

# Problems in Quantum Field Theory

**Set 1**

**Problem 1.1**

Prove the following expansion

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$$

Prove also the Baker-Hausdorff operator identity

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} = e^{-\frac{1}{2}[A, B]} e^A e^B,$$

holding for operators  $A$  and  $B$  commuting with  $[A, B]$ . Finally, prove the general formula,<sup>1</sup>

$$e^{t(A+B)} = e^{tA} \text{T exp} \left( \int_0^t d\tau e^{-\tau A} B e^{\tau A} \right),$$

valid for any two operators  $A$  and  $B$ , in which T denotes the “time” ordered product.

**Hints:** To prove the expansion solve iteratively the differential equation satisfied by the operator function  $C(\lambda) = e^{\lambda A} B e^{-\lambda A}$ . Similarly, to prove the Baker-Hausdorff formula consider the function  $F(\lambda) = e^{-\lambda B} e^{-\lambda A} e^{\lambda(A+B)}$  and simplify the differential equation it satisfies by using the fact that owing to the assumption, in the expansion of  $e^{-\lambda B} A e^{\lambda B}$  in powers of the operator  $\lambda B$  only two first terms are nonvanishing and, moreover,  $F(\lambda)$  commutes with  $A + B$  (show this).

**Problem 1.2**

Let  $|\Psi(t)\rangle_S$  be an eigenvector with the eigenvalue  $a(t)$  of the Schrödinger picture operator  $A^S$ . Show that  $|\Psi\rangle_H$  representing the same state in the Heisenberg picture (defined with respect to  $t = 0$ ) is the eigenvector of  $A^H(t)$  with the same eigenvalue  $a(t)$ . Prove also that if  $[A^H(t_0), B^H(t_0)] = C^H(t_0)$ , then the same holds for any  $t$ .

**Problem 1.3**

Find the Heisenberg picture operators  $\hat{x}_H(t)$  and  $\hat{p}_H(t)$  of a particle of mass  $M$  moving in one dimension if

- a) it is a free particle ( $H = \hat{p}^2/2M$ ),
- b)  $H = \hat{p}^2/2M - \hat{x}F(t)$ , where  $F(t)$  is an external, time dependent force,
- c)  $H = \hat{p}^2/2M + M\omega^2 \hat{x}^2/2$ .

In all these cases compute the commutators

$$[\hat{x}_H(t), \hat{x}_H(t')], \quad [\hat{p}_H(t), \hat{p}_H(t')], \quad [\hat{x}_H(t), \hat{p}_H(t')].$$

Using the Heisenberg picture operators compute in cases a) and c) the dispersion of the particle’s position at the instant  $t$  expressing it through matrix elements of some combinations of the position and momentum operators at  $t = 0$ .

---

<sup>1</sup>The Baker-Hausdorff formula is its special case corresponding to  $t = 1$  and  $[A, [A, B]] = 0$ ,  $[B, [A, B]] = 0$ .

**Problem 1.4**

Justify the identity<sup>2</sup>

$$a^\dagger a = \sum_{n=0}^{\infty} |n\rangle n \langle n|,$$

in which  $|n\rangle$  are the normalized eigenvectors of the operator  $a^\dagger a$ , where  $a$  and  $a^\dagger$  are the standard annihilation and creation operators.

**Problem 1.5**

Find the Heisenberg picture operators  $\hat{x}_H(t)$  and  $\hat{p}_H(t)$  of the one-dimensional harmonic oscillator the dynamics of which is set by the time dependent Hamiltonian

$$H(t) = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2 - \hat{x}F(t),$$

in which the force  $F(t)$  is a given  $c$ -number function of time, using the solution<sup>3</sup> of the corresponding classical equations of motion with the initial conditions  $x(0) = x_0$  and  $p(0) = p_0$ . To this end, recalling that  $x_0$  and  $p_0$  are also canonical variables related to the standard ones,  $x(t)$  and  $p(t)$ , by the canonical transformation (the generating function of which is just the properly understood action  $I$ ), promote them to operators  $\hat{x}_0$  and  $\hat{p}_0$  on which the standard commutation rules  $[\hat{x}_0, \hat{p}_0] = i\hbar$ , etc. are imposed and represent them in the standard way in terms of the creation and annihilation operators. Since the classical Hamiltonian written in terms of the canonical variables  $x_0$  and  $p_0$  vanishes (this is precisely what is ensured by solving the Hamilton-Jacobi equation, but one does not need to do it explicitly here), the operators  $\hat{x}(t)$  and  $\hat{p}(t)$  obtained from the classical solution in which the operators  $\hat{x}_0$  and  $\hat{p}_0$  are substituted for  $x_0$  and  $p_0$  (expressed, in turn, through the creation and annihilation operators) are just the Heisenberg picture operators. The Heisenberg picture operators  $a_H(t)$  and  $a_H^\dagger(t)$  can be then read off from the form of  $\hat{x}_H(t)$  and  $\hat{p}_H(t)$ .

**A reassuring remark:** the description of the problem is long but the steps to do are entirely trivial. After doing it, you will have, perhaps, a better understanding of what “quantization” means.

**Problem 1.6**

A particle of mass  $m$  and electric charge  $q$  (in units of  $e > 0$ ) moves in the constant magnetic field  $\mathbf{B} = \mathbf{e}_z B$ . Find the Heisenberg picture operators  $\hat{x}_H(t)$ ,  $\hat{y}_H(t)$  and  $\hat{z}_H(t)$  and compute the commutators  $[\hat{x}_H(t), \hat{x}_H(t')]$ ,  $[\hat{y}_H(t), \hat{y}_H(t')]$ ,  $[\hat{x}_H(t), \hat{y}_H(t')]$  and  $[\hat{x}_H(t), \hat{z}_H(t')]$ . Do these commutators depend on the choice of the potential  $\mathbf{A}$  (the choice of the gauge)? Consider also the operators  $\hat{p}_H^x(t)$ ,  $\hat{p}_H^y(t)$ ,  $\hat{p}_H^z(t)$  and their commutators. Do they depend on the gauge?

**Hint:** If it is too difficult to work without specifying explicitly a gauge, set e.g.  $\mathbf{A} = \mathbf{e}_y \xi B x - \mathbf{e}_x (1 - \xi) B y$  with an arbitrary parameter  $\xi$  in order to follow the gauge (in)dependence at least within a restricted class of gauges. To construct the Heisenberg picture

<sup>2</sup>This is taken from the BMW, but there the problem is formulated with a misprint...

<sup>3</sup>The solution can be found e.g. in my notes to Classical Mechanics (in polish).

operators  $\hat{x}_H(t)$ ,  $\hat{y}_H(t)$ ,  $\hat{p}_H^x(t)$ ,  $\hat{p}_H^y(t)$ , take the inspiration from Problem 1.5. Remember that the canonical momenta  $p^x$  and  $p^y$  are not simply given by  $m\dot{x}$  and  $m\dot{y}$ .

**Problem 1.7**

A particle of mass  $M$  and electric charge  $q$  (in units of  $e > 0$ ) moves in the electric and magnetic fields represented by the potentials  $\varphi(t, \mathbf{r})$  and  $\mathbf{A}(t, \mathbf{r})$ . Find the equation of motion satisfied by the Heisenberg picture operator  $\hat{\mathbf{r}}_H(t)$ , that is compute  $d^2\hat{\mathbf{r}}_H(t)/dt^2$ . Establish how this derivative differs from the classical formula (written here in the Gauss system of units)

$$M \frac{d^2\mathbf{r}(t)}{dt^2} = qe \left[ \mathbf{E}(t, \mathbf{r}) + \frac{\mathbf{v}}{c} \times \mathbf{B}(t, \mathbf{r}) \right].$$

**Problem 1.8** (Sucher formula)

Express the difference  $E_\Omega - E_{\Omega_0}$  of ground state energies of the Hamiltonians  $H = H_0 + \lambda V_{\text{int}}$  and  $H_0$  through the derivative with respect to the parameter  $\lambda$  of the operator<sup>4</sup>

$$S_0^\varepsilon \equiv U_I^{-\varepsilon}(+\infty, 0) U_I^\varepsilon(0, -\infty) = [U_I^{-\varepsilon}(0, +\infty, )]^\dagger U_I^\varepsilon(0, -\infty),$$

that is, prove the so-called Sucher formula

$$E_\Omega - E_{\Omega_0} = \frac{1}{2} i\hbar \varepsilon \lambda \frac{\partial}{\partial \lambda} \ln \langle \Omega_0 | S_0^\varepsilon | \Omega_0 \rangle,$$

**Problem 1.9**

By considering the differential equation satisfied by it, find the complete evolution operator  $U^\varepsilon(t, 0)$ , including its phase, corresponding to the Gell-Mann - Low modification  $V_{\text{int}} \rightarrow e^{\varepsilon t} V_{\text{int}}$  of the Hamiltonian ( $\Delta_\omega = \hbar\omega/2$ )

$$H = H_0 + V_{\text{int}} = \hbar\omega a^\dagger a + \Delta_\omega + \lambda a^\dagger + \lambda^* a,$$

of the linearly perturbed harmonic oscillator. Show then by an explicit computation that the expression

$$\lim_{T \rightarrow \infty} \left( \lim_{\varepsilon \rightarrow 0^+} \langle \Omega_0 | U_I^{-\varepsilon}(T, 0) [U_I^\varepsilon(-T, 0)]^\dagger | \Omega_0 \rangle \right) \equiv \lim_{T \rightarrow \infty} \langle \Omega_0 | U_I(T, -T) | \Omega_0 \rangle,$$

in which  $U_I^{\pm\varepsilon}(t, 0)$  are the interaction picture evolution operators corresponding to the interaction term adiabatically switched on and off and the limit  $\varepsilon \rightarrow 0^+$  is taken first, behaves as

$$\exp \left\{ -i \frac{2T}{\hbar} (E_\Omega - E_{\Omega_0}) \right\}.$$

---

<sup>4</sup> $U_I^{-\varepsilon}(+\infty, 0)$  is the interaction picture evolution operator corresponding to replacing the original time independent interaction  $\lambda V_{\text{int}}$  by  $\lambda V_{\text{int}} e^{-\varepsilon t}$ . The limit  $\varepsilon \rightarrow 0^+$  is implicit.

**Hint:** In order to ensure the proper transformation to the Heisenberg picture of the basic operators  $a$  and  $a^\dagger$ , the sought evolution operator must have the form

$$U^\varepsilon(t, 0) = e^{i\varphi(t)} e^{-iH_0 t/\hbar} e^{h(t)} a^\dagger - h^*(t) a, \quad h(t) = -\frac{i}{\hbar} \int_0^t d\tau \lambda e^{(\varepsilon+i\omega)\tau},$$

so only the phase  $\varphi(t)$  has to be determined.

**Problem 1.10**

The harmonic oscillator of mass  $M$  and frequency  $\omega$  on which acts an external force  $F(t)$  vanishing as  $t \rightarrow \mp\infty$  (i.e. the Hamiltonian as in Problem 1.5; one can consider concrete forces like  $F(t) = F_0/(1+t^2/\tau^2)$  or  $F(t) = F_0 \exp(-t^2/\tau^2)$ ) was at  $t = -\infty$  in the ground state of  $H_0$ . Compute the mean oscillator energy  $\overline{E}$  at  $t = \infty$  and the energy dispersion squared  $\overline{E^2} - \overline{E}^2$  using the formalism of the *in* and *out* operators. Obtain the same result using the  $S$ -matrix elements  $S_{kl} = \langle k \text{ out} | l \text{ in} \rangle$ .

**Problem 1.11**

The work  $W$  done on the harmonic oscillator of frequency  $\omega$  and mass  $M$ , the classical motion of which as  $t \rightarrow -\infty$  is given by  $x(t) = A \cos(\omega t + \delta)$ , by a force  $F(t)$  vanishing as  $t \rightarrow \mp\infty$  can be computed as the difference of the oscillator energies at  $t = \infty$  and  $t = -\infty$ . If  $F(t) = F_0 \exp(-t^2/\tau^2)$  this work is equal (see Kotkin & Serbo Problem 5.12 or my notes to classical mechanics, Problem in Section 2)

$$W = \frac{\pi F_0^2}{2M\omega^2} \omega^2 \tau^2 e^{-\frac{1}{2}\omega^2 \tau^2} - \sqrt{\pi} F_0 A \omega \tau e^{-\frac{1}{4}\omega^2 \tau^2} \sin \delta.$$

Find the quantum mechanics analog of this result, that is compute the mean value  $\overline{W}$  of the work done by the force  $F(t)$  on the oscillator the (Schrödinger picture) state-vector  $|\psi(t)\rangle$  of which was at  $t \rightarrow -\infty$  such that

$$\langle \psi(t) | \hat{x} | \psi(t) \rangle = A \cos(\omega t + \delta).$$

**Hint:** Express the matrix element  $\langle \psi(t) | \hat{x} | \psi(t) \rangle$  in the Heisenberg picture and use the formalism of the *in* and *out* operators.

**Problem 1.12**

Using the commutation rules of the rotation group generators  $[J^x, J^y] = iJ^z$  etc., show that

$$e^{-i\phi J^z} e^{-i\theta J^y} e^{+i\phi J^z} = e^{-i\theta(J^y \cos \phi - J^x \sin \phi)}.$$

Write down also other similar relations with the generators  $J^x$ ,  $J^y$  and  $J^z$ .

**Problem 1.13**

Show that if the (active) rotation represented by the  $3 \times 3$  orthogonal matrix  $O$  is generated through the formula<sup>5</sup>

$$M \cdot (\sigma_i r^i) \cdot M^\dagger = \sigma_j (O^j_i r^i),$$

by the  $2 \times 2$  matrix  $M$  belonging to the  $SU(2)$  group and if the rotation matrix  $\tilde{O}$  is related in the same way to another  $SU(2)$  matrix  $\tilde{M}$ , then the  $SU(2)$  matrix  $\tilde{M} \cdot M$  generates in this way the rotation matrix  $\tilde{O} \cdot O$ .

**Problem 1.14**

A vector  $\mathbf{V}$  rotated by the angle  $\phi$  around the axis  $\mathbf{n}$  (where  $|\mathbf{n}| = 1$ ) can be written as

$$\begin{aligned} \mathbf{V}' &= \mathbf{V} \cos \phi + \mathbf{n} (\mathbf{n} \cdot \mathbf{V}) (1 - \cos \phi) + \mathbf{n} \times \mathbf{V} \sin \phi \\ &\approx \mathbf{V} + \phi \times \mathbf{V} - \frac{1}{2} \phi^2 \mathbf{V} + \frac{1}{2} \phi (\phi \cdot \mathbf{V}) + \dots \end{aligned}$$

where  $\phi \equiv \phi \mathbf{n}$ . Justify this formula. Show that if  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are two vectors perpendicular to one another and perpendicular to  $\mathbf{n}$ , then  $(O \cdot \mathbf{V})$  denotes the action of the rotation  $O$  on the vector  $\mathbf{V}$ )

$$O \cdot (\mathbf{V}_1 \pm i \mathbf{V}_2) = e^{\pm i \phi} (\mathbf{V}_1 \pm i \mathbf{V}_2),$$

i.e. that (since obviously  $O \cdot \mathbf{n} = \mathbf{0}$ ) the rotation  $O$  treated as a linear mapping has the eigenvalues 0 and (in the complexified vector space)  $e^{\pm i \phi}$ . Find the vector  $\phi$  corresponding to the composition of two successive infinitesimal rotations characterized by  $\phi_1$  and  $\phi_2$  of a vector  $\mathbf{V}$ . Using the result find the structure constants of the rotation group. Show also that the matrix<sup>6</sup>

$$[O_{\text{vec}}(\phi, \mathbf{n})]_j^i = \delta^{ij} \cos \phi + (1 - \cos \phi) n^i n^j + \epsilon^{ikj} n^k \sin \phi,$$

such that  $V'^i = [O_{\text{vec}}(\phi, \mathbf{n})]_j^i V^j$ , is just the matrix  $\exp(-i \phi^k \mathcal{J}_{\text{vec}}^k)$ , where  $(\mathcal{J}_{\text{vec}}^k)_j^i = i \epsilon^{ikj}$  are the rotation group generators in the defining (vector) representation.

---

<sup>5</sup>In view of notation used in the applications of the  $SL(2, C)$  group representations to spinors, it is convenient to operate with two sets of the Pauli matrices  $\sigma^i$  and  $\bar{\sigma}^i$  which can be written in the “co-” and “contravariant forms:  $\sigma^i = -\sigma_i = -\bar{\sigma}^i = \bar{\sigma}_i$ , where  $\sigma^i$  are the three “standard” Pauli matrices.

<sup>6</sup>Since it is desirable to denote differently active rotations (which are linear mappings of the vector space into itself and, hence, their matrices are written in a fixed basis) and passive rotations (matrices of which are matrices of changes of bases i.e. matrices of the identity mapping but written in two different bases - see my famous Algebra notes), we choose to denote the active ones by  $O$  (from polish - let proud Poland getting up from knees contribute also to physics - “obrót” - written under the regime of pis-patriots).

**Problem 1.15**

Let  $O(\theta, \mathbf{n})$  with  $\mathbf{n}^2 = 1$  be the (active) rotation around the direction  $\mathbf{n}$  by the angle  $\theta$ . Show that ( $\mathbf{k}^2 = 1$ )

$$O(\theta, \mathbf{n}) \cdot O(\psi, \mathbf{k}) \cdot O^{-1}(\theta, \mathbf{n}) = O(\psi, O_{\text{vec}}(\theta, \mathbf{n}) \cdot \mathbf{k}),$$

where  $O_{\text{vec}}$  means the rotation realized on vectors (in this formula  $O(\theta, \mathbf{n})$  stand for an abstract rotation which can be realized in any vector space, in particular in a Hilbert space, by an appropriate symmetry operator).

**Problem 1.16**

Using the result of Problem 1.15 show that the (active) rotation  $O(\alpha, \beta, \gamma)$  parametrized by three Euler angles and composed of three successive rotations: first by the angle  $\alpha$  around the axis  $\mathbf{n}_1 \equiv \mathbf{e}_z$ , then by the angle  $\beta$  around the axis  $\mathbf{n}_2 \equiv -\mathbf{e}_x \sin \alpha + \mathbf{e}_y \cos \alpha$  and finally by  $\gamma$  around the axis  $\mathbf{n}_3 \equiv \mathbf{e}_x \cos \alpha \sin \beta + \mathbf{e}_y \sin \alpha \sin \beta + \mathbf{e}_z \cos \beta$  (these are the famous three moves of the paw - who attended my Classical Mechanics course, knows what I mean) is equivalent to the composition of three other successive rotations: first by  $\gamma$  around  $\mathbf{e}_z$ , then by  $\beta$  around  $\mathbf{e}_y$  and finally by  $\alpha$  again around  $\mathbf{e}_z$ :

$$O(\gamma, \mathbf{n}_3) \cdot O(\beta, \mathbf{n}_2) \cdot O(\alpha, \mathbf{n}_1) = O(\alpha, \mathbf{e}_z) \cdot O(\beta, \mathbf{e}_y) \cdot O(\gamma, \mathbf{e}_z).$$

Show also formally, that is treating the matrices  $O$  (of the active rotations) as matrices of the linear mappings of the vector space into itself which are given in the fixed basis  $\mathbf{e}_i$  and matrices of the passive rotations as matrices of the changes of the bases (that is matrices of the identity mappings but written in different bases), that (what should be obvious) in the reference frame rotated by the angles  $\alpha$ ,  $\beta$  and  $\gamma$  the components of the rotated vector are the same as the components of the original vector in the original reference frame.

**Problem 1.17**

A left-invariant measure  $d\mu(g)$  on a group  $G$  has the property

$$\int d\mu(g) f(g) = \int d\mu(g) f(g'g)$$

( $g$  denotes an element of  $G$  and  $f(g)$  is a function defined on the group  $G$ ). In a concrete parametrization  $g = g(\boldsymbol{\theta})$  of the group elements by some parameters  $\theta_a$  with  $a = 1, \dots, o$ , where  $o$  is the dimension of the Lie algebra of  $G$  the measure is given by  $d\mu(g) = d^o \boldsymbol{\theta} \rho(\boldsymbol{\theta})$ . Using the general formula

$$\rho(\boldsymbol{\theta}) = \rho(\mathbf{0}) \det^{-1} \left( \frac{\partial h_a(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})}{\partial \tilde{\theta}_b} \right)_{\tilde{\theta}_b=0},$$

in which  $h(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})$  is the group composition function appropriate for the chosen parametrization  $g = g(\boldsymbol{\theta})$  of the group elements, find the left-invariant measure (i.e. the density  $\rho$ ) on the rotation group  $SO(3)$  in the parametrization given by the components of the vector

$\phi = (\phi^x, \phi^y, \phi^z)$  defined in Problem 1.14. Compute the rotation group volume adopting the usual convention according to which  $\rho(\mathbf{0}) = 1$ .

**Problem 1.18**

Let the two-parameter group  $G$  of transformations of the real axis  $\mathbb{R}$  be defined by the formula

$$x' = (1 + \xi_1)x + \xi_2,$$

Using the general formula quoted in Problem 1.17 and its counterpart appropriate for right-invariant measures, find both these measures on the group  $G$ . Are they identical?

**Problem 1.19**

Prove that if the dimension  $n$  of the group  $G$  is odd, the formula  $d\mu(g) \equiv d^n \xi \rho(\xi)$  with

$$\rho(\xi^1, \dots, \xi^o) \propto \epsilon^{i_1 i_2 \dots i_o} \text{tr} \left( O^{-1} \cdot \frac{\partial O}{\partial \xi^{i_1}} \cdot O^{-1} \cdot \frac{\partial O}{\partial \xi^{i_2}} \cdot \dots \cdot O^{-1} \cdot \frac{\partial O}{\partial \xi^{i_o}} \right),$$

where  $O(\xi)$  is a matrix representation of the group element parametrized by the parameters  $\xi^i$ , defines on  $G$  a left-invariant measure (if  $n$  is even, the measure defined in this way vanishes as a result of the antisymmetry of  $\epsilon^{i_1 i_2 \dots i_o}$  and the cyclicity of the trace). Use this result to find explicitly the density  $\rho(\alpha, \beta, \gamma)$  of the left-invariant measure on the  $SO(3)$  and  $SU(2)$  groups parametrized by the three Euler angles  $\alpha$ ,  $\beta$  and  $\gamma$ .

**Problem 1.20**

Show that the left-invariant measure on a compact group  $G$  parametrized by the parameters  $\theta_a$ ,  $a = 1, \dots, o$

$$g(\theta) = \exp(-i\theta_a Q^a),$$

where  $Q^a$ ,  $a = 1, \dots, o$ , are the group generators in some representation, takes, infinitesimally close to the identity transformation, the simple form  $d^o \theta$ .