matrix $R(a) = (R^\nu_\mu(a))$ by

$$\widetilde{\mathrm{Ad}}(a) e_\mu = e_\nu R^\nu_\mu(a). \tag{28}$$

If (V, g) is complex, then the algebras $\mathcal{C}\ell(V, g)$ and $\mathcal{C}\ell(V, -g)$ are isomorphic; this induces an isomorphism of the groups $\mathrm{Pin}(V, g)$ and $\mathrm{Pin}(V, -g)$. If $V = \mathbb{C}^m$, then this group is denoted by $\mathrm{Pin}_m(\mathbb{C})$. If $V = \mathbb{R}^{k+l}$ and g is of signature (k, l), then one writes $\mathrm{Pin}(V, g) = \mathrm{Pin}_{k,l}$. A similar notation is used for the groups spin, see below.

Spin groups

The *spin group* $\mathrm{Spin}(V,g) = \mathrm{Pin}(V,g) \cap \mathcal{C}\ell^0(V,g)$ is generated by products of all sequences of an even number of unit vectors. Since the algebras $\mathcal{C}\ell^0(V,g)$ and $\mathcal{C}\ell^0(V,-g)$ are isomorphic, so are the groups $\mathrm{Spin}(V,g)$ and $\mathrm{Spin}(V,-g)$. Since $\alpha(a) = a$ for $a \in \mathrm{Spin}(V,g)$, the twisted adjoint representation reduces to the adjoint representation and yields the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V,g) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V,g) \to 1. \tag{29}$$

For $V = \mathbb{C}^m$, the spin group is denoted by $\mathrm{Spin}_m(\mathbb{C})$. Since $\mathrm{Spin}_m(\mathbb{C}) \subset \mathcal{G}_1(\beta)$, the bilinear form B is invariant with respect to the action of this group.

Spin^0 groups

The connected component $\mathrm{Spin}^0(V,g)$ of the group $\mathrm{Spin}(V,g)$ coincides with $\mathrm{Spin}(V,g)$ if either the quadratic space (V,g) is complex or real and $kl = 0$. In signature (k,l), the connected group $\mathrm{Spin}^0_{k,l}$ is generated in $\mathcal{C}\ell^0_{k,l}$ by all products of the form $u_1 \dots u_{2p} v_1 \dots v_{2q}$ such that $u_i^2 = -1$ and $v_j^2 = 1$. The connected groups $\mathrm{Spin}_{m.0}$ and $\mathrm{Spin}_{0,m}$ are isomorphic and denoted by Spin_m. Since $\mathrm{Spin}^0_{k,l} \subset \mathcal{G}_1(\beta)$, the Hermitian form A and the bilinear form B are invariant with respect to the action of this group. Moreover, for $k+l$ even, from (24) and (28) there follows the transformation law of multivectors formed from pairs of spinors,

$$A_{\mu_1\dots\mu_p}(\gamma(a)s, \gamma(a)t) = \\ A_{\nu_1\dots\nu_p}(s,t) R^{\nu_1}_{\mu_1}(a^{-1}) \dots R^{\nu_p}_{\mu_p}(a^{-1}).$$

Consider $\mathrm{Spin}^0(V,g)$ and assume that either V is complex of dimension $\geqslant 2$ or real with k or $l \geqslant 2$. There then are two unit orthogonal vectors $e_1, e_2 \in V$ such that $(e_1e_2)^2 = -1$. The vector $u(t) = e_1 \cos t + e_2 \sin t$ is obtained from e_1 by rotation in the plane $\mathrm{span}\{e_1, e_2\}$ by the angle $t \in \mathbb{R}$. The curve $t \mapsto e_1 u(t)$, $0 \leqslant t \leqslant \pi$, connects the elements 1 and -1 of $\mathrm{Spin}^0(V,g)$. Its image in $\mathrm{SO}^0(V,g)$, i.e. the curve $t \mapsto \mathrm{Ad}(e_1 u(t))$, $0 \leqslant t \leqslant \pi$, is closed: $\mathrm{Ad}(1) = \mathrm{Ad}(-1)$. This fact is often expressed by saying that "a spinor undergoing a rotation by 2π changes sign". There is no homomorphism—not even a continuous map—$f : \mathrm{SO}^0(V,g) \to \mathrm{Spin}^0(V,g)$ such that $\mathrm{Ad} \circ f = \mathrm{id}$.

Spin^c groups

For the purposes of physics, to describe charged fermions, and in the theory of the Seiberg–Witten invariants, one needs the spin^c groups that are spinorial extensions of the real orthogonal groups by the group U_1 of 'phase factors'. Assume V to be real and g of signature (k,l) so that the sequence (29) can be written as

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_{k,l} \to \mathrm{SO}_{k,l} \to 1.$$

Define the action of $\mathbb{Z}_2 = \{1, -1\}$ in $\mathrm{Spin}_{k,l} \times \mathrm{U}_1$ so that $(-1)(a, z) = (-a, -z)$. The quotient $(\mathrm{Spin}_{k,l} \times \mathrm{U}_1)/\mathbb{Z}_2 = \mathrm{Spin}^c_{k,l}$ yields the extensions

$$1 \to \mathrm{U}_1 \to \mathrm{Spin}^c_{k,l} \to \mathrm{SO}_{k,l} \to 1$$

and

$$1 \to \mathrm{Spin}_{k,l} \to \mathrm{Spin}^c_{k,l} \to \mathrm{U}_1 \to 1.$$

For example, $\mathrm{Spin}_3 = \mathrm{SU}_2$ and $\mathrm{Spin}^c_3 = \mathrm{U}_2$.

Spin groups in dimensions $\leqslant 6$

The connected components of spin groups associated with orthogonal spaces of dimension $\leqslant 6$ are isomorphic to classical groups. They can be explicitly described starting from the following observations.

Consider the four-dimensional vector space (of twistors) T over K, with a volume element $\mathrm{vol} \in \wedge^4 T$. The six-dimensional vector space $V = \wedge^2 T$ has a scalar product g defined by $g(u, v)\,\mathrm{vol} = 2u \wedge v$ for $u, v \in V$. The quadratic form $g(u, u)$ is the Pfaffian, $\mathrm{Pf}(u)$. If $u \in V$ is represented by the corresponding isomorphism $T^* \to T$ and $a \in \mathrm{End}\, T$, then $\mathrm{Pf}(aua^*) = \det a\, \mathrm{Pf}(u)$. The last formula shows $\mathrm{Spin}^0(V, g) = \mathrm{SL}(T)$ so that $\mathrm{Spin}_6(\mathbb{C}) = \mathrm{SL}_4(\mathbb{C})$. For $K = \mathbb{R}$, the Pfaffian is of signature $(3, 3)$ so that $\mathrm{Spin}^0_{3,3} = \mathrm{SL}_4(\mathbb{R})$. A non-null vector $v \in V$ defines a symplectic form on T^*. The five-dimensional vector space $v^\perp \subset V$ is invariant with respect to the symplectic group $\mathrm{Sp}(T^*, v) = \mathrm{Spin}^0(v^\perp, \mathrm{Pf}\,|v^\perp)$. This shows $\mathrm{Spin}_5(\mathbb{C}) = \mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{Spin}^0_{2,3} = \mathrm{Sp}_4(\mathbb{R})$. Spin groups for other signatures in real dimensions 6 and 5 are obtained by considering appropriate real subspaces of $\mathbb{C}^6$ and $\mathbb{C}^5$, respectively. For example, (6) is used to show $\mathrm{Spin}^0_{1,5} = \mathrm{SL}_2(\mathbb{H})$.

Spin groups in dimensions 4 and lower are similarly obtained from the observation that det is a quadratic form on the four-dimensional space $K(2)$ and $\mathcal{C}\ell^0(K(2), \det) = K(2) \oplus K(2)$.

The complex spin groups

$$\begin{aligned}
&\mathrm{Spin}_2(\mathbb{C}) = \mathbb{C}^\times,\\
&\mathrm{Spin}_3(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}),\\
&\mathrm{Spin}_4(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}),\\
&\mathrm{Spin}_5(\mathbb{C}) = \mathrm{Sp}_4(\mathbb{C}),\\
&\mathrm{Spin}_6(\mathbb{C}) = \mathrm{SL}_4(\mathbb{C}).
\end{aligned}$$

The real, compact spin groups

$$\begin{aligned}\mathrm{Spin}_2 &= \mathrm{U}_1,\\ \mathrm{Spin}_3 &= \mathrm{SU}_2,\\ \mathrm{Spin}_4 &= \mathrm{SU}_2 \times \mathrm{SU}_2,\\ \mathrm{Spin}_5 &= \mathrm{Sp}_2(\mathbb{H}),\\ \mathrm{Spin}_6 &= \mathrm{SU}_4\,.\end{aligned}$$

The groups $\mathrm{Spin}^0_{k,l}$ *for* $1 \leqslant k \leqslant l$ *and* $k+l \leqslant 6$

$$\begin{aligned}\mathrm{Spin}^0_{1,1} &= \mathbb{R},\\ \mathrm{Spin}^0_{1,2} &= \mathrm{SL}_2(\mathbb{R}),\\ \mathrm{Spin}^0_{1,3} &= \mathrm{SL}_2(\mathbb{C}),\\ \mathrm{Spin}^0_{2,2} &= \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}),\\ \mathrm{Spin}^0_{1,4} &= \mathrm{Sp}_{1,1}(\mathbb{H}),\\ \mathrm{Spin}^0_{2,3} &= \mathrm{Sp}_4(\mathbb{R}),\\ \mathrm{Spin}^0_{1,5} &= \mathrm{SL}_2(\mathbb{H}),\\ \mathrm{Spin}^0_{2,4} &= \mathrm{SU}_{2,2},\\ \mathrm{Spin}^0_{3,3} &= \mathrm{SL}_4(\mathbb{R}).\end{aligned}$$

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