# 13 Green's functions and the LSZ prescription

In the preceding Chapters we have presented two different formulations of quantum field theory based on two different "ontologies". Firstly, within the framework of quantum mechanics of many particle systems (written in the language of second quantization of Chapter 5) we have formulated a prescription for constructing interactions $V_{\rm int}$ leading to a relativistically covariant $S$-matrix for an assumed set of particles the free Hamiltonian $H_0$ of which (i.e. in the absence of interactions) has the form of a sum of the terms (6.113). Satisfying by the $S$-matrix the cluster decomposition principle requires $V_{\rm int}$ to be built out of the creation and annihilation operators associated with the particles while its Poincaré covariance is ensured by combining these operators into free field operators constructed in Chapter 8.

Next, in Chapter 11, we have presented an alternative approach based on quantization of classical fields transforming as regular representations of the Lorentz group under changes of inertial reference frames. In this approach one starts with a local, Lorentz scalar Lagrangian density $\mathcal{L}$ and, by appropriately choosing the canonical variables, one obtains the Hamiltonian $H$ expressed through the Heisenberg picture (canonical or rescaled, i.e. renormalized - see Section 11.10) field operators called also *interacting field operators* (i.e. the operators which, unlike the ones introduced in Section 8, cannot, except for an arbitrarily chosen one moment of time, usually $t = 0$, be written as simple linear combinations of the time independent creation and annihilation operators). Formal equivalence of this approach to the first one is obtained by splitting the Hamiltonian $H$ into $H_0$ and $V_{\rm int}$, expressing the interacting field operators taken at $t = 0$ through the creation and annihilation operators, so that $H_0$ is (in most cases) diagonalized, and going over to the interaction picture (see Section 11.9). The free field operators out of which one constructs the interaction $V_{\rm int}$ in the first approach acquire in this way the interpretation of the interaction picture operators obtained by giving time dependence dictated by $H_0$ to the interacting operators (taken at $t = 0$) arising from quantization of the corresponding classical field theory.

Owing to the formal identity of the mathematical structure obtained when the canonical commutation relation of field operators are represented in an appropriate Fock space with the structure of the theory of interacting particles, the explicit expression (7.63) for $S$-matrix elements corresponding to the Hamiltonian $H = H_0 + V_{\rm int}$ can be derived in both of them under the same assumptions (see Section 7): *i)* that the true spectrum of $H$ can be interpreted in terms of particles, i.e. there exist *in* and *out* states which have Lorentz transformation properties appropriate for a states representing a set of free relativistic particles and *ii)* that these *in* and *out* states are in the strict one-to-one correspondence with the free particle states created and annihilated by the interaction picture free field operators out of which the interaction $V_{\rm int}$ is built. The prescription for evaluating the formula (7.63) for $S$-matrix elements by using Feynman diagrams assumes, in addition, that their perturbative expansion in powers of the coupling constant(s) is a reliable tool.

From the "technical" point of view, satisfying the assumptions underlying the expression (7.63) for $S$-matrix elements forces us in the first approach to include in the interaction operator $V_{\rm int}$ judiciously chosen terms which cancel effects of self-interactions of isolated particles (see Section 9.7). As explained in Section 11.10, in the approach based on quantization of fields the same assumptions single out one particular choice of canonical variables which, upon quantization, become physically renormalized field operators and a particular splitting of the mass terms (nonderivative terms quadratic in fields) between $H_0$ and $V_{\rm int}$. The two prescriptions (pertaining to two conceptually different approaches) are in fact equivalent to each other because the extra terms which have to be added to $V_{\rm int}$ in the first approach have precisely the same form as the terms generated in the interaction term if the corresponding field theory is quantized in the canonical variables singled out by the assumptions of Section 7.3. Thus, adjusting $V_{\rm int}$ in the first approach (by adding to it terms needed to cancel self-interaction of isolated particles) is equivalent[1] to rewriting the original Hamiltonian in terms of the canonical (called also *bare*) Heisenberg picture operators and in terms of the *bare mass parameters* instead in terms of the physical operators and masses (and, as will be discussed in Chapter 14, also in terms of *bare coupling constants*). This change is therefore like a canonical transformation which should not alter the character of the spectrum of the Hamiltonian (and, because it is merely a rescaling of field operators, it preserves also its original form).

It is important, however, to stress (what was already said in Section 7.3), that the assumptions *i)* and *ii)* need not be always satisfied: the true spectrum of the complete Hamiltonian $H$, even if interpretable in terms of particles, may correspond to particles which are very different than the ones created by the free field operators. Such a situation is common in condensed matter physics or nuclear physics where quantum states of systems the Hamiltonians of which are constructed out of operators corresponding to some well known "elementary physical" particles like electrons and ions or nucleons must be interpreted in terms of the so-called collective excitations such as phonons, magnons etc. (which, moreover, being usually unstable, are not true eigenstates of the respective Hamiltonians), rather than in terms of states of original particles ("put in" into the system). From the point of view of the second approach it is not difficult to imagine that quantum states of some systems of fields, represented in principle (in the big nonseparable Hilbert space) by wave functionals $\Psi[\phi(\mathbf{x}), t] \equiv \langle\phi(\mathbf{x})|\Psi(t)\rangle$, may correspond to particles different than the ones in terms of which the $H_0$ eigenvectors are interpreted or could even not correspond to particles at all (as happens in conformally invariant field theories) i.e. the particle-like *in* and *out* eigenvectors of $H$ may not exist. Finally, computing $S$-matrix elements (assuming its existence, i.e. assuming $H$ does possess particle-like *in* and *out* states) as power series in coupling constants may not be reliable (or may be reliable only in a limited sense).

[1] As already mentioned, this statement is true only in a certain, rather narrow, class of theories called *renormalizable*. In generic theories, as will be discussed in Chapter 14, to remove all infinities (related to the UV behaviour of loop diagrams) infinitely many terms must be added to the original interaction, but this is true in both approaches to quantum field theory and does not spoil their equivalence.

In view of all this, it is important to develop a formulation of the quantum field theory *not* relying on the one-to-one relation between the eigenstates of $H_0$ and of $H$ (and on the perturbative expansion). We give such a formulation in this chapter. It is based on Green's functions which allows, at least in principle, to investigate the true spectrum of the full Hamiltonian $H$ (both in the first and in the secong approach) and compute $S$-matrix elements (if the spectrum of $H$ is interpretable in terms of particles). From the field theory point of view this formulation restores also the full equivalence of all possible choices of canonical variables which is superficially lost (due to the distinguished role played by physically normalized operators) in the previous formulation.

To this end, still within the assumptions of Section 7.3 and using the perturbative expansion, we shall first introduce Green's functions - called also off-shell amplitudes - as a generalization of $S$-matrix elements (the on-shell amplitudes, all external line momenta of which satisfy the mass-shell conditions $p_i^2 = m_i^2$ required for physical particles, and are "closed up" with the "wave function" factors listed in (9.29)). We will then demonstrate the relation of Green's functions to matrix elements between the *in* and *out* states of chronologically ordered products of Heisenberg picture field operators (constructed out of physically renormalized - see Section 11.10 - elementary field operators). This will allow us to give Green's function a meaning outside the scheme developed up to now and outside the perturbative expansion. Reverting then the logic we shall argue that the vacuum Green's functions - vacuum matrix elements of chronological products of arbitrary Heisenberg picture interacting operators - can be taken for the basic quantum field theory objects from which the true spectrum of $H$ (of particles in terms of which it can be interpreted if it is particle-like) can be read off by investigating their pole structure. This will be demonstrated rigorously (without appealing to the perturbative expansion) in Section 13.2. Before doing this, we will show that the perturbative expansion of vacuum Green's functions takes formally the same form as the perturbative expansion based on restrictive assumptions of Section 7.3 despite the fact that these assumptions may not be satisfied: it remains true even if the true spectrum of $H$ differs from that of its part taken for $H_0$ (the splitting of $H$ into $H_0$ and $V_{\text{int}}$ must, however, be done in the appropriate variables).

In this more general formulation of quantum field theory, Green's functions which are defined as matrix elements of chronological products of Heisenberg picture (interacting) operators between arbitrary *in* and *out* states and in particular, $S$-matrix elements, can be extracted with the help of the Lehmann-Symanzik-Zimmermann prescription which is explained in Section 13.4 (and 13.6). When applied within the perturbative expansion to obtain $S$-matrix elements, the LSZ prescription reduces (if no complications related to mixing of field operators is involved) to adding one simple extra (compared to the ones formulated in Chapter9) rule for external lines of Feynman diagrams.

## 13.1 Off-shell amplitudes

Suppose we formally modify a given interaction picture Hamiltonian density $\mathcal{H}^I_{\rm int}(x)$ (obtained either in the framework of relativistic quantum mechanics of particles by using the prescriptions of Chapters 7 and 8 or by performing the transition to the interaction picture on the quantum Hamiltonian of a system of interacting relativistic fields expressed in terms of physically normalized field operators $\varphi_{\rm ph}$ - see Section 11.10) by adding to it terms of the form $-\sum_k J_k(x)O^I_k(x)$:

$$\mathcal{H}^I_{\rm int}(x) \to \mathcal{H}^I_{{\rm int}\ J}(x) \equiv \mathcal{H}^I_{\rm int}(x) - \sum_k J_k(x)O^I_k(x)\,, \tag{13.1}$$

where $J_k(x)$ are some arbitrary $c$-number functions,[2] which we will call *external sources*, and $O^I_k(x)$ are the operators constructed out of the free field operators introduced in Chapter 8 and their derivatives. In what follows Lorentz group representation indices $l_i$ of the operators $O_{l_i}$ are also meant to denote the type of the operator: $O_{l_1}$ can for instance stand for $\phi$ and $O_{l_2}$ for $\psi_\alpha$, etc. The time dependence of $O^I_k(x)$ is therefore dictated by $H_0$:

$$O^I_k(t,\mathbf{x}) = e^{iH_0t}\,O_k(0,\mathbf{x})\,e^{-iH_0t}. \tag{13.2}$$

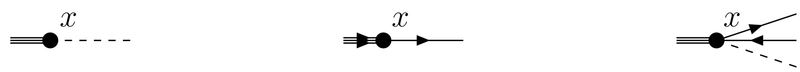
All the Feynman rules resulting from $\mathcal{H}^I_{\rm int}(x)$ follow from $\mathcal{H}^I_{{\rm int}\ J}(x)$ as well, but there are also additional rules like the ones shown in Figure 13.1, since the terms $J_k(x)O_k(x)$ in (13.1) are now treated as ordinary interaction vertices.[3] These rules dictate that in a new vertex to which an internal line (or multiple internal lines) of the diagram is (are) attached, the source $J_k(x)$ is integrated with the "free" end(s) of the propagator(s). For example, if the dashed line shown in Figure 13.1a, comes from an ordinary vertex at $y$, the expression associated with this part of the Feynman diagram is $\int d^4x\,iJ(x)\,i\Delta^F(x-y)$ (the line coming into the vertex 13.1a can also correspond directly to an initial or a final state particle or can come from another new vertex).

With the additional Feynman rules $S$-matrix elements can still be computed as previously but become dependent on the arbitrary functions $J_k(x)$:

$$S_{\beta\alpha} \to S_{\beta\alpha}[J]\,. \tag{13.3}$$

[2]If the operator $O^I_k(x)$ is fermionic, i.e. constructed out of an odd number of half-integer spin particle operators, then $J_k(x)$ assume values which are not $c$-numbers but rather Grassman variables $\xi_k$, i.e. anticommuting numbers $\xi_i\xi_j = -\xi_j\xi_i$. In addition, $\xi_i$ are assumed to anticommute also with fermionic field operators, so that the new terms added to $\mathcal{H}^I_{\rm int}(x)$ effectively commute under the chronological ordering in the formula (7.63).

[3]If the operators $O^I_k(x)$ involve time derivatives, it is understood that they are coupled to the sources together with appropriate noncovariant terms, so that noncovariant terms in propagators connecting the vertices arising from $O^I_k(x)$ are canceled. This means that in both sides of the formula (13.6) derivatives on the operators $O^I_k(x)$ should be effectively treated as standing outside the $T$-product.




rules: $i\int d^4x\, J(x)$ $\quad$ $i\int d^4x\, J_\alpha(x)$ $\quad$ $i\int d^4x\, J(x)$

a) $\quad$ b) $\quad$ c)

Figure 13.1: Additional Feynman rules: a) $O^I(x) = \varphi^I(x)$, b) $O^I_\alpha(x) = \bar\psi^I_\alpha(x)$, c) $O^I(x) = \varphi^I(x)\bar\psi^I_\alpha(x)\psi^I_\alpha(x)$.

Therefore $S_{\beta\alpha}[J]$ can be functionally differentiated with respect to $J_k(x)$ and derivatives of $S_{\beta\alpha}[J]$ with respect to the sources $J_k$ taken at $J_k = 0$ define the amplitudes which we will call the Green's functions. The theorem which we now prove can be written as follows

$$\left(\frac{1}{i}\right)^r \frac{\delta^r S_{\beta\alpha}[J_k]}{\delta J_{l_1}(x_1)\dots\delta J_{l_r}(x_r)}\bigg|_{J_k=0} = \langle\beta_-|\mathrm{T}\big[O^H_{l_1}(x_1)\dots O^H_{l_r}(x_r)\big]|\alpha_+\rangle\,. \tag{13.4}$$

That is, the $r$-th functional derivative of $S_{\beta\alpha}[J_k]$ taken at vanishing external sources gives the matrix element between the *in* and *out* states $|\alpha_+\rangle$ and $|\beta_-\rangle$ of the chronologically ordered product of the Heisenberg picture operators $O^H_k(x)$ defined by

$$O^H_l(x) = e^{iHt}\,O_l(0,\mathbf{x})\,e^{-iHt}\,, \tag{13.5}$$

where $H = H_0 + V_{\rm int} = H_0 + \int d^3\mathbf{x}\ \mathcal{H}_{\rm int}(0,\mathbf{x})$ is the complete Hamiltonian and $\mathcal{H}_{\rm int}$ involves all the terms necessary to satisfy the assumptions on which the calculation of the $S$-matrix elements is based (see Sections 9.7 and 11.10). Performing the functional differentiations in (13.4) we get the theorem in the form

$$\langle\beta_0|\mathrm{T}\left[O^I_{l_1}(x_1)\dots O^I_{l_r}(x_r)\exp\left(-i\int d^4x\ \mathcal{H}^I_{\rm int}(x)\right)\right]|\alpha_0\rangle \\ = \langle\beta_-|\mathrm{T}\big[O^H_{l_1}(x_1)\dots O^H_{l_r}(x_r)\big]|\alpha_+\rangle\,. \tag{13.6}$$

with no reference to auxiliary external sources $J_k(x)$ whatsoever. In this form it applies also to fermionic operators $O_l(x)$. The theorem thus states that Green's functions obtained with the help of the recipe given above are equal to matrix elements between the *in* and *out* states of the Heisenberg picture operators $O^H_l$ which, because the recipe is up to now formulated within the assumptions of Section 7.3, have to be interpreted as being constructed out of physically renormalized canonical field operators $\varphi_{\rm ph}$ and their derivatives (that is $O_k(0,\mathbf{x})$ in (13.2) are constructed out of $\varphi_{\rm ph}$ and their derivatives taken at $t = 0$). This means that if one wants to compute (with the scheme developed up to now) e.g. the Green's function $\langle\beta_-|\mathrm{T}\,[\varphi_H(x_1)\dots\varphi_H(x_n)]\,|\alpha_+\rangle$ of the canonical (bare) field operators one has, before going over to the interaction picture, to write it first as $Z^{n/2}\langle\beta_-|\mathrm{T}\,[\varphi_{\rm ph}(x_1)\dots\varphi_{\rm ph}(x_n)]\,|\alpha_+\rangle$ with the appropriate $Z$ factors (which ought to be fixed order by order computed separately by computing the relevant two-point Green's functions) and compute $\langle\beta_-|\mathrm{T}\,[\varphi_{\rm ph}(x_1)\dots\varphi_{\rm ph}(x_n)]\,|\alpha_+\rangle$ using (13.6).

The proof of the theorem goes as follows. Consider the left hand side of (13.6):

$$\sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int_{-\infty}^{+\infty}d\tau_1\ldots\int_{-\infty}^{+\infty}d\tau_N\langle\beta_0|\mathrm{T}\big[O^I_{l_1}(x_1)\ldots O^I_{l_r}(x_r)V^I_{\rm int}(\tau_1)\ldots V^I_{\rm int}(\tau_N)\big]|\alpha_0\rangle\,,\quad(13.7)$$

and assume that $x^0_1\geq\ldots\geq x^0_r$. Performing the integrals one has to sum over different orderings of the times $\tau_1$, ..., $\tau_N$ with respect to $x^0_1$, ..., $x^0_r$. We consider therefore partitions of the set of instances $\tau_1$, ..., $\tau_N$ into $r+1$ clusters, such that

$$\tau_{01},\ldots,\tau_{0N_0}\geq x^0_1\geq\tau_{11},\ldots,\tau_{1N_1}\geq x^0_2\geq\ldots\geq x^0_r\geq\tau_{r1},\ldots,\tau_{rN_r}\,,$$

and sum over all possible such partitions. Since the operators $V^I_{\rm int}(\tau_i)$ commute under the chronological ordering, it does not matter which of the $\tau$'s belong to which partitions. All that matter is how many $\tau$'s there are in the $k$-th cluster. Thus, (13.7) can be rewritten in the form

$$\begin{aligned}&\sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\sum_{N_0,\ldots,N_r}\frac{N!}{N_0!\ldots N_r!}\,\delta_{N,N_0+\ldots+N_r}\\&\quad\times\int_{x^0_1}^{+\infty}d\tau_{01}\ldots d\tau_{0N_0}\int_{x^0_2}^{x^0_1}d\tau_{11}\ldots d\tau_{1N_1}\ldots\int_{-\infty}^{x^0_r}d\tau_{r1}\ldots d\tau_{rN_r}\\&\qquad\times\langle\beta_0|\mathrm{T}[V_{\rm int}(\tau_{01})\ldots V_{\rm int}(\tau_{0N_0})]O_{l_1}(x_1)\mathrm{T}\,[V_{\rm int}(\tau_{11})\ldots V_{\rm int}(\tau_{1N_1})]\,O_{l_2}(x_2)\ldots\\&\qquad\qquad\ldots O_{l_r}(x_r)\mathrm{T}[V_{\rm int}(\tau_{r1})\ldots V_{\rm int}(\tau_{rN_r})]|\alpha_0\rangle\,.\end{aligned}$$

The factor $N!/N_0!\ldots N_r!$ accounts for the number of ways the $N$ $\tau$'s can be distributed among $r+1$ clusters containing $N_0$, $N_1$, ..., $N_r$ time variables. $\delta_{N,N_0+\ldots+N_r}$ ensures that there are only $N$ $\tau$'s. However, since

$$\sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\sum_{N_0,\ldots,N_r}\frac{N!}{N_0!\ldots N_r!}\,\delta_{N,N_0+\ldots+N_r}=\sum_{N_0=0}^{\infty}\frac{(-i)^{N_0}}{N_0!}\ldots\sum_{N_r=0}^{\infty}\frac{(-i)^{N_r}}{N_r!}\,,$$

we get in this way that the left hand side of (13.6) equals to

$$\langle\beta_0|U_I(\infty,x^0_1)O^I_{l_1}(x_1)U_I(x^0_1,x^0_2)O^I_{l_2}(x_2)\ldots O^I_{l_r}(x_r)U_I(x^0_r,-\infty)|\alpha_0\rangle\,,$$

where $U_I(t',t)$ is the interaction picture evolution operator (7.26)

$$\begin{aligned}U_I(t',t)&=e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}\\&=\sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int_t^{t'}d\tau_1\ldots\int_t^{t'}d\tau_N\,\mathrm{T}\big[V^I_{\rm int}(\tau_1)\ldots V^I_{\rm int}(\tau_N)\big]\,.\end{aligned}$$

Recalling now the relation (7.39)

$$|\alpha_\pm\rangle=\lim_{t\to\mp\infty}e^{iHt}e^{-iH_0t}|\alpha_0\rangle\equiv\lim_{t\to\mp\infty}U_I(0,t)|\alpha_0\rangle\,,\qquad(13.8)$$

and taking into account (13.2) and (13.5) we obtain the right hand side of (13.6). From the proof it is clear that the assumed ordering $x_1^0 \geq \ldots \geq x_r^0$ is by no means special, and the same steps can be performed for any other ordering of $x_1^0, \ldots, x_r^0$ with the same result.

The Green's functions (13.6) have been introduced above using the interaction picture formulation of quantum field theory which up to now was based on the assumptions of Section 7.3 which enforce using physically normalized field operators only (i.e. the On-Shell renormalization scheme). It should be clear, however, that these objects can be defined without any reference to this picture as they involve only matrix elements of Heisenberg picture operators $O_l^H(x)$ between true eigenstates of the full Hamiltonian. In fact, *vacuum Green's functions* of the form

$$iG^{(n)}_{l_1\ldots l_n}(x_1,\ldots,x_n) = \langle\Omega|\,\mathrm{T}\big[O^H_{l_1}(x_1)\ldots O^H_{l_n}(x_n)\big]\,|\Omega\rangle\,, \tag{13.9}$$

are the quantum field theory most basic objects: their definition requires only the existence of the ground state $|\Omega\rangle$, which any physically sensible system should possess, but makes otherwise no a priori assumptions about the spectrum of the Hamiltonian. In particular, neither the kind of particles in terms of which the *in* and *out* states are interpreted, nor even the mere possibility of interpretation in terms of particles of the Hamiltonian eigenvectors, is presupposed here. As a matter of facts, vacuum Green's functions (13.9) carry themselves the full information about the spectrum of the Hamiltonian and allow to determine also the $S$ matrix elements (if the Hamiltonian does posses particle-like *in* and *out* states).

Before proving in Sections 13.2 and 13.4 the assertions made above, we want to formulate the perturbative expansion for computing the vacuum Green's functions (13.9). This reduces to noticing, that while the derivation of the formula (13.6) with arbitrary *in* and *out* states $|\alpha_+\rangle$ and $\langle\beta_-|$ relies on the one-to-one correspondence (13.8), which in turn requires a special splitting of $H$ into $H_0$ and $V_{\rm int}$ (or in the language of fields, working with physically renormalized operators and physical mass parameters in $H_0$), for the vacuum Green's functions (13.9) the formula (13.6) remains formally valid even if the splitting of $H$ into $H_0$ and $V_{\rm int}$ is arbitrary (if one works with arbitrarily renormalized operators and arbitrary mass parameter in $H_0$) and even if the spectra of $H$ and $H_0$ are different, i.e. when the eigenvectors $|\alpha_0\rangle$ of $H_0$ constructed using the creation and annihilation operators in terms of which is expressed the operator $V^I_{\rm int}(t)$ are not in the one-to-one correspondence with the true *in* and *out* eigenstates of $H$ (this is true even if the $H$ spectrum is not interpretable in terms of particles; the only assumption needed is that the true ground state $|\Omega\rangle$ belongs to the discrete part of the $H$ spectrum). Justification of this statement is most straightforwardly obtained using the path integral formulation (Section 16.3) of the perturbative expansion for Green's functions (13.9) which turns out to be identical with the one obtained from the formula (13.6), but does not rely on the one-to-one correspondence of the $|\alpha_0\rangle$ and the *in* and *out* states. In the operator formalism developed up to this point the formula (13.6) with $|\Omega\rangle$ replacing the states $|\alpha_+\rangle$ and $|\beta_-\rangle$ has been demonstrated in Section 5.7. It directly relies of the Gell-Mann - Low construction given in Section 1.2 of the lowest energy eigenvector of $H$ and is therefore subject to the same restrictions: the splitting of the complete Hamiltonian into $H_0$ and the interaction must be such that the ground state $|\Omega\rangle$ of $H$ can be reached from the ground state $|\Omega_0\rangle$ of $H_0$ adiabatically. This in particular means (especially in cases in which systems exhibit spontaneous - parametrical or genuinely dynamical - breaking of some symmetries) that the classical theory of fields must be quantized using the appropriately chosen field variables (to realize the canonical commutation relations in the right Fock space); in the case of the approach of Chapters 7-9 performing an appropriate Bogolyubov-type transformation may be necessary before setting the perturbative expansion.

With this justification, the vacuum Green's functions (13.9) can be computed perturbatively using the formula (13.6) without imposing too strict restrictions (except for those explained above) on the actual splitting of $H$ into $H_0$ and $V_{\rm int}$ (because vacuum Green's functions do not have on-shell external lines the question of canceling corrections to them does not appear). This means that (in the language of fields) the theory can be quantized in almost *arbitrarily chosen* variables $\tilde{\varphi}$, $\tilde{\Pi}$ (see the discussion around the formula (11.433)). The interaction picture free field operators can be then introduced as in (11.438), so that they (in most cases) diagonalize $H_0$ (11.436) obtained for arbitrarily split bare (Lagrangian) mass parameters (masses of free particles which are represented by the $H_0$ eigenvectors are then not equal to the physical masses of particles the states of which are repressented by the *in* and *out* eigenvectors of $H$). To evaluate perturbatively the formula (13.9), the operators $O^H_l(x)$ in it, which can be arbitrarily renormalized canonical (interacting) field operators or even *composite operators* like $(\tilde{\varphi}^2)(x)$, $(\tilde{\varphi}\partial_\mu\tilde{\varphi})(x)$, $(\overline{\tilde{\psi}}\tilde{\psi})(x)$, etc. constructed out of arbitrarily renormalized elementary field operators and their derivatives, have of course to be expressed through the elementary field operators renormalized in the same way as the operators $\tilde{\varphi}$ in $H_0$ (11.436) which in turn, upon transition to the interaction picture and on account of the formula (13.6) for vacuum matrix elements, are expressed through the free field (interaction picture) operators. E.g. if $O^H_l(x)$ are built out of the bare canonical operators $\varphi_H$ and the perturbative expansion is set by replacing by $\varphi_I$ the "physically" renormalized operators $\varphi_{\rm ph}$, one has to write

$$O^H_l(x) \equiv O_l(\varphi_H(x), \partial) = O_l(Z^{1/2}\varphi_{\rm ph}(x), \partial) \to O_l(Z^{1/2}\varphi_I(x), \partial)\,, \qquad (13.10)$$

where $\varphi_I(x) = e^{iH_0t}\varphi_{\rm ph}(0, \mathbf{x})\, e^{-iH_0t}$. In Section 13.4 we will show how the more general Green's functions (13.6) and in particular $S$-matrix elements can be obtained from the appropriate vacuum Green's functions (13.9).

The practical prescription for computing the Green's function (13.9) within the perturbative expansion is then given by the Wick's theorem discussed in Section 5.9 which allows to reduce the right had sides of the expressions like

$$\langle\Omega_0|\mathrm{T}\left[\varphi^I_{i_1}(x_1)\dots\varphi^I_{i_k}(x_k)\exp\left(-i\int d^4y\,\mathcal{H}_{\rm int}(y)\right)\right]|\Omega_0\rangle$$
$$= \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int d^4y_1\dots\int d^4y_N\langle\Omega_0|\mathrm{T}\big[\varphi^I_{i_1}(x_1)\dots\varphi^I_{i_k}(x_k)\mathcal{H}_{\rm int}(y_1)\dots\mathcal{H}_{\rm int}(y_N)\big]|\Omega_0\rangle\,,$$

to unrestricted integrals of products of elementary contractions

$$\langle\varphi^I_{i_k}(x_k)\varphi^I_{i_l}(x_l)\rangle \equiv \langle\Omega_0|\mathrm{T}\big[\varphi^I_{i_k}(x_k)\varphi^I_{i_l}(x_l)\big]|\Omega_0\rangle\,, \tag{13.11}$$

which are just the appropriate free field propagators (9.32), (9.33), (9.60), (9.65) or (9.68) for different kinds of interaction picture operators. The resulting expansion can be represented in the form of Feynman diagrams and is equivalent to the previously formulated technique for calculating amplitudes by drawing all possible Feynman diagrams with a fixed number of external vertices corresponding to the operators $\varphi_{i_k}(x_k)$ and ascribing to each element of a diagram an appropriate analytical expression. The only difference with the rules formulated in Section 9 there are no integrations over the positions of vertices corresponding to the operators $\varphi_{i_k}(x_k)$ and that no coupling factors are associated with these vertices. If the operators $\varphi_{i_k}(x_k)$ are elementary one usually calls the lines ending in the corresponding vertices *external lines*.

Taking the Green's functions (13.9) for the basis of the quantum field theory is very important conceptually (it allows to get rid of the restrictive assumptions of Section 7) but is also convenient from the practical point of view. Firstly, within the perturbative expansion, using the Green's functions approach allows for a flexible formulation of the renormalization programme (see Chapter 14). One can then work with arbitrarily normalized field operators and with parameters which are not necessarily directly identified with observables (observables are then computed as functions of these parameters). This, in turn, enables one to formulate the powerful renormalization group methods (see Chapter 18). Secondly, the most effective methods for investigating the structure of quantum field theory models without assuming the validity of the perturbative expansion are provided by the path integral approach based on functional integrals (Chapter 16). These methods give most easily just the vacuum Green's functions (13.9). As in the path integral formulation of quantum field theory one deals with ordinary functions, rather than with operators acting in an unknown Hilbert space, lattice methods can be employed to compute vacuum Green's functions numerically and to extract information about the field theory structure nonperturbatively.[4]

## 13.2 Poles of the Green's functions

As already said, the full information about predictions of a given field theory can in principle be obtained from its vacuum Green's functions. We will now argue that if the field theory Hamiltonian does have a particle-like spectrum, Green's functions have simple

[4]In practice, the results which are obtained to date in the most interesting case of quantum chromodynamics with the help of numerical lattice computations are limited to static properties (like masses, charge distributions, etc.) of hadrons and to matrix elements between one particle states and the vacuum or other one particle states of some operators which are relevant for transitions induced by the weak interactions.

poles corresponding to these physical particles. The results of this section will be used in Section 13.4 to obtain the $S$-matrix.

Consider the connected[5] Green's function[6]

$$G_c^{(n)}(q_n, \dots, q_1) = \int d^4x_n \dots \int d^4x_1 \, e^{-iq_n \cdot x_n} \dots e^{-iq_1 \cdot x_1} \\ \times \langle \Omega | \mathrm{T}[O_n(x_n) \dots O_1(x_1)] | \Omega \rangle_{\mathrm{con}} \,, \tag{13.12}$$

in which $O_i(x_i)$ are some Heisenberg picture field operators (since in this section all operators will be taken in the Heisenberg picture, we suppress the subscript $H$ as well as possible indices related to their Lorentz transformation properties). As explained in the preceding section, $O_i(x)$ are not necessarily elementary fields operators of the theory - they may also be composite operators like $\bar{\psi}\psi$ or $\bar{\psi}_i A^a_\mu T^a_{ij} \psi_j$ in gauge theories, etc. We are interested in simple poles of $G_c^{(n)}(q_n, \dots, q_1)$ treated as a function of a sum $p$ of a subset of its four-momenta:

$$p \equiv (q_1 + \dots + q_r) = -(q_{r+1} + \dots + q_n) \,, \qquad 1 \leq r \leq n-1 \,. \tag{13.13}$$

We will argue that simple poles are located at $p^2 = m^2_{\mathrm{ph}}$ where $m^2_{\mathrm{ph}}$ is the mass squared (the eigenvalue of the $P_\mu P^\mu$ operator) of a physical one-particle state $|\mathbf{p}, \sigma\rangle$ (the full Hamiltonian $H$ eigenstate) which has a nonvanishing scalar products (in the Hilbert space) with the state-vectors

$$O^\dagger_{r+1}(x_{r+1}) \dots O^\dagger_n(x_n) |\Omega\rangle \,, \\ O_r(x_r) \dots O_1(x_1) |\Omega\rangle \,. \tag{13.14}$$

More precisely, we will show, that in the vicinity of such a pole $G_c^{(n)}(q_n, \dots, q_1)$ behaves as

$$G_c^{(n)}(q_n, \dots, q_1) \approx \frac{i}{p^2 - m^2_{\mathrm{ph}} + i0} (2\pi)^4 \delta^{(4)}(q_n + \dots + q_1) \\ \times \sum_\sigma \mathcal{A}(q_n, \dots, q_{r+1} | \mathbf{p}\sigma) \mathcal{A}(\mathbf{p}\sigma | q_r, \dots, q_1) \,, \tag{13.15}$$

[5]In the language of Feynman graphs, i.e. within the perturbative expansion, connected part of a Green's function is given by diagrams that do not contain disconnected pieces. More generally, the Fourier transform $G_c^{(n)}(q_n, \dots, q_1)$ of a connected Green's function is proportional to the single delta function expressing the overall conservation of the four-momentum. A connected $n$-point Green's function can be also defined recursively starting from the 1-point one, see Section 17.1, without any reference to diagrams.

[6]As the consideration of this and the next sections do not rely on the perturbative expansion, we will assume that the *in* and *out* vacuum states, as well as the *in* and *out* one-particle states, coincide. Hence we omit the subscripts $+$ and $-$. In the perturbative expansion these states differ by a phase factor given by disconnected vacuum graphs which are simply omitted.

where[7]

$$(2\pi)^4\delta^{(4)}(q_n+\ldots+q_{r+1}+p)\mathcal{A}(q_n,\ldots,q_{r+1}|\mathbf{p}\sigma) \equiv \int d^4x_n \ldots \int d^4x_{r+1}$$
$$e^{-iq_n\cdot x_n}\ldots e^{-iq_{r+1}\cdot x_{r+1}}\,\langle\Omega|\mathrm{T}[O_n(x_n)\ldots O_{r+1}(x_{r+1})]|\mathbf{p}\sigma\rangle_{\mathrm{con}}\,,$$
$$(13.16)$$
$$(2\pi)^4\delta^{(4)}(q_r+\ldots+q_1-p)\mathcal{A}(\mathbf{p}\sigma|q_r,\ldots,q_1) \equiv \int d^4x_r \ldots \int d^4x_1$$
$$e^{-iq_r\cdot x_r}\ldots e^{-iq_1\cdot x_1}\,\langle\mathbf{p}\sigma|\mathrm{T}[O_r(x_r)\ldots O_1(x_1)]|\Omega\rangle_{\mathrm{con}}\,.$$

In other words, in the vicinity of the pole $G^{(n)}(q_n,\ldots,q_1)$ can be written in the form

$$G_c^{(n)}(q_n,\ldots,q_1) \approx \int\frac{d^4k}{(2\pi)^4}\sum_\sigma (2\pi)^4\delta^{(4)}(q_n+\ldots+q_{r+1}+k)\mathcal{A}(q_n,\ldots,q_{r+1}|\mathbf{k}\sigma)$$
$$\times\frac{i}{k^2-m_{\mathrm{ph}}^2+i0}\,(2\pi)^4\delta^{(4)}(q_r+\ldots+q_1-k)\,\mathcal{A}(\mathbf{k}\sigma|q_r,\ldots,q_1)\,,$$

which, compared with the Feynman rules, looks (see Figure 13.2) as if there was a contribution of a propagator of a particle of mass $m_{\mathrm{ph}}$ connecting, as an internal line, two multi-leg vertices (with the appropriate momentum space delta functions). However, the point is that the particle associated with the pole may not correspond to any of the elementary fields out of which the Hamiltonian is built and need not have its counterpart in the spectrum of $H_0$. Note also, that even if the particle can be associated with an elementary field operator in the original Lagrangian, the mass $m_{\mathrm{ph}}$ of this particle determined by poles of Green's functions may not be identical with the bare mass parameter $m$ of this field in the original Lagrangian nor with the mass parameters in $H_0$.

---

[7]Alternative definition of $\mathcal{A}(q_n,\ldots,q_{r+1}|\mathbf{p}\sigma)$ reads

$$\mathcal{A}(q_n,\ldots,q_{r+1}|\mathbf{p}\sigma) = \int d^4y_{n-1}\ldots\int d^4y_{r+1}$$
$$e^{-iq_{n-1}\cdot y_{n-1}}\ldots e^{-iq_{r+1}\cdot y_{r+1}}\langle\Omega|\mathrm{T}[O_n(0)O_{n-1}(y_{n-1})\ldots O_{r+1}(y_{r+1})]|\mathbf{p}\sigma\rangle_{\mathrm{con}}\,.$$

To demonstrate the equivalence with the definition given in the text we introduce $y_{n-1} = x_{n-1} - x_n,\ldots,y_{r+1} = x_{r+1} - x_n$ and write the right hand side of (13.16) in the form

$$\int d^4x_n\int d^4y_{n-1}\ldots\int d^4y_{r+1}\,e^{-i(q_n+\ldots+q_{r+1}+p)\cdot x_n}$$
$$\times e^{-iq_{n-1}\cdot y_{n-1}}\ldots e^{-iq_{r+1}\cdot y_{r+1}}\,\langle\Omega|\mathrm{T}[O_n(0)O_{n-1}(y_{n-1})\ldots O_{r+1}(y_{r+1})]|\mathbf{p}\sigma\rangle_{\mathrm{con}}\,.$$

We have used here the fact that $O_i(x_i) = e^{i\hat{P}\cdot x_i}\,O_i(0)\,e^{-i\hat{P}\cdot x_i}$ and that

$$\mathrm{T}[O_n(x_n)\ldots O_{r+1}(x_{r+1})] = e^{i\hat{P}\cdot x_n}\mathrm{T}\,[O_n(0)\ldots O_{r+1}(y_{r+1})]e^{-i\hat{P}\cdot x_n}\,,$$

as can be seen by writing down $\mathrm{T}\,[O_n(x_n)O_{n-1}(x_{n-1})\ldots O_{r+1}(x_{r+1})]$ as the sum of different orderings of the operators multiplied by the appropriate $\theta$ functions. Taking the integral over $d^4x_n$ yields then the alternative definition. $\mathcal{A}(\mathbf{p}\sigma|q_r,\ldots,q_1)$ can be written similarly.

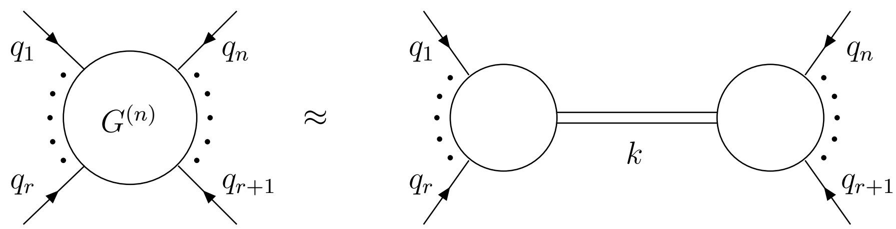



Figure 13.2: Factorization of a pole of a Green's function.

The proof is as follows. Among all orderings of $x_n^0, \dots, x_1^0$ contributing to the multiple integral in (13.12) there are $n!/r!(n-r)!$ orderings such that

$$x_n^0, \dots, x_{r+1}^0 > x_r^0, \dots, x_1^0 .$$

Assuming that in the complete set of the full Hamiltonian eigenstates there are one-particle states, the contribution of these orderings to $G_c^{(n)}(q_n, \dots, q_1)$ definded in (13.12) is

$$\int d^4x_n \dots \int d^4x_1 \, e^{-iq_n\cdot x_n} \dots e^{-iq_1\cdot x_1} \, \Theta\big(\min(x_n^0, \dots, x_{r+1}^0) - \max(x_r^0, \dots, x_1^0)\big)$$
$$\times \int d\Gamma_{\mathbf{k}} \sum_\sigma \langle\Omega|\mathrm{T}[O_n(x_n)\dots O_{r+1}(x_{r+1})]|\mathbf{k}\sigma\rangle_{\rm con}$$
$$\times \langle\mathbf{k}\sigma|\mathrm{T}[O_r(x_r)\dots O_1(x_1)]|\Omega\rangle_{\rm con} \; + \text{remainder} .$$

The "remainder" stands for other terms arising as a result of inserting the unit operator in the form

$$\hat{1} = |\Omega\rangle\langle\Omega| + \int d\Gamma_{\mathbf{k}} \sum_\sigma |\mathbf{k}\sigma\rangle\langle\mathbf{k}\sigma| + \dots$$

between the first $r$ and the last $n-r$ operators $O_k(x_k)$ as well as for terms arising from other orderings of $x_n^0, \dots, x_1^0$. One can now write the operators $O_n(x_n)$ and $O_r(x_r)$ using the relation

$$O_i(x_i) = e^{i\hat{P}\cdot x_i} \, O_i(0) \, e^{-i\hat{P}\cdot x_i} ,$$

which follow from the assumed translational invariance. This allows us to introduce the new variables

$$\begin{aligned}
&y_i = x_i - x_r \,, \qquad \text{if} \quad i = 1, \dots, r-1 \,, \\
&y_i = x_i - x_n \,, \qquad \text{and} \quad i = r+1, \dots, n-1 \,, \\
&\min(x_n^0, \dots, x_{r+1}^0) - \max(x_r^0, \dots, x_1^0) \\
&\qquad\qquad = x_n^0 - x_r^0 + \min(0, y_{n-1}^0, \dots, y_{r+1}^0) - \max(0, y_{r-1}^0, \dots, y_1^0)
\end{aligned}$$

and, representing the Heaviside step function $\Theta(\tau)$ as

$$\Theta(\tau) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega\ \frac{e^{-i\omega\tau}}{\omega + i0}\,, \tag{13.17}$$

write

$$\begin{aligned}
G_c^{(n)}(q_n,\dots,q_1) = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega+i0}\int d\Gamma_{\mathbf{k}}\sum_\sigma \\
\int d^4x_n\int d^4y_{n-1}\dots\int d^4y_{r+1}\int d^4x_r\int d^4y_{r-1}\dots\int d^4y_1 \\
e^{-i(q_n+\dots+q_{r+1})\cdot x_n}\, e^{-ik\cdot x_n}\, e^{-i(q_r+\dots+q_1)\cdot x_r}\, e^{+ik\cdot x_r} \\
\times\exp\left(-i\omega\left[x_n^0 - x_r^0 + \min(0,y_{n-1}^0,\dots,y_{r+1}^0) - \max(0,y_{r-1}^0,\dots,y_1^0)\right]\right) \\
\times e^{-iq_{n-1}\cdot y_{n-1}}\dots e^{-iq_{r+1}\cdot y_{r+1}}\,\langle\Omega|\mathrm{T}[O_n(0)O_{n-1}(y_{n-1})\dots O_{r+1}(y_{r+1})]|\mathbf{k}\sigma\rangle_{\rm con} \\
\times\langle\mathbf{k}\sigma|\mathrm{T}[O_r(0)O_{r-1}(y_{r-1})\dots O_1(y_1)]|\Omega\rangle_{\rm con}\, e^{-iq_{r-1}\cdot y_{r-1}}\dots e^{-iq_1\cdot y_1} \\
+\,\text{remainder},
\end{aligned}$$

where the factors $e^{-ik\cdot x_n}$, $e^{+ik\cdot x_r}$ in the third line arose from the action of $e^{-i\hat{P}\cdot x_n}$ and $e^{i\hat{P}\cdot x_r}$ on $|\mathbf{k}\sigma\rangle$ and $\langle\mathbf{k}\sigma|$, respectively (see the footnote, two pages back). Performing the integrals over $x_n$ and $x_r$ one finds

$$\begin{aligned}
G_c^{(n)}(q_n,\dots,q_1) = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega+i0}\int d\Gamma_{\mathbf{k}}\sum_\sigma \mathcal{A}(q_n,\dots,q_{r+1}|\mathbf{k}\sigma)\,\mathcal{A}(\mathbf{k}\sigma|q_r,\dots,q_1) \\
\times(2\pi)^4\delta^{(3)}(\mathbf{q}_n+\dots+\mathbf{q}_{r+1}+\mathbf{k})\,\delta(q_n^0+\dots+q_{r+1}^0+E_{\mathbf{k}}+\omega) \\
\times(2\pi)^4\delta^{(3)}(\mathbf{q}_r+\dots+\mathbf{q}_1-\mathbf{k})\,\delta(q_r^0+\dots+q_1^0-E_{\mathbf{k}}-\omega) \\
\times\exp\left(-i\omega\left[\min(0,y_{n-1}^0,\dots,y_{r+1}^0)-\max(0,y_{r-1}^0,\dots,y_1^0)\right]\right) \\
+\text{remainder},
\end{aligned}$$

where $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m_{\rm ph}^2}$. Neither the "remainder" nor the exponent contribute at the pole.[8] The exponent can be therefore approximated by 1. The integrals over $\mathbf{k}$ and $\omega$ can be then done with the result

$$\begin{aligned}
G_c^{(n)}(q_n,\dots,q_1) = \frac{i}{2\sqrt{\mathbf{p}^2+m_{\rm ph}^2}\left(p^0-\sqrt{\mathbf{p}^2+m_{\rm ph}^2}+i0\right)} \\
\times(2\pi)^4\delta^{(4)}(q_n+\dots+q_1)\sum_\sigma\mathcal{A}(q_n,\dots,q_{r+1}|\mathbf{p}\sigma)\mathcal{A}(\mathbf{p}\sigma|q_r,\dots,q_1) \\
+\text{ the remainder.}
\end{aligned} \tag{13.18}$$

where $\mathbf{p} = \mathbf{q}_r+\dots+\mathbf{q}_1$ and $p^0 = q_r^0+\dots+q_1^0$. Since for $p^0\approx\sqrt{\mathbf{p}^2+m_{\rm ph}^2}$ one has $2\sqrt{\mathbf{p}^2+m_{\rm ph}^2}\approx p^0+\sqrt{\mathbf{p}^2+m_{\rm ph}^2}$ this is just the result (13.15). By appealing to the

[8]If $\exp(-i\omega[\dots])$ is expanded as $1-i\omega[\dots]+\dots$ only the first term will remain singular.

Feynman rules it can also be expected that in (13.18) each of the factors $\mathcal{A}$ in the product $\mathcal{A}(q_n, \dots, q_{r+1}|\mathbf{p}\sigma)\mathcal{A}(\mathbf{p}\sigma|q_r, \dots, q_1)$ contains the appropriate "wave" function $u(\mathbf{p}, \sigma)$ and $u^*(\mathbf{p}, \sigma)$ of the on-shell particle (or the functions $v^*(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ if in the formalism this particle is treated as an antiparticle), which in the perturbative expansion come from the free field operators (8.2). These, when summed over the spin states $\sigma$ for $p^2 = m^2_{\rm ph}$, should reproduce the numerator of the Feynman propagator.

## 13.3 Two-point functions: the Källen-Lehman representation

Let us now consider the connected two-point Green's function[9]

$$G^{(2)}_{l_1 l_2}(p_1, p_2) = \int d^4x_1 d^4x_2 \, e^{+ip_1\cdot x_1} \, e^{-ip_2\cdot x_2} \, \langle\Omega|\mathrm{T}[O_{l_1}(x_1) O^\dagger_{l_2}(x_2)]|\Omega\rangle_{\rm con} \,. \tag{13.19}$$

of an operator $O_l(x)$ and its Hermitian conjugate ($l$ is the index which under changes of the inertial frame transforms as some regular representation of the Lorentz group). According to the analysis of the preceding section, for $p_1^2 \approx m^2_{\rm ph}$ where $m_{\rm ph}$ is the mass of a physical particle $G^{(2)c}_{l_1 l_2}(p_1, p_2)$ takes the form

$$G^{(2)}_{l_1 l_2}(p_1, p_2) \approx (2\pi)^4 \delta^{(4)}(p_1 - p_2) \sum_\sigma \langle\Omega|O_{l_1}(0)|\mathbf{p}_1\sigma\rangle \, \frac{i}{p_1^2 - m^2_{\rm ph} + i0} \langle\mathbf{p}_1\sigma|O^\dagger_{l_2}(0)|\Omega\rangle \,. \tag{13.20}$$

Poincaré covariance implies that

$$\langle\Omega|O_l(x)|\mathbf{p}_1\sigma\rangle = \mathcal{Z}_O^{1/2} \, u_l(\mathbf{p}_1, \sigma) \, e^{-ip_1\cdot x} \,, \tag{13.21}$$

with the functions $u_l(\mathbf{p}_1, \sigma)$ appropriate for this particle (or $v^*_l(\mathbf{p}_1, \sigma)$ if the particle is taken for an antiparticle) and some factor $\mathcal{Z}_O^{1/2}$ specific for the operator $O_l(x)$ and depending on the dynamics generated by the Hamiltonian $H$. It will be called the residue factor associated with the operator $O_l(x)$. Of course, if the particle corresponding to the pole has its canonical field and $O_l(x)$ is just the corresponding physically normalized (in the sense specified in Section 11.10) field operator $\phi_{i,\rm ph}$ of this particle, then $\mathcal{Z}_O^{1/2} = 1$ (possibly up to an irrelevant phase factor). It follows that in the vicinity of the pole the two-point function has the form

$$\begin{aligned} G^{(2)c}_{l_1 l_2}(p_1, p_2) &\equiv (2\pi)^4 \delta^{(4)}(p_1 - p_2) \, \tilde{G}^{(2)c}_{l_1 l_2}(p_1) \\ &\approx (2\pi)^4 \delta^{(4)}(p_1 - p_2) \, \frac{i \, \mathcal{Z}_O}{p_1^2 - m^2_{\rm ph} + i0} \sum_\sigma u_{l_1}(\mathbf{p}_1, \sigma) u^*_{l_2}(\mathbf{p}_1, \sigma) \,, \end{aligned} \tag{13.22}$$

(from now on we denote with a tilde Green's functions with the overall delta function removed) i.e. it has the form of the free field propagator (notice that the sum over spin

[9] As in the preceding section we suppress the subscript $H$ on Heisenberg picture operators $O_l$.

projections $\sigma$ of the particle "wave functions" $u_{l_1}(\mathbf{p}_1,\sigma)u^*_{l_2}(\mathbf{p}_1,\sigma)$, or $v^*_{l_1}(\mathbf{p}_1,\sigma)v_{l_2}(\mathbf{p}_1,\sigma)$ for an antiparticle, reproduces the numerator of the propagator) with an extra factor $\mathcal{Z}_O$.

On the other hand, for a two-point Green's function of an operator $O_l$ and its Hermitian conjugate (which may be identical to $O_l$ itself) it is possible to derive an exact expression. We discuss it taking as $O_l$ the canonical (bare, i.e. the one corresponding to the canonically normalized kinetic term in the Lagrangian) field operator $\phi_H(x)$ transforming as a Lorentz scalar. Consider first the matrix element

$$\langle\Omega|\phi_H(x)\phi^\dagger_H(y)|\Omega\rangle = \int d\gamma\, e^{-ip_\gamma\cdot(x-y)}\, |\langle\Omega|\phi_H(0)|\gamma\rangle|^2\,, \tag{13.23}$$

where we have used $\phi_H(x) = e^{i\hat{P}\cdot x}\phi_H(0)e^{-i\hat{P}\cdot x}$, and $\hat{P}^\mu|\gamma\rangle = p^\mu_\gamma|\gamma\rangle$ (if the Hamiltonian spectrum is particle-like, the states $|\gamma\rangle$ can be the *in* or the *out* basis of states). We now introduce the spectral density $\rho(p^2)$

$$\theta(p^0)\rho(p^2) = \int d\gamma\, \delta^{(4)}(p_\gamma - p)\,(2\pi)^3\,|\langle\Omega|\phi_H(0)|\gamma\rangle|^2\,, \tag{13.24}$$

which if the operator $O(x)$ is a Lorentz scalar as in (13.24), can, by Poincaré covariance, depend only on $p^2$ and vanishes for $p^2 < 0$ and $p^0 < 0$ (we assume that the Hamiltonian $H$ has a ground state corresponding to $p^0_\Omega \geq 0$, so that all other $H$ eigenstates $|\gamma\rangle$ have $p^2_\gamma \geq 0$ and $p^0_\gamma \geq 0$). With the help of the spectral density the matrix element (13.23) can be rewritten as

$$\begin{aligned}\langle\Omega|\phi_H(x)\phi^\dagger_H(y)|\Omega\rangle &= \int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot(x-y)}\,\theta(p^0)\,\rho(p^2)\\ &= \int_0^\infty d\mu^2\,\rho(\mu^2)\int\frac{d^4p}{(2\pi)^4}\,2\pi\,\delta(p^2-\mu^2)\,\theta(p^0)\,e^{-ip\cdot(x-y)}\\ &= \int_0^\infty d\mu^2\,\rho(\mu^2)\,\Delta_+(x-y;\mu^2)\,,\end{aligned} \tag{13.25}$$

where $\Delta_+(x-y;\mu^2)$ is the function defined in Section 8.2 except that now also its dependence on the mass parameter $\mu^2$ is explicitly indicated. Similarly,

$$\langle\Omega|\phi^\dagger_H(y)\phi_H(x)|\Omega\rangle = \int_0^\infty d\mu^2\,\tilde{\rho}(\mu^2)\,\Delta_+(y-x;\mu^2)\,,$$

with an, a priori, different spectral function $\tilde{\rho}(\mu^2)$ determined by $|\langle\Omega|\phi^\dagger_H(0)|\gamma\rangle|^2$. However, local causality requires[10] that the commutator $[\phi^\dagger_H(y),\ \phi_H(x)]$ vanishes when $(x-y)^2 < 0$. This is ensured if $\tilde{\rho}(\mu^2) = \rho(\mu^2)$ (recall that $\Delta_+(y-x;\mu^2) = \Delta_+(x-y;\mu^2)$ for $(x-y)^2 < 0$). Similar representation can be written down also for vacuum matrix element

[10]In Chapter 8 this condition was imposed on the free field (interaction picture) operators. It should, however, be satisfied by all local Heisenberg picture operators.

$\langle\Omega|O_{l_1}(x)O^\dagger_{l_2}(y)|\Omega\rangle$ of arbitrary operators $O_l(x)$. It then follows that the connected two-point function (13.19) has the *Källen-Lehman spectral representation*:

$$G^{(2)}_c(x-y) = \int_0^\infty d\mu^2\,\rho(\mu^2)\,i\Delta^F(x-y;\mu^2)\,, \tag{13.26}$$

or, in the momentum space,

$$\tilde{G}^{(2)}_c(p^2) = \int_0^\infty d\mu^2\,\rho(\mu^2)\,\frac{i}{p^2-\mu^2+i0}\,, \tag{13.27}$$

which follows from adding the two matrix elements considered above with the appropriate theta functions according to the definition of the chronological product of two operators. From this representation it follows that the two-point function, i.e. the full propagator, cannot vanish for $p^2\to\infty$ faster than $1/p^2$. On this basis one can argue that Lagrangians containing terms quadratic in fields and more than two derivatives must be interpreted as effective Lagrangians, arising from integrating out some high energy degrees of freedom (heavy particles, strings or anything else), valid only for processes with particle energies not exceeding a certain cut-off scale. Otherwise the asymptotic vanishing of a propagator faster than $1/p^2$ would conflict with basic principles of quantum mechanics.

The spectral representation allows to constrain the $\mathcal{Z}$ factor of the canonical (bare) Heisenberg field operator $\phi_H$ the canonical momentum $\Pi_H(x)$ of which following from the first of the equations (11.125) is $\Pi_H(x)=\dot{\phi}^\dagger_H(x)$. Differentiating the equality

$$\langle\Omega|[\phi_H(x),\ \phi^\dagger_H(y)]\Omega\rangle = \int_0^\infty d\mu^2\,\rho(\mu^2)\left[\Delta_+(x-y;\mu^2)-\Delta_+(y-x;\mu^2)\right],$$

with respect to $y^0$ and setting $x^0=y^0$ one gets

$$\langle\Omega|[\phi_H(\mathbf{x},t),\ \dot{\phi}^\dagger_H(\mathbf{y},t)]|\Omega\rangle = \int_0^\infty d\mu^2\,\rho(\mu^2)\,i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$

upon using the relation

$$\partial_{x^0}\Delta_+(x;\mu^2)\big|_{x^0=0} = -\frac{i}{2}\,\delta^{(3)}(\mathbf{x})\,.$$

Since $[\phi_H(t,\mathbf{x}),\ \Pi_H(t,\mathbf{y})] = [\phi_H(t,\mathbf{x}),\ \dot{\phi}^\dagger_H(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$ (cf. (11.124)), it follows that

$$\int_0^\infty d\mu^2\,\rho(\mu^2) = 1\,. \tag{13.28}$$

Furthermore, if there is a one-particle state $|\mathbf{k}\rangle$ such that $\langle\Omega|\phi_H(0)|\mathbf{k}\rangle=\mathcal{Z}^{1/2}$, it produces a simple pole in the Fourier transform of $G^{(2)}_c(x-y)$. Indeed, comparing (13.27) with (13.20) we see that the contribution of such a one-particle state to (13.24) must give

$$\rho(\mu^2) = \mathcal{Z}\,\delta(\mu^2-m^2_{\rm ph})+\sigma(\mu^2)\,, \tag{13.29}$$

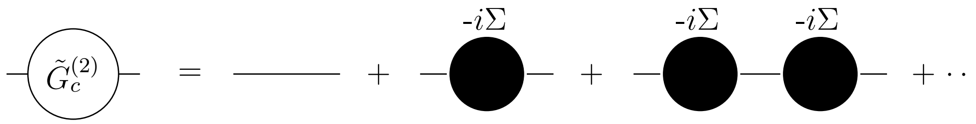



Figure 13.3: Graphical representation of the connected two-point function $\tilde{G}_c^{(2)}(p)$ in terms of one particle irreducible (1PI) self energy insertions $-i\Sigma(p)$ introduced in Section 9.7.

so that[11]

$$1 = \mathcal{Z} + \int_0^\infty d\mu^2\, \sigma(\mu^2)\ . \tag{13.30}$$

Since $\sigma(\mu^2) \geq 0$, it follows that $0 < \mathcal{Z} < 1$. Let us also remark that if the spectrum of the Hamiltonian $H$ is not particle-like, the spectral functions of operators of such a theory do not contain isolated delta-like contributions of the form (13.29). Similar exact results can also be derived for spectral functions of other canonical (bare) operators transforming as more complicated representations under changes of the Lorentz frame.

In the perturbative expansion full connected two-point Green's functions $\tilde{G}_c^{(2)}(p)$ of canonical or renormalized canonical field operators, represented in Figure 13.3 by the open circle, are given by infinite sums of Feynman diagrams which can be organized in subsets (also consisting of infinitely many graphs) corresponding to one-particle irreducible (1PI) self-energy insertions $-i\Sigma(p)$ (1PI two-point Green's functions with removed external simple propagators). These are represented in Figure 13.3 by the black blobs, connected by simple (i.e. free field) propagators. By definition the 1PI Green's functions are the sums of Feynman graphs which cannot be decomposed into two disconnected parts by cutting just one line of the graph. By summing the resulting geometric series (see (13.52) for an example) such a two-point Green's function can be brought into the form $[\tilde{G}_c^{(2)}(p)]^{-1} = [\tilde{G}_{\rm tree}^{(2)}(p)]^{-1} + i\Sigma(p)$ (this will be shown more rigorously in in Chapter 17). For example, the two-point Green's function of a scalar field operators can be written in the form

$$\tilde{G}_c^{(2)}(p) = \frac{i}{p^2 - m^2 - \Sigma(p^2, m)}\,,$$

[11]Considering the two-point Green's function of arbitrarily renormalized elementary field operators $\tilde{\phi}_H = Z^{-1/2}\phi_H$ one would work with the spectral function $\tilde{\rho}(\mu^2)$ satisfying (in agreement with the canonical commutation rule $[\tilde{\phi}_H,\ \tilde{\Pi}_H] = Z[\tilde{\phi}_H,\ \dot{\tilde{\phi}}^\dagger{}_H] = i\delta$) the relation

$$Z^{-1} = \tilde{\mathcal{Z}} + \int_0^\infty d\mu^2\, \tilde{\sigma}(\mu^2)\,,$$

with $\tilde{\mathcal{Z}} = Z^{-1}\mathcal{Z}$ being the residuum at $p^2 = m^2_{\rm ph}$ of the two-point Green's function of the $\tilde{\phi}_H$ field operators and $\tilde{\sigma} = Z^{-1}\sigma$. Of course, if $\tilde{\phi}_H$ is renormalized in the physical way (see Section 11.10) then $\tilde{\mathcal{Z}} = 1$, which means that the factor $\mathcal{Z}$ of the canonical field equals the $Z$ renormalization factor of the physically renormalized operator.

where $m$ is the mass parameter in the free Hamiltonian $H_0$.

From this representation of the two-point Green's functions in terms of the 1PI self-energy insertions $\Sigma(p)$ it should be clear that if the interaction is adjusted (either by adding appropriate interactions like (9.103) - in the approach based on quantum mechanics of relativistic particles - or by working with physically normalized canonical field operator $\tilde{\varphi}_H(x) = \varphi_{\rm ph}(x)$ and appropriately splitting $m^2$ into $m^2_{\rm ph} + \delta m^2$ - in the approach based on classical field quantization) so that[12] that the first two terms of the Taylor expansion of $\Sigma(p)$ around the value $p^2 = m^2_{\rm ph} = m^2$ (where $m^2$ enters $H_0$) vanish (as was required in Section 9.7) the complete two-point connected function $\tilde{G}^{(2)}_c(p)$ has, for $p^2 \approx m^2_{\rm ph} = m^2$, the form of the free field propagator with $\mathcal{Z} = 1$ which ensures that the relation between the free particle states $|\alpha_0\rangle$ and the *in* and *out* states is given by (7.39), i.e. the rescaled (renormalized) elementary field operator $\tilde{\varphi}_H$ is the "physical" one, i.e. it is just the one denoted $\varphi_{\rm ph}$ in Section 11.10. But as we have argued, working with physically renormalized operators $\varphi_{\rm ph}$ is not mandatory: perturbative expansion of vacuum Green's function is valid with arbitrary operators (and almost arbitrary splitting of the total Hamiltonian between $H_0$ and $V_{\rm int}$, subject only to the conditions under which the Gell-Mann - Low construction is valid) and $S$-matrix elements can be extracted from vacuum Green's functions of arbitrary operators with the help of the LSZ prescription which we formulate in the next section.

Two-point vacuum Green's functions (13.19) of other local composite operators $O_l(x)$ and the corresponding factors $\mathcal{Z}_O$ can be defined analogously as those of elementary operators. The factors $\mathcal{Z}_O$ can be also computed using the perturbation expansion. The only difference compared to the two-point Green's functions of (renormalized) elementary field operators is that the factors $\mathcal{Z}_O$ of composite operators differ usually from unity already in the lowest (zero-th in the coupling constants) order. It may also happen that the two-point Green's function of a given operator $O_l(x)$ does not have a one particle pole, that is, $\mathcal{Z}_O = 0$, (at least in the perturbation expansion).

## 13.4 The LSZ reduction

We now show how $S$-matrix elements can be extracted from vacuum Green's functions (13.9). The prescription in principle does not require introducing free particle states $|\alpha_0\rangle$ nor the free field operators. Moreover, Green's functions of arbitrary operators, elementary or composite, can be used for this purpose equally well.

The basic tool is the Lehman-Symanzik-Zimmermann (LSZ) asymptotic condition which in the usual formulation states that if a local operator $O_l(x)$ has the matrix element between the vacuum and a one-particle state $|\mathbf{p}, \sigma\rangle$ given by (13.21), then in the limit

[12] This is of course possible only if $\tilde{G}^{(2)}_c(p)$ has a simple pole at a real value of $p^2$.

$x^0 \to -\infty(+\infty)$

$$O_l(x) \longrightarrow \mathcal{Z}_O^{1/2}\,\phi_l^{\rm in(out)}(x)\,, \tag{13.31}$$

where $\phi_l^{\rm in(out)}(x)$ is the *in* (*out*) field operator transforming under changes of inertial frames in the same way as does $O_l(x)$ and constructed out of the *in* or *out* creation and annihilation operators of the particle represented by the $H$ eigenvector $|\mathbf{p},\sigma\rangle$ (as discussed in Section 8.7).

The limit (13.31) cannot be a strong operator limit. For example, if $O_l(x)$ is the canonical Heisenberg picture field operator $\varphi_H(x)$ so that $\Pi_H(x) = \dot{\varphi}_H(x)$, then at any instant $[\varphi_H(t,\mathbf{x}),\ \dot{\varphi}_H(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$, which, if (13.31) with $\mathcal{Z}\neq 1$ were treated as a strong operator limit, would conflict with the relation $[\varphi^{\rm in,out}(t,\mathbf{x}),\ \dot{\varphi}^{\rm in,out}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$ trivially satisfied by the opertators $\varphi^{\rm in,out}$. The limit (13.31) holds only between normalizable *in* (*out*) states $\int d\alpha\, g(\alpha)|\alpha_\pm\rangle$ and for operators $O_l(x)$ appropriately smoothed and localized in space with normalizable functions of $\mathbf{x}$. It expresses the crucial property of a *local* quantum field theory that if an operator $O_l(t) = \int d^3\mathbf{x}\, f(\mathbf{x})\, O_l(t,\mathbf{x})$, localized in space with the help of a normalizable function $f(\mathbf{x})$, acts on a state representing particles which (in the limit $t\to-\infty$ or $t\to\infty$) are also localized and well separated in space,[13] it annihilates the particle corresponding to it only if the latter is localized within the operator's support (or creates the corresponding antiparticle localized within the operator's support) and in the limit $x^0\to-\infty(+\infty)$ it does it just as if there were no other particles (because other particles are then spatially well apart). Hence, under these conditions a matrix element of such an operator between *in* (*out*) states representing localized particles factorizes in the limit $x^0\to-\infty(+\infty)$ into its matrix element between the vacuum and the (localized) one-particle state times the scalar product of the "remainder" (see (13.34) for an example). Taking into account (13.21) it follows that the action of such an operator $O_l$ on localized states is, up to the $\mathcal{Z}_O^{1/2}$ factor, the same as the action of the (localized) *in* or *out* field operators (with the same Lorentz transformation properties as $O_l$) which create and destroy individual particles independently of the presence of other particles.

Consider now an $S$-matrix element corresponding to a transition from an *in* state of $r$ particles $|(\mathbf{p}_1\sigma_1,\dots,\mathbf{p}_r\sigma_r)_+\rangle$ to an *out* state of $n-r$ particles $|(\mathbf{p}'_{r+1}\sigma'_{r+1},\dots,\mathbf{p}'_n\sigma'_n)_-\rangle$ and a connected Green's function $G_c^{(n)}(q_n,\dots,q_1)$ (13.12) with the operators $O_i$ chosen so that to each particle in the *in* state there corresponds one operator having nonvanishing matrix element $\langle\mathbf{p}_k,\sigma_k|O_k|\Omega\rangle$ and to each particle in the *out* state corresponds one operator $O_j$ such that $\langle\Omega|O_j|\mathbf{p}'_j,\sigma'_j\rangle\neq 0$. Applying to such a Green's function the theorem (13.15) with $r=1$, $p\equiv p_1=q_1$ (assuming without loss of generality that it is the operator $O_1$ that corresponds to the first particle of momentum $\mathbf{p}_1$ in the *in* state) we obtain for $p_1^2\approx m^2_{\rm ph,1}$

$$G^{(n)}(q_n,\cdots,q_1) \approx (2\pi)^4\delta^{(4)}(q_n+\ldots+q_2+p_1)$$

[13]The assumption that the particles can be separated in space is in fact equivalent to the assumption that the considered Hamiltonian $H$ possesses the spectrum interpretable in terms of particles that is, that its *in* and *out* eigenvectors transform under changes of inertial frames as collections of free particles.

$$\times \sum_{\sigma_1} \mathcal{A}(q_n, \cdots, q_2|\mathbf{p}_1\sigma_1) \times \frac{i}{p_1^2 - m_{\text{ph},1}^2 + i0} \langle \mathbf{p}_1\sigma_1|O_1(0)|\Omega\rangle . \quad (13.32)$$

It is straightforward to apply the theorem (13.15) once again to the factor $\mathcal{A}(q_n, \cdots, q_2|\mathbf{p}_1\sigma_1)$ given by (13.16), this time assuming that the operator $O_n$ corresponds to the particle of momentum $\mathbf{p}'_n$ in the *out* state. For $q_n = -p'_n$, with $p'^2_n \approx m^2_{\text{ph},n}$ this gives

$$\begin{aligned} G^{(n)}(q_n, \cdots, q_1) \approx (2\pi)^4 \delta^{(4)}(-p'_n + q_{n-1} + \ldots + q_2 + p_1) \\ \times \sum_{\sigma'_n} \sum_{\sigma_1} \frac{i}{p'^2_n - m^2_{\text{ph},n} + i0} \mathcal{A}(\mathbf{p}'_n\sigma'_n|q_{n-1}, \cdots, q_2|\mathbf{p}_1\sigma_1) \frac{i}{p_1^2 - m^2_{\text{ph},1} + i0} \\ \times \langle \Omega|O_n(0)|\mathbf{p}'_n\sigma'_n\rangle \times \langle \mathbf{p}_1\sigma_1|O_1(0)|\Omega\rangle , \qquad (13.33) \end{aligned}$$

where

$$\begin{aligned} (2\pi)^4 \delta^{(4)}(-p'_n + q_{n-1} + \ldots + q_2 + p_1) \mathcal{A}(\mathbf{p}'_n\sigma'_n|q_{n-1}, \ldots, q_2|\mathbf{p}_1\sigma_1) \\ = \int d^4x_{n-1} \ldots \int d^4x_2 \, e^{-iq_{n-1}x_{n-1}} \ldots e^{-iq_2x_2} \\ \times \langle \mathbf{p}'_n\sigma'_n|T[O_{n-1}(x_{n-1}) \ldots O_2(x_2)]|\mathbf{p}_1\sigma_1\rangle . \end{aligned}$$

Suppose now it is the operator $O_2$ that corresponds to a particle of momentum $\mathbf{p}_2$ in the *in* state. We consider the contribution to the multiple integral above of the ordering $x^0_{n-1}, \ldots, x^0_3 > x^0_2$ and insert the complete set of the *in* state-vectors between $O_2$ and the other operators. This gives (among others) a term[14]

$$\begin{aligned} \int d^4x_{n-1} \ldots \int d^4x_2 \, e^{-iq_{n-1}x_{n-1}} \ldots e^{-iq_2x_2} \, \Theta\left(\min(x^0_{n-1}, \ldots, x^0_3) - x^0_2\right) \\ \int d\Gamma_{\mathbf{k}_1} \int d\Gamma_{\mathbf{k}_2} \sum_{\tilde\sigma_1, \tilde\sigma_2} \langle \mathbf{p}'_n\sigma'_n|\mathrm{T}[O_{n-1}(x_{n-1}) \ldots O_3(x_3)]|(\mathbf{k}_1\tilde\sigma_1, \mathbf{k}_2\tilde\sigma_2)_+\rangle \\ \times \langle (\mathbf{k}_1\tilde\sigma_1, \mathbf{k}_2\tilde\sigma_2)_+|O_2(x_2)|(\mathbf{p}_1\sigma_1)_+\rangle + \text{remainder.} \end{aligned}$$

After changing the integration variables to $y_{n-1} = x_{n-1} - x_3, \ldots, y_4 = x_4 - x_3$, representing the theta function by the integral (13.17) and performing the integral over $x_3$ this is

$$\begin{aligned} \int d^4y_{n-1} \ldots \int d^4y_4 \, \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega + i0} \, e^{-i\omega \min(y^0_{n-1}, \ldots, y^0_4, 0)} \, e^{-iq_{n-1}y_{n-1}} \ldots e^{-iq_4y_4} \\ \int d\Gamma_{\mathbf{k}_1} \int d\Gamma_{\mathbf{k}_2} \sum_{\tilde\sigma_1, \tilde\sigma_2} \langle \mathbf{p}'_n\sigma'_n|\mathrm{T}[O_{n-1}(y_{n-1}) \ldots O_3(0)]|(\mathbf{k}_1\tilde\sigma_1, \mathbf{k}_2\tilde\sigma_2)_+\rangle \\ \times (2\pi)^4 \delta^{(3)}(-\mathbf{p}'_n + \mathbf{q}_{n-1} + \ldots + \mathbf{q}_3 + \mathbf{k}_2 + \mathbf{k}_1) \\ \times \delta(-E_{\mathbf{p}'_n} + q^0_{n-1} + \ldots + q^0_3 + E_{\mathbf{k}_2} + E_{\mathbf{k}_1} + \omega) \\ \times \int d^4x_2 \, e^{i\omega x^0_2} \langle (\mathbf{k}_1\tilde\sigma_1, \mathbf{k}_2\tilde\sigma_2)_+|O_2(x_2)|\mathbf{p}_1\sigma_1\rangle \, e^{-iq_2x_2} + \text{remainder.} \end{aligned}$$

[14]If the particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are identical the integrals over $d\Gamma_{\mathbf{k}_1} d\Gamma_{\mathbf{k}_2}$ have to be divided by the factor 2! - see (7.4) but there will be then also two terms in the formula (13.34).

If $\mathcal{A}(\mathbf{p}'_n\sigma'_n|q_{n-1},\ldots,q_2|\mathbf{p}_1\sigma_1)$ has for $q_2\approx p_2$ with $p_2^2=m^2_{\rm ph,2}$ a one-particle pole, this pole must be due to the $x_2^0\to-\infty$ lower limit of the integral over $dx_2^0$ (this integral is obviously cut-off from the other side by the theta function). Therefore, near the pole, the matrix element of $O_2(x_2)$ factorizes on account of the LSZ asymptotic condition (13.31):

$$\langle(\mathbf{k}_1\tilde{\sigma}_1,\mathbf{k}_2\tilde{\sigma}_2)_+|O_2(x_2)|(\mathbf{p}_1\sigma_1)_+\rangle\to \langle(\mathbf{k}_1\tilde{\sigma}_1)_+|(\mathbf{p}_1\sigma_1)_+\rangle\times\langle(\mathbf{k}_2\tilde{\sigma}_2)_+|O_2(x_2)|\Omega\rangle\,. \tag{13.34}$$

Incidentally, the same factorization has the effect that the other terms resulting from inserting the complete set of *in* states between $O_2(x_2)$ and the remaining operators vanish at the pole just by the orthogonality of $|(\mathbf{p}_1\sigma_1)_+\rangle$ with respect to all other *in* states with the number of particles different then one.

Representing then $O_2(x_2)$ in the form $e^{i\hat{P}\cdot x_2}O_2(0)e^{-i\hat{P}\cdot x_2}$ allows to perform the integration over $dx_2$. This produces a pole for $k_2^2\approx m^2_{\rm ph,2}$ in the same way as in (13.18) so that one gets

$$\begin{aligned}G^{(n)}(q_n,\cdots,q_1)\approx(2\pi)^4\delta^{(4)}(-p'_n+q_{n-1}+\ldots+q_3+p_2+p_1)\sum_{\sigma'_n}\sum_{\sigma_2\sigma_1}\\ \frac{i}{p'^2_n-m^2_{\rm ph,n}}\,\mathcal{A}(\mathbf{p}'_n\sigma'_n|q_{n-1},\cdots,q_3|\mathbf{p}_2\sigma_2,\mathbf{p}_1\sigma_1)\,\frac{i}{p_2^2-m^2_{\rm ph,2}}\,\frac{i}{p_1^2-m^2_{\rm ph,1}}\\ \times\langle\Omega|O_n(0)|\mathbf{p}'_n\sigma'_n\rangle\,\times\,\langle\mathbf{p}_2\sigma_2|O_2(0)|\Omega\rangle\times\,\langle\mathbf{p}_1\sigma_1|O_1(0)|\Omega\rangle\,,\end{aligned} \tag{13.35}$$

with $\mathcal{A}(\mathbf{p}'_n\sigma'_n|q_{n-1},\cdots,q_3|\mathbf{p}_2\sigma_2,\mathbf{p}_1\sigma_1)$ being the matrix element of the T-product of the remaining operators between the *in* state $|(\mathbf{p}_1\sigma_1,\mathbf{p}_2\sigma_2)_+\rangle$ and the *out* state $\langle\mathbf{p}'_n\sigma'_n|$. Similarly, "reducing" operators corresponding to particles in the *out* state we insert the complete sets of *out* states because the poles arise then from the limit $x_j^0\to+\infty$ $(r<j\leq n)$ and in this limit the condition (13.31) allows to find the matrix elements of the operators $O_j(x_j)$ between the *out* states. At the last step of the procedure one is left with the matrix element

$$\int d^4x\,e^{-iq\cdot x}\,\langle(\mathbf{p}'_n\sigma'_n,\ldots)_-|O(x)|(\mathbf{p}_1\sigma_1,\ldots)_+\rangle\,.$$

To exhibit the pole, the integral over $dx^0$ has to be split by inserting $1=\Theta(x^0)+\Theta(-x^0)$. If the operator $O$ is to be associated with a particle in the *in* state it is $\Theta(-x^0)$ which produces the pole and one inserts the complete set of *in* states between $\langle(\mathbf{p}'_n\sigma'_n,\ldots)_-|$ and the operator $O$ the matrix elements of which between the *in* states can, on account of the condition (13.31), be computed.

After "reducing" all the operators and using the formula (13.21) we obtain[15]

$$G_c^{(n)}(-p'_n,\ldots,-p'_{r+1},p_r\ldots,p_1)\approx(2\pi)^4\delta^{(4)}(-p'_n\ldots-p'_{r+1}+p_r+\ldots+p_1)$$

[15]If some of the particles in the *in* (*out*) state are treated as antiparticles then the corresponding $u^*$ $(u)$ functions should be replaced by $v$ $(v^*)$.

$$\times \prod_{j=r+1}^{n} \left[ \sum_{\sigma'_j} \frac{i\, \mathcal{Z}_{O_j}^{1/2}}{p'^2_j - m^2_{\mathrm{ph},j} + i0}\, u(\mathbf{p}'_j, \sigma'_j) \right] \tag{13.36}$$
$$\times \prod_{k=1}^{r} \left[ \sum_{\sigma_k} u^*(\mathbf{p}_k, \sigma_k)\, \frac{i\, \mathcal{Z}_{O_k}^{1/2}}{p^2_k - m^2_{\mathrm{ph},k} + i0} \right] (-i)\mathcal{A}(\mathbf{p}'_n\sigma'_n, \dots, \mathbf{p}_1\sigma_1)\,,$$

where $(-i)\mathcal{A}(\mathbf{p}'_n\sigma'_n, \dots, \mathbf{p}_1\sigma_1)$ is related to the connected part of the $S$-matrix element as in (7.87):

$$\langle(\mathbf{p}'_n\sigma'_n, \dots, \mathbf{p}'_{r+1}\sigma'_{r+1})_-|(\mathbf{p}_1\sigma_1, \dots, \mathbf{p}_r\sigma_r)_+\rangle_{\mathrm{con}} \tag{13.37}$$
$$= (2\pi)^4\delta^{(4)}(-p'_n + \dots - p'_{r+1} + p_r + \dots + p_1)(-i)\mathcal{A}(\mathbf{p}'_n\sigma'_n, \dots, \mathbf{p}_1\sigma_1)\,.$$

The formula (13.36) gives the required relation between the momentum space connected Green's function $G_c^{(n)}(q_n, \dots, q_1)$ and the $S$-matrix elements that can be extracted from it. It is important that this relation is exact, i.e. does not rely on the perturbative expansion nor on the assumptions of Section 7.3. In particular, since the operators $O_l$ do not need to be the elementary field operators, (13.36) allows to find the $S$-matrix also (at least in principle) in theories in which particles in terms of which the asymptotic states (the *in* and *out* eigenstates of the full Hamiltonian) are interpreted are not the "quanta" of the elementary fields used to build the Hamiltonians $H$ and $H_0$.

Extraction of $S$-matrix elements with the help of the LSZ procedure from vacuum Green's functions computed perturbatively does not require in principle that the free particle states $|\alpha_0\rangle$ created by the interaction picture operators are related to the *in* and *out* states $|\alpha_\pm\rangle$ by (7.39) nor that the mass parameters of $H_0$ be equal to masses of physical particles in the *in* and *out* states but in practice, setting the perturbative expansion, one makes a simplifying assumption that all true *in* and *out* states $|\alpha_\pm\rangle$ do have their counterparts (as far as their quantum numbers and Lorentz transformation properties are concerned) in the $|\alpha_0\rangle$ states because one assumes that the all one-particle poles factorized in the LSZ procedure from $n$-point Green's functions are only those poles which are found in two-point Green's functions of the elementary (however suitably rescaled) operators of the considered theory.[16] With this (weak) assumption it is of course most convenient to use for the operators $O_l(x)$ just the elementary operators of the theory,

[16] With loop corrections included, some of the two-point Green's functions of elementary field operators can have poles only for complex values of $q^2$. Such poles, which manifest themselves phenomenologically as unstable particles, do not, strictly speaking, correspond to any particles in the asymptotic states. Thus, even in the perturbative expansion, not all elementary field operators building the Hamiltonian generate the true asymptotic one-particle states. On the other hand, even perturbatively computed Green's functions $G_c^{(n)}$ for $n \geq 4$ can exhibit poles, either for complex or for real values of $p^2 = (p_1 + \dots + p_r)^2$, where $p_1, \dots, p_r$ is some subset of the external four-momenta, which are not found in two-point Green's functions of the elementary field operators. Such poles correspond respectively to unstable and stable bound states. For instance four-point (and higher) Green's functions computed in quantum electrodynamics of electrons and muons should have poles at complex values of $p^2 = (p_1 + p_2)^2$ (where $p_1$ and $p_2$ are the four-momenta brought in by the appropriate two elementary operators) corresponding to the unstable $e^-e^+$ or $\mu^-\mu^+$ bound states and poles at real values of $p^2 = (p_1 + p_2)^2$ corresponding to

the canonical ones $\phi_H(x)$ or arbitrarily rescaled (renormalized) ones $\tilde{\phi}_H(x) = Z^{-1/2}\phi_H(x)$, but it is also possible to chose composite operators. (For example, in the $\varphi^4$ theory it is possible to use the operator $\varphi_H^3$). Below we will analyze the LSZ prescription within the perturbative approach working with the elementary field operators. (Similar steps should, of course be performes if composite operators are used). It will turn out that in this case the prescription reduces to a small modification of the Feynman rules for external lines formulated in Section 9.4. The required modification can be obtained as follows.

As will be demonstrated rigorously using the functional methods (path integral formulation of quantum field theory) in Chapter 17, for $n > 2$ any connected $n$-point Green's function $\tilde{G}_c^{(n)}(q_n, \ldots, q_1)$ of the elementary[17] (however rescaled) field operators $\tilde{\phi}_H$ can be written in such a way that on each of its external legs the full propagator $\tilde{G}_c^{(2)}(q_l)$ - the (Fourier transform of the) two-point function of $\tilde{\phi}_H$ and $\tilde{\phi}_H^\dagger$ - corresponding to this line is factorized out.[18] More generally, any connected $n$-point function $\tilde{G}_c^{(n)}(q_n, \ldots, q_1)$ with $n > 2$ can be represented as a sum of block diagrams composed from 1PI Green's functions $i\tilde{\Gamma}^{(r)}$ with $2 < r < n$, connected by the full propagators $\tilde{G}_c^{(2)}$ (the $n$-point function is then represented as a sum of tree diagrams whose vertices are given by the 1PI functions $i\tilde{\Gamma}^{(r)}$ and internal lines are the full propagators $\tilde{G}_c^{(2)}$). Moreover, from the derivation presented in Chapter 17 it will be clear that this representation of vacuum Green's function is valid also outside the perturbative expansion.

Consider now one of the external lines, say the first one, of a connected $n$-point Green's function $G_c^{(n)}(q_n, \ldots, q_1)$. Factorizing out the full propagator $\tilde{G}_c^{(2)}(q_1)$ corresponding to it and using the formulae (13.20), (13.21) with $O_l = \tilde{\phi}_H$ and $q_1 \approx p_1$ where $p_1^2 = m_{\text{ph}1}^2$ one can write $G_c^{(n)}(q_n, \ldots, q_1)$ in the form[19]

$$\begin{aligned} G_c^{(n)}(q_n, \ldots, q_1) &= (2\pi)^4 \delta^{(4)}(\sum_i q_i)\, \tilde{G}_c^{(2)}(q_1)\, \mathcal{N}_c(q_n, \ldots, q_1) \\ &\approx (2\pi)^4 \delta^{(4)}(p_1 + \sum_{i=2}^n q_i) \sum_{\sigma_1} i\tilde{\mathcal{Z}}\, \frac{u(\mathbf{p}_1, \sigma_1) \otimes u^*(\mathbf{p}_1, \sigma_1)}{p_1^2 - m_{\text{ph},1}^2 + i0}\, \mathcal{N}_c(q_n, \ldots, p_1)\,, \end{aligned} \tag{13.38}$$

the stable $e^-\mu^+$ and $e^+\mu^-$ bound states. These, in principle, do have the corresponding asymptotic *in* and *out* states. In practice it is technically difficult to extract $S$-matrix elements corresponding to these asymptotic states using the perturbative expansion, because the corresponding poles appear only after an infinite subset of Feynman diagrams is resummed and finding the $S$-matrix element corresponding to, say, electron (elastic or inelastic) scattering on the muonic atom would require investigating the six-point Green's function.

[17] The same applies also to composite operators.

[18] We assume here for simplicity that the fields do not mix, i.e. that the mixed two-point functions $G_{c\ ij}^{(2)} = \langle\Omega|T[\tilde{\phi}_j^H(x)\tilde{\phi}_i^{H\dagger}(x)]|\Omega\rangle$ with $i \neq j$ all vanish (because of a mismatch of quantum numbers of $\phi_i^H$ and $\phi_j^H$). The LSZ prescriptions remains valid also if the fields can mix (there are several operators$\tilde{\phi}_j^H(x)$ with the same quantum numbers); in such a case anyone of the fields $\tilde{\phi}_j^H(x)$ can be used in Green's functions to extract $S$-matrix elements with the considered particle in the asymptotic state.

[19] If in the position space this two-point function is $\langle\Omega|T[\tilde{\phi}_H^\dagger(z)\tilde{\phi}_H(x_1)]|\Omega\rangle$ with $x_1$ being the argument of $G_c^{(n)}(x_n, \ldots, x_1)$, the factor $u(\mathbf{p}_1, \sigma_1) \otimes u^*(\mathbf{p}_1, \sigma_1)$ should be replaced by $v^*(\mathbf{p}_1, \sigma_1) \otimes v(\mathbf{p}_1, \sigma_1)$.

where $\mathcal{N}_c(q_n, \ldots, p_1)$ denotes the rest of the (connected) diagram (we suppress all Lorentz indices on the functions $u(\mathbf{p}, \sigma)$, amplitudes, etc.) and $\tilde{\mathcal{Z}} \equiv \mathcal{Z}_{\tilde{\phi}_H}$. Similar factors can be isolated on other external lines of $G_c^{(n)}(q_n, \ldots, q_1)$ (we continue to denote $\mathcal{N}_c(p'_n, \ldots, p_1)$ the remaining part of $G_c^{(n)}(-p'_n, \ldots, p_1)$).

In the limits $q_k \to p_k$, where $p_k^2 = m^2_{\mathrm{ph}\ k}$, $1 \geq k \geq r$, and $q_j \to -p'_j$, with $p'^2_j = m^2_{\mathrm{ph}\ j}$, $r + 1 \geq j \geq n$ one can therefore give two different expressions for $G_c^{(n)}(q_n, \ldots, q_1)$: one with the factors as displayed in (13.38) for each external line and with $\mathcal{N}_c(p'_n, \ldots, p_1)$ being the sum of Feynman diagrams without propagators on external lines (i.e. without any corrections to these lines), and another one given by (13.36). Comparing them, we get the final result

$$-i\mathcal{A}(\mathbf{p}'_m\sigma'_m, \ldots, \mathbf{p}'_1\sigma'_1,\, \mathbf{p}_n\sigma_n, \ldots, \mathbf{p}_1\sigma_1) = \tilde{\mathcal{Z}}_m^{1/2} \ldots \tilde{\mathcal{Z}}_1^{1/2} \tilde{\mathcal{Z}}_n^{1/2} \ldots \tilde{\mathcal{Z}}_1^{1/2} \qquad (13.39)$$
$$\times \sum_{l'_m, \ldots, l'_1} \sum_{l_m, \ldots, l_1} \mathcal{N}^c_{l'_m, \ldots, l'_1, l_n, \ldots, l_1}(\mathbf{p}'_m, \ldots, \mathbf{p}_1)\, u^*_{l'_m}(\mathbf{p}'_m, \sigma_n) \ldots u_{l_1}(\mathbf{p}_1, \sigma_1) \ldots,$$

where the momenta of the external lines are put on their respective mass-shells (we have restored the Lorentz indices). Thus, the prescription for calculating the $S$ matrix elements reads: compute the sum of all Feynman diagrams contributing to connected Green's functions $\tilde{G}_c^{(n)}(q_n, \ldots, q_1)$ but *without* diagrams contributing to full propagators on external lines and take the momenta on external lines on their respective mass-shells. Instead of the external lines propagators insert the appropriate wave functions $u_{l_i}(\mathbf{p}_i, \sigma_i)$ and $u^*_{l'_j}(\mathbf{p}'_j, \sigma'_j)$ (or, for antiparticles, $v^*_{l_i}(\mathbf{p}_i, \sigma_i)$ and $u_{l'_j}(\mathbf{p}'_j, \sigma'_j)$) and include for each external line an appropriate factor $\tilde{\mathcal{Z}}_i^{1/2}$. The necessary factors $\tilde{\mathcal{Z}}_i$ should in turn be extracted by computing the full propagators, i.e. the connected Green's functions $G_c^{(2)}(p_i)$ of the operators $\tilde{\phi}_H$ as in (13.20) and investigating their behavior near the poles at $p_i^2 \approx m^2_{\mathrm{ph},i}$.

Of course, working in the On-Shell scheme, i.e. using the physically (re)normalized elementary field operators $\tilde{\phi}_H = \phi_{\mathrm{ph}} = Z^{-1/2}\phi_H$ and the physical masses $m_{\mathrm{ph}}$ included in $H_0$ one has $\tilde{\mathcal{Z}} = 1$ (and $Z = \mathcal{Z}$ where $\mathcal{Z}$ are the factors corresponding to the bare canonical field operators $\phi_H$) and one recovers the rules of Section 9.4 with the prescriptions for treating external lines (corresponding to particles in the initial or final states) formulated in Sections 9.7 and 11.10. However, with the rules for obtaining $S$-matrix elements established above one can work with arbitrarily rescaled elementary field operators $\tilde{\phi}_H(x)$ and with arbitrary mass parametrs included in $H_0$; moreover it is also possible to replace the elementary operators like $\tilde{\phi}_H(x)$ by composite operators:[20] as long as one includes for external lines the appropriate $\tilde{\mathcal{Z}}^{1/2}$ factors of the employed operators (determined as in Section 13.3) and expresses the final results in terms of physical masses (determined by positions of the Green's function's poles) the physical predictions will be the same.

[20]In principle such operators must be used if a physical particle in the initial or final state is composite. For instance, to obtain $S$-matrix elements corresponding to strong interactiosn of hadrons one should consider in QCD Green's functions of composite operators which are singlets w.r.t. the color $SU(3)$ group and having appropriate flavour quantum numbers.

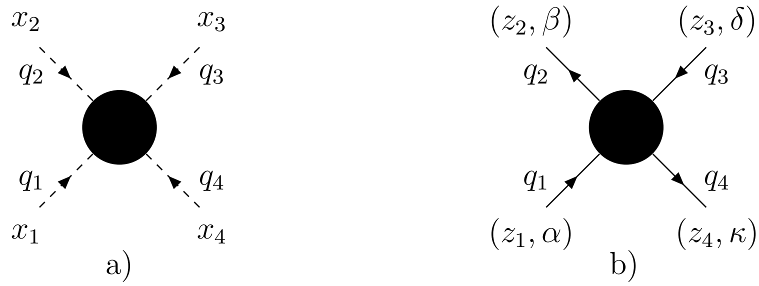



Figure 13.4: Four-point amplitudes: a) in the $\varphi^4$ theory; the arrows show the flow of the four-momenta as defined by the Fourier transform (13.40), b) in the Fermi theory of weak interactions (the arrows show the flow of the four-momenta defined by (13.47) and of the fermion number).

## 13.5 Simple examples

We shall now illustrate the LSZ reduction on the simplest examples. Consider first the $\varphi^4$ theory with the Lagrangian density specified by (11.472) and (11.471) and quantized using the canonical field $\varphi$ as the dynamical variable. The corresponding Heisenberg picture operator will be the canonical (bare) one, $\varphi_H(x)$. Taking as the interaction term only the quartic term (so that the mass squared parameter in $H_0$ will be the original, "bare" parameter $M^2$) one arrives in the interaction picture at $\mathcal{H}^I_{\rm int}(x) = (\lambda/4!)\varphi^4_I(x)$. The theory cast in the interaction picture will be therefore equivalent to the theory of interacting spin 0 particles which can be constructed directly in the approach of Chapter 9 with the simplest interaction term $V^I_{\rm int}(t) = (\lambda/4!)\int d^3\mathbf{x}\, \varphi^4_I(t,\mathbf{x})$. Direct computation of $S$-matrix elements would therefore lead to a catastrophe due to singularities related to external lines. Nothing however prevents computing the Green's function represented graphically in Figure 13.4a of four canonical field operators $\varphi_H$. It is formally given by (we omit $I$ on the interaction picture operators)

$$\begin{aligned}
G^{(4)}(x_4,\dots,x_1) &= \langle\Omega_-|\mathrm{T}\left[\varphi_H(x_4)\varphi_H(x_3)\varphi_H(x_2)\varphi_H(x_1)\right]|\Omega_+\rangle \\
&= \left(\frac{1}{i}\right)^4 \frac{\delta^4}{\delta J(x_4)\dots\delta J(x_1)}\,\langle\Omega_0|\mathrm{T}\exp\left(-i\!\int\! d^4x\left[\frac{\lambda}{4!}\,\varphi^4(x) - J(x)\varphi(x)\right]\right)|\Omega_0\rangle\Bigg|_{J=0} \\
&= \langle\Omega_0|T\left[\varphi(x_4)\varphi(x_3)\varphi(x_2)\varphi(x_1)\exp\left(-i\,\frac{\lambda}{4!}\int d^4x\,\varphi^4(x)\right)\right]|\Omega_0\rangle\,.
\end{aligned}$$

As will be clear (see Chapter 17), the four-point Green's function can be split into the disconnected and connected parts (much in the same way as were split $S$-matrix elements in Section 7.8. The lowest order contribution to the connected part reads

$$\begin{aligned}
G^{(4)}_c(x_4,\dots,x_1) &= -i\lambda\int d^4x\; i\Delta^F(x-x_1)\,i\Delta^F(x-x_2)\,i\Delta^F(x-x_3)\,i\Delta^F(x-x_4) \\
&= \int\frac{d^4q_4}{(2\pi)^4}\dots\int\frac{d^4q_1}{(2\pi)^4}\,e^{+iq_4\cdot x_4}\dots e^{+iq_1\cdot x_1}\,G^{(4)}_c(q_4,\dots,q_1)\,, \qquad (13.40)
\end{aligned}$$

and its Fourier transform is

$$G_c^{(4)}(q_4,\dots,q_1) = -i\lambda\,(2\pi)^4\delta^{(4)}(q_4+\ldots+q_1)\,\frac{i}{q_4^2-M^2+i0}\cdots\frac{i}{q_1^2-M^2+i0}\,. \quad (13.41)$$

Thus, in this order, the poles of the Green's function $G_c^{(4)}(q_4,\dots,q_1)$ are located at $q_i^2 = M^2$, so that in this order the parameter $M^2$ is to be identified with the physical mass squared. This is consistent with position of the pole of the Fourier transform of the lowest order approximation to the the two-point function $G^{(2)}(x-y)$

$$\int d^4x\, e^{ip\cdot x}\,\langle\Omega_-|T[\varphi_H(x)\varphi_H(0)]|\Omega_+\rangle = \int d^4x\, e^{ip\cdot x}\left[i\Delta^F(x) + \text{higher orders}\right]$$
$$= \frac{i}{p^2-M^2+i0} + \text{higher orders}, \quad (13.42)$$

(the free propagator $i\Delta^F(x-y)$ is given by (9.32)). It is also clear that, as advocated, four two-point functions can be factorized out of the connected Green's function $G_c^{(4)}(q_4,\dots,q_1)$. From the two-point function we learn however, that to this order the factor $\mathcal{Z}$ of the canonical (bare) field operator $\varphi_H(x)$ equals unity, because the the residuum of the simple pole at $p^2 = M^2$ is just $i$. From (13.39) it follows, therefore, that to this order the element of the $S$-matrix corresponding to the elastic $2\to 2$ scattering of the spinless particles of mass $M$ having the initial and final four-momenta $k_1 = (E_{\mathbf{k}_1},\mathbf{k}_1)$, $k_2 = (E_{\mathbf{k}_2},\mathbf{k}_2)$ and $p_1 = (E_{\mathbf{p}_1},\mathbf{p}_1)$, $p_2 = (E_{\mathbf{p}_2},\mathbf{p}_2)$ is (because the $u$ functions of spinless particles are equal 1) directly obtained by stripping off from $G_c^{(4)}(q_4,\dots,q_1)$ the four propagators and by substituting for (or, more precisely, by analytically continuing) the four four-momenta $q_1,\dots,q_4$ (to) the values

$$q_1\to k_1\,,\qquad q_2\to k_2\,,\qquad q_3\to -p_1\,,\qquad q_4\to -p_2\,. \quad (13.43)$$

This gives the lowest order scattering amplitude $-i\mathcal{A}(\mathbf{p}_2,\mathbf{p}_2,\mathbf{k}_2,\mathbf{k}_1) = -i\lambda$, as previously.

It is instructive to consider also a different treatement of the same theory. If quantization is performed taking as the field variable $\tilde\varphi = Z^{-1/2}\varphi$ one is led, as in Section 11.10, to the Hamiltonian density (11.433). Suppose however that one does not take for $H_0$ the expression (11.436) but instead splits (11.433) so that the interaction term is just $-(\lambda/4!)Z^2\tilde\varphi^4$. The free momentum space propagator will then take the form[21]

$$\frac{i}{Z(q^2-M^2+i0)}\,,$$

[21]This is obvious in the path integral formulation which will be discussed in Section 16.3. Another argument is that the Green's function $\langle\Omega|T[\tilde\varphi_H(x)\tilde\varphi_H(y)]|\Omega\rangle$ (in the perturbative expansion of which the free propagator is the first term) is related to $\langle\Omega|T[\varphi_H(x)\varphi_H(y)]|\Omega\rangle$ which leads to (13.42) just by the factor $Z^{-1}$. It is however a good exercise to perform the canonical steps of Sections 11.2 and 11.9 and to obtain this free propagator using the definition (9.32), that is as the Fourier transform of

$$\theta(x^0-y^0)[\tilde\varphi_I^{(+)}(x),\,\tilde\varphi_I^{(-)}(y)] + \theta(y^0-x^0)[\tilde\varphi_I^{(+)}(y),\,\tilde\varphi_I^{(-)}(x)],$$

where $\tilde\varphi_I(x)$ is the interaction picture operator corresponding through the adopted $H_0$ to $\tilde\varphi_H(x)$.

with the residue $\mathcal{Z}_{\tilde{\varphi}} = Z^{-1}$, and the Fourier transform of the lowest order approximation to the Green's function $\langle\Omega|T[\tilde{\varphi}(x_1)\dots\tilde{\varphi}(x_4)|\Omega\rangle$ will take the form

$$G_c^{(4)}(q_4,\dots,q_1) = -iZ^2\lambda\,(2\pi)^4\delta^{(4)}(q_4+\dots+q_1)\,\frac{i\,Z^{-1}}{q_4^2-M^2+i0}\cdots\frac{i\,Z^{-1}}{q_1^2-M^2+i0}\,, \quad (13.44)$$

and the prescription (13.39) with $\mathcal{N}^c = -iZ^2\lambda = -i\mathcal{Z}_{\tilde{\varphi}}^{-2}\lambda$ will again lead to the scattering amplitude $-i\mathcal{A}(\mathbf{p}_2,\mathbf{p}_2,\mathbf{k}_2,\mathbf{k}_1) = -i\lambda$. We will extend this analysis up to one-loop order after considering another example.

As the second example we consider the four-Fermi interaction of leptons (again, it is assumed that the corresponding field theory has been quantized using the canonical field variables and transition to the interaction picture has been performed equating the canonical, that is bare, Heisenberg picture operators at $t=0$ to the interaction picture operators and that all terms bilinear in field operators - mass terms - have been left in $H_0$)

$$\mathcal{H}_I^{\text{weak}} = \frac{G_F}{\sqrt{2}}\left(\bar{\psi}_{(\nu_\mu)}\Gamma^\lambda\psi_{(\mu)}\right)\left(\bar{\psi}_{(e)}\Gamma_\lambda\psi_{(\nu_e)}\right) + \text{H.c.} \quad (13.45)$$

where $\Gamma^\lambda = \gamma^\lambda(1-\gamma^5)$, and compute in the lowest order the amplitude

$$G^{(4)}_{\kappa\delta\beta\alpha}(z_4,\dots,z_1) = \langle\Omega_-|T\left[\psi^H_{(e)\kappa}(z_4)\bar{\psi}^H_{(\nu_e)\delta}(z_3)\psi^H_{(\nu_\mu)\beta}(z_2)\bar{\psi}^H_{(\mu)\alpha}(z_1)\right]|\Omega_+\rangle\,, \quad (13.46)$$

shown in Figure 13.4b. In the lowest order (because all the field operators are different this Green's function has no disconnected part, that is $G^{(4)} = G^{(4)}_c$)

$$G^{(4)}_{\kappa\delta\beta\alpha}(z_4,\dots,z_1) = -i\frac{G_F}{\sqrt{2}}\int d^4x\left[iS^{(e)}_{\kappa\varphi}(z_4-x)\Gamma_{\varphi\gamma}iS^{(\nu_e)}_{\gamma\delta}(x-z_3)\right]\left[iS^{(\nu_\mu)}_{\beta\rho}(z_2-x)\Gamma_{\rho\chi}iS^{(\mu)}_{\chi\alpha}(x-z_1)\right].$$

We have explicitly indicated which propagator corresponds to which particle[22] (at the price of suppressing the superscript "F"). The considered function can be written as the Fourier transform

$$G^{(4)}_{\kappa\delta\beta\alpha}(z_4,\dots,z_1) = \int\frac{d^4q_1}{(2\pi)^4}\cdots\int\frac{d^4q_4}{(2\pi)^4}\,e^{-iz_4\cdot q_4}e^{+iz_3\cdot q_3}e^{-iz_2\cdot q_2}e^{+iz_1\cdot q_1}\times G^{(4)}_{\kappa\delta\beta\alpha}(q_4,\dots,q_1)\,, \quad (13.47)$$

and the lowest order contribution reads

$$G^{(4)}_{\kappa\delta\beta\alpha}(q_4,\dots,q_1) = (2\pi)^4\delta^{(4)}(q_1-q_2+q_3-q_4)\left(-i\frac{G_F}{\sqrt{2}}\right)\times\left[i\frac{\not{q}_4+m_e}{q_4^2-m_e^2}\,\Gamma_\lambda\,i\frac{\not{q}_3}{q_3^2}\right]_{\kappa\delta}\left[i\frac{\not{q}_2}{q_2^2}\,\Gamma^\lambda\,i\frac{\not{q}_1+m_\mu}{q_1^2-m_\mu^2}\right]_{\beta\alpha}. \quad (13.48)$$

[22]Strictly speaking, since as we have argued, the correspondence between elementary field operators and (physical) particles (to which *in* and *out* $H$ eigenstates correspond) need not be direct nor one-to-one, one should rather speak of propagators of (interaction picture) field operators.

The exponential factors have been written in (13.47) in such a way that the flow of the four-momenta $q_1$, …, $q_4$ agrees with the flow of the fermion number (i.e. with the flow of particles and not antiparticles) as shown by arrows in Figure 13.4b. The poles (at $m_e^2$, $m_\mu^2$ and zero in the respective channels) of $G^{(4)}_{\kappa\delta\beta\alpha}(q_4,\dots,q_1)$ are again explicit and their positions agree with the positions of the poles of the appropriate two-point Green's functions evaluated in the lowest order. These two-point Green's functions indicate that, to this order, the $\mathcal{Z}$ factors of all operators entering the Green's function (13.46) are equal to unity. These four two-point Green's functions can again be neatly factorized out of Green's function (13.48). Therefore, in order to obtain the $S$-matrix element corresponding to the muon decay $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ considered in Section 9.2 it is sufficient to replace the external propagators in $G^{(4)}_{\kappa\delta\beta\alpha}(q_4,\dots,q_1)$ by the appropriate wave functions $u$ and $v$ as in (13.39) and to set (in the notation defined in Figure 9.1) $q_1 = (E_{\mathbf{q}},\mathbf{q})$, $q_2 = (E_{\mathbf{k}_2},\mathbf{k}_2)$, $q_3 = (-E_{\mathbf{k}_1},-\mathbf{k}_1)$ - because the flow of physical energy and momentum through this line is opposite to the flow of the fermion number - and $q_4 = (E_{\mathbf{p}},\mathbf{p})$. The resulting amplitude is then, perhaps up to a sign[23], identical to (9.27). Moreover, when the four-momenta of the Green's function are continued to the mass shells, as indicated, that is, when $q_1$ is continued to $q_1 = (E_{\mathbf{q}},\mathbf{q})$, the numerator $\not{q}_1 + m_\mu$ of the two-point Green's function corresponding to this line becomes equal to $\sum_{\sigma_q} u(\mathbf{q},\sigma_q)\otimes\bar{u}(\mathbf{q},\sigma_q)$; similarly, when $q_3$ is continued to $q_3 = (-E_{\mathbf{k}_1},-\mathbf{k}_1)$ the numerator $\not{q}_3$ can be written as $\sum_{\sigma_1} v(\mathbf{k}_1,\sigma_1)\otimes\bar{v}(\mathbf{k}_1,\sigma_1)$. This is how the wave function factors present in the general form (13.36) of Green's functions arise.

This example shows also that the Green's function $G^{(4)}_{\kappa\delta\beta\alpha}(q_4,\dots,q_1)$ is a more general object than the mere $S$-matrix element corresponding to the $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ decay: $S$-matrix elements of all "crossed" processes like $\mu^-e^+ \to \nu_\mu\bar{\nu}_e$, or $\mu^-\nu_e \to e^-\nu_\mu$, or $\nu_e\bar{\nu}_\mu \to \mu^+e^-$ etc. can be also extracted from it by continuing the external momenta $q_1, q_2, q_3, q_4$ to different physical domains and by providing the appropriate wave functions.

In order to illustrate the considerations of the previous section in the one-loop order we return to the $\varphi^4$ theory and, in the same setting as specified at the beginning of this section, compute the amplitude of the elastic scattering to the one-loop accuracy. Unfortunately, the $\varphi^4$ theory is slightly pathological - as we will see to the one-loop order still $\mathcal{Z} = 1$ - but this will not be a serious obstacle for illustrating the main points.

The tree level (lowest order) contribution to the connected off-shell Green's function $G^{(4)}_c(q_4,\dots,q_1) = (2\pi)^4\delta^{(4)}(\sum_i q_i)\,\tilde{G}^{(4)}_c(q_4,\dots,q_1)$, represented by the diagram of Figure 13.4a, is given by (13.41). We write $\tilde{G}^{(4)}_c(q_4,\dots,q_1)$ here in the form

$$\tilde{G}^{(4)}_c(q_4,\dots,q_1) = -i\lambda \times (\text{ext. prop.})\,. \tag{13.49}$$

[23]Performing the LSZ reduction of the Green's functions of the fermionic operators one has in principle to keep track of the minus signs present in the definition of the chronological product of such operators. This is however in most cases not important as the overall sign of the complete amplitude is irrelevant for physical quantities.

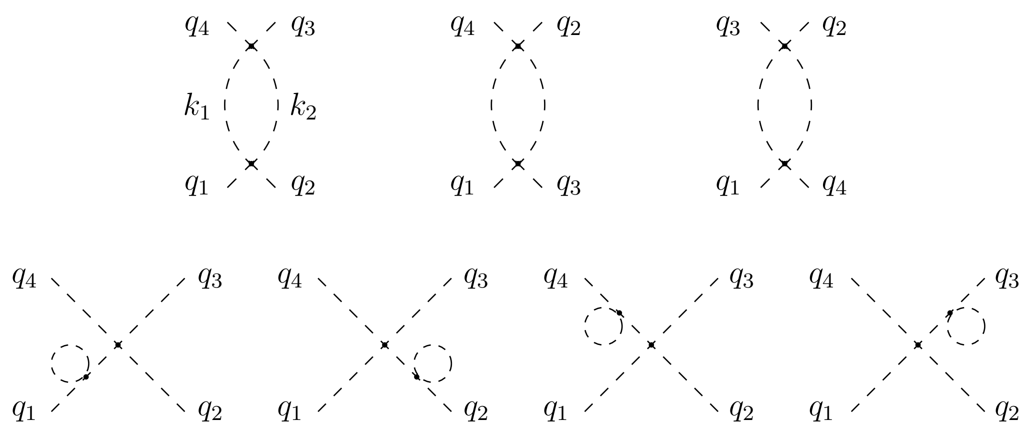



Figure 13.5: One-loop contributions to $\tilde{G}_c^{(4)}(p_1, \ldots, p_4)$ in $\varphi^4$ theory. All momenta are incoming.

At the one-loop order (in the case of this Green's function identical with the order $\lambda^2$) there are seven additional contributions to $\tilde{G}_c^{(4)}(q_4, \ldots, q_1)$. They are shown in Figure 13.5. According to the Feynman rules, the first upper diagram in Figure 13.5 contributes to $G_c^{(4)}(q_4, \ldots, q_1)$ the expression

$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - M^2 + i0} \frac{i}{k_2^2 - M^2 + i0}$$
$$\times (2\pi)^4 \delta(q_4 + q_3 + q_2 + q_1) \times (2\pi)^4 \delta(q_3 + q_4 - k_1 - k_2) \times (\text{ext. prop.}) \,,$$

(1/2 is the combinatoric factor). Therefore the contribution to $\tilde{G}_c^{(4)}(q_4, \ldots, q_1)$ is

$$-i\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - M^2 + i0][(k + q_1 + q_2)^2 - M^2 + i0]} \times (\text{ext. prop.}) \,.$$

The integral over $d^4k$ is logarithmically divergent because in the limit $|k^\mu| \to \infty$ it behaves as $\int dk/k$. As will be shown in the next Chapter, the divergent part of this integral is real and independent of the external momenta $q_1$, $q_2$. Therefore, we write this contribution as

$$-i\frac{\lambda^2}{2(4\pi)^2} \left[I_{\text{div}} + f(s)\right] \times (\text{ext. prop.}) \,, \tag{13.50}$$

where we have factorized $1/(4\pi)^2$ typical for one-loop integrals and introduced the Mandelstam variable $s = (q_1 + q_2)^2$: since the integral is Lorentz invariant it can depend only on $s$; similarly, the loop integrals associated with the second and third upper diagrams in Figure 13.5 depend only on $t = (q_1 + q_3)^2$ and $u = (q_1 + q_4)^2$, respectively.

The contribution to $\tilde{G}_c^{(4)}(q_4, \ldots, q_1)$ of the first one of the four diagrams shown in the lower part of Figure 13.5 can be written in the form

$$(-i\lambda) \frac{i}{q_1^2 - M^2 + i0} \left[-i\Sigma(q_1)\right] \times (\text{ext. prop.}) \,,$$

in which the self-energy insertion $\Sigma(p)$ is given by the (quadratically divergent) integral

$$-i\Sigma(p) = -i\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M^2+i0}\,, \tag{13.51}$$

and is, in this order in $\lambda$, just the 1PI part of the one-loop contribution to $\tilde{G}_c^{(2)}(p)$ (by definition, $-i\Sigma(p)$ is the sum of all 1PI Feynman diagrams contributing to $\tilde{G}_c^{(2)}(p)$ with simple propagators on external lines removed). It is then clear that the full propagator, as represented in Figure 13.3 can be written (suppressing $i0$) in the form

$$\begin{aligned}\tilde{G}_c^{(2)}(p) &= \frac{i}{p^2-M^2} + \frac{i}{p^2-M^2}\,[-i\Sigma(p)]\,\frac{i}{p^2-M^2}\\ &+ \frac{i}{p^2-M^2}\,[-i\Sigma(p)]\,\frac{i}{p^2-M^2}\,[-i\Sigma(p)]\,\frac{i}{p^2-M^2} + \dots\\ &= \frac{i}{p^2-M^2-\Sigma(p)}\,. \end{aligned} \tag{13.52}$$

Tn the last line the geometric series has been resummed. For future use it is also convenient to define the 1PI two-point function $\tilde{\Gamma}^{(2)}(p)$ as

$$i\tilde{\Gamma}^{(2)}(p) = i\left(p^2-M^2-\Sigma(p^2)\right), \tag{13.53}$$

so that $i\tilde{\Gamma}^{(2)}(p) = -[\tilde{G}_c^{(2)}(p)]^{-1}$. Combining together the contributions to $\tilde{G}_c^{(4)}(q_4,\dots,q_1)$ corresponding to all diagrams shown in Figure 13.5 we get

$$\begin{aligned}\tilde{G}_c^{(4)}(q_4,\dots,q_1) &= (-i\lambda)\times(\text{ext. prop.})\\ &- i\frac{\lambda^2}{2(4\pi)^2}\,[3I_{\text{div}}+f(s)+f(t)+f(u)]\times(\text{ext. prop.})\\ &- i\lambda\left[\frac{1}{q_4^2-M^2+i0}\,\Sigma(q_4)+\dots+\frac{1}{q_1^2-M^2+i0}\,\Sigma(q_1)\right]\times(\text{ext. prop.})\,.\end{aligned}$$

Since $\Sigma(p)$ is of order $\lambda$, it is easy to see, that up to terms $\mathcal{O}(\lambda^3)$ the above expression for $\tilde{G}^{(4)}(q_4,\dots,q_1)$ can be rewritten as

$$\begin{aligned}\tilde{G}_c^{(4)}(q_4,\dots,q_1) &= -i\left(\lambda+\frac{\lambda^2}{2(4\pi)^2}\,[3I_{\text{div}}+f(s)+f(t)+f(u)]\right)\\ &\times\ \frac{i}{q_4^2-M^2-\Sigma(q_4)+i0}\cdots\frac{i}{q_1^2-M^2-\Sigma(q_1)+i0}\,.\end{aligned}$$

This formula illustrates the statement made in the paragraph preceding the one containing the formula (13.38): the Green's function of the four (bare) operators $\varphi_H$ can be represented in the form[24]

$$\tilde{G}_c^{(4)}(q_4,\dots,q_1) = i\tilde{\Gamma}^{(4)}(q_4,\dots,q_1)\,\tilde{G}_c^{(2)}(q_4)\dots\tilde{G}_c^{(2)}(q_1) \tag{13.54}$$

[24]Due to the $\varphi\to-\varphi$ symmetry of the Lagrangian density (11.472) and (11.471)the three-point 1PI function $\tilde{\Gamma}^{(3)}$ vanishes.

in which corrections to its external legs are factorized in the form of the full two-point functions.

The position of the pole (which is to be identified with the mass squared $M^2_{\rm ph}$ of the physical particle in terms of which the *in* and *out* eigenstates are interpreted) of the full two-point function (13.52) is given by the solution of the equation

$$M^2_{\rm ph} - M^2 - \Sigma(M^2_{\rm ph}, M) = 0 \tag{13.55}$$

in which we have explicitly indicated the dependence of $\Sigma$ also on the mass parameter $M^2$ (the one present in $H_0$). To the one loop accuracy $M$ can be replaced in $\Sigma(p^2, M)$ by $M_{\rm ph}$, so that the equation (13.55) takes the form $M^2 = M^2_{\rm ph} - \Sigma(M^2_{\rm ph}, M_{\rm ph})$. In order to find the factor $\mathcal{Z}$ associated with the employed bare operators $\varphi_H$, we expand the denominator in the general form (13.52) of their two-point function (the Green's function of the two bare field operators $\varphi_H$) in the Taylor series around $p^2 = M^2_{\rm ph}$

$$\begin{aligned} p^2 - M^2 - \Sigma(p^2) &= 0 + (p^2 - M^2_{\rm ph})\, \frac{d}{dp^2}\left[p^2 - M^2 - \Sigma(p)\right]\Big|_{p^2=M^2_{\rm ph}} + \dots \\ &= (p^2 - M^2_{\rm ph})\left[1 - \Sigma'(M^2_{\rm ph})\right] + \dots \end{aligned} \tag{13.56}$$

Comparison with (13.20) shows that

$$\mathcal{Z} = \frac{1}{1 - \Sigma'(M^2_{\rm ph})} \approx 1 + \Sigma'(M^2_{\rm ph}) + \dots \tag{13.57}$$

The pathology of the $\varphi^4$ theory is that at one-loop order $\Sigma(p)$ does not depend on $p$ (see (13.51)) and therefore $\mathcal{Z} = 1$. This is of course no longer true in higher orders of the perturbative expansion.

Summarizing, the amplitude of the elastic scattering in the $\varphi^4$ theory is in order $\lambda^2$ given by

$$-i\mathcal{A}(\mathbf{p}'_4, \mathbf{p}'_3, \mathbf{p}_2, \mathbf{p}_1) = -i\mathcal{Z}^2\left(\lambda + \frac{\lambda^2}{2(4\pi)^2}\left[3I_{\rm div} + f(s) + f(u) + f(t)\right]\right), \tag{13.58}$$

(we keep $\mathcal{Z}^2$ although it is 1 to this order) with the Mandelstamm variables given now by $s = (p_1 + p_2)^2$, $t = (p_1 - p'_3)^2$ and $u = (p_1 - p'_4)^2$. It is still infinite, but the coupling $\lambda$ is not yet related to anything physical. Removing the divergence by relating $\lambda$ to something measurable will be disscussed in the next chapter.

Before we discuss renormalization, it is useful to understand the origin of the $\mathcal{Z}^{1/2}$ factors in physical amplitudes in a different way, by using the effective Lagrangian technique. This technique is widely exploited in modern approach to quantum field theory problems. Usually it is used in the context of "integrating out" heavy degrees of freedom (heavy particles).[25] Here we use the effective Lagrangian in a simpler context: we construct $\mathcal{L}_{\rm eff}$, so that it reproduces already at the tree-level the four-point Green's function $G_c^{(4)}$ calculated in the full $\varphi^4$, theory including loop contributions. Of course, because the full off-shell Green's function has a complicated dependence on its external momenta, the tree level diagrams derived from the *local* (i.e. having a finite number of terms constructed from causal field operators and their derivatives of finite order) effective Lagrangian cannot reproduce it exactly for all values of the external line momenta. Tree level diagrams generated by the local effective Lagrangian

$$\mathcal{L}_{\rm eff} \;=\; \frac{1}{2}(1+\delta z)\,\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\Delta M^2\varphi^2 - \frac{1}{4!}\Delta\lambda\,\varphi^4 \qquad (13.59)$$

can, however, reproduce the full Green's function $G_c^{(4)}$ at some (arbitrarily chosen) fixed kinematical point in the $(q_1,\dots,q_4)$ space, in particular for $p_i^2 \approx M^2_{\rm ph}$ and fixed values of the Mandelstam variables $s_0$, $t_0$ and $u_0$. From the representation (13.54) of $G_c^{(4)}$ it follows that it is sufficient to reproduce $\tilde{G}_c^{(2)}$ and $\tilde{\Gamma}^{(4)}$ in this kinematical regime. Reproducing $\tilde{\Gamma}^{(4)}$ is simple: it is sufficient to replace in the Lagrangian (13.59) the coupling $\Delta\lambda$ by $-\tilde{\Gamma}^{(4)}(s_0,t_0,u_0)$. Furthermore, treating the parts with $\delta z$ and $\Delta M^2 - M^2$ of $\mathcal{L}_{\rm eff}$ as interactions and using the Taylor expansion of $\Sigma(p^2)$ around $p^2 = M^2_{\rm ph}$

$$\Sigma(p^2) \approx \Sigma(M^2_{\rm ph}) + (p^2 - M^2_{\rm ph})\Sigma'(M^2_{\rm ph}) \qquad (13.60)$$

it is easy to see that in order to reproduce the full self energy for $p^2 \approx M^2_{\rm ph}$ already at the tree level, that is to have

$$\frac{i}{p^2-M^2}\left[i\,\delta z\,p^2 - i\left(\Delta M^2 - M^2\right)\right]\frac{i}{p^2-M^2} = \frac{i}{p^2-M^2}[-i\Sigma(p^2)]\frac{i}{p^2-M^2}\,,$$

it is sufficient to take $\delta z = -\Sigma'(M^2_{\rm ph})$ and $\Delta M^2 - M^2 = \Sigma(M^2_{\rm ph}) - M^2_{\rm ph}\Sigma'(M^2_{\rm ph})$. The effective Lagrangian takes then the form

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\left(1 - \Sigma'(M^2_{\rm ph})\right)\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\Delta M^2\varphi^2 + \frac{1}{4!}\Gamma^{(4)}(s_0,t_0,u_0)\,\varphi^4. \qquad (13.61)$$

The Lagrangian has a non-canonical form because the residue of the simple tree-level propagator derived by inverting its quadratic part is not equal to $i$. To bring it to the

[25] If the mass scale $M$ of some particles in a quantum field theory model is much larger than the energy scale $E$ of light particle processes we are interested in, the low energy effective Lagrangian can be constructed out of fields of light particles only in such a way that (using the Feynman rules derived from it) one reproduces Green's functions relevant for low energy processes (i.e. those with external lines corresponding to light particles only) of the original theory up to some fixed power of the ratio $E/M$. For example, the Fermi theory of weak interactions discussed in section 12 reproduces the Green's functions of the full electroweak theory (to be discussed later) up to $E^4/M_W^4$, where $M_W = 80.4$ GeV is the mass of the $W^\pm$ vector bosons. Transition from the full to the effective theory is frequently referred to as integrating out of heavy particles. The name derives from the path integral formulation of quantum field theory in which this indeed corresponds to performing the functional integration over the fields of heavy particles.

canonical form it is necessary to rescale the field

$$\varphi \to \varphi' = \left[1 - \Sigma'(M^2_{\rm ph})\right]^{1/2} \varphi \ . \tag{13.62}$$

The kinetic term of the effective Lagrangian becomes then canonical and for the physical mass $M^2_{\rm ph}$ we get[26]

$$M^2_{\rm ph} = \frac{\Delta M^2}{1+\delta z} = \frac{M^2 + \Sigma(M^2_{\rm ph}) - M^2_{\rm ph}\Sigma'(M^2_{\rm ph})}{1 - \Sigma'(M^2_{\rm ph})} \tag{13.63}$$

which is the same as (13.55). Moreover, after rescaling the field $\varphi$ the interaction Hamiltonian becomes

$$\mathcal{H}_{\rm int} = -\frac{1}{4!}\left[1 - \Sigma'(M^2_{\rm ph})\right]^{-2} \Gamma^{(4)}(s_0, t_0, u_0)\ \varphi^4 \ . \tag{13.64}$$

Using then the ordinary Feynman rules we obtain the connected off-shell Green's function $G^{(4)}_{c\ {\rm eff\ th}}(q_4 \dots, q_1)$ (of the effective theory) for $q_1^2 \approx M^2_{\rm ph}$ and $(q_1 + q_2)^2 \approx s_0$, etc. at the tree level in the form

$$\tilde{G}^{(4)}_{\rm eff\ th}(q_4 \dots, q_1) = i\left[1 - \Sigma'(M^2_{\rm ph})\right]^{-2} \Gamma^{(4)}(s_0, t_0, u_0)\, \frac{i}{q_4^2 - M^2_{\rm ph}} \cdots \frac{i}{q_1^2 - M^2_{\rm ph}} \ .$$

Stripping off the external propagators we obtain directly at the tree level the elastic scattering amplitude in the form (13.58) including the correct $\mathcal{Z}^2$ factor ($\mathcal{Z}^{1/2}$ per each leg). This effective Lagrangian technique for properly obtaining $\mathcal{Z}$ factors in the physical amplitudes is particularly useful in the case of fermionic external lines where the matrix structure of the propagators introduces some complications.

Let us also remark, that the off-shell Green's functions obtained from the effective Lagrangian after rescaling the field $\varphi$ as in (13.62) are *not* the same as the off-shell Green's functions obtained from the effective Lagrangian (13.61). Nevertheless, physical, on-shell amplitudes are unchanged by the rescaling (13.62) which is just equivalent to working with a differently renormalized canonical field operator. This is the special case of a general result that the $S$ matrix elements are not modified if one makes arbitrary redefinitions of the Lagrangian fields (the dependence of the off-shell Green's functions on external momenta may be, for nonlinear redefinitions, drastically altered).

## 13.6 The LSZ reduction in the position space

For some applications it is useful to formulate the LSZ reduction formula for $S$ matrix elements directly in terms of the position space Green's functions. Unlike the derivation

[26] Note: we first expand $\Sigma(p^2)$ around unknown $M^2_{\rm ph}$ and only at the end get the equation determining $M^2_{\rm ph}$!

in the momentum space presented in Section 13.4, this requires considering different types of field operators (corresponding to particles of different spins) separately.

We consider first the case of spinless particles in the asymptotic states and take the operators corresponding to them to transform as Lorentz scalars. We start from an $S$ matrix element $S_{\beta\alpha} = \langle\beta_-|\alpha_+\rangle$ and "reduce" first one spinless particle e.g. from the state $|\alpha_+\rangle$:

$$S_{\beta\alpha} = \langle\beta_-|a^\dagger_{\rm out}(\mathbf{p}_1)|\alpha'_+\rangle - \langle\beta_-|\left[a^\dagger_{\rm out}(\mathbf{p}_1) - a^\dagger_{\rm in}(\mathbf{p}_1)\right]|\alpha'_+\rangle ,$$

where we have replaced the first particle with momentum $\mathbf{p}_1$ in the *in* state (denoting what remains by $|\alpha'_+\rangle$) by the corresponding *in* creation operator (and have also added and subtracted the corresponding *out* creation operator). The first term is zero if the same particle is not present in the *out* state. If it is present, this term contributes to the non-connected part of the $S$ matrix and will be denoted below by "disc." The remaining term can be rewritten using the formula (8.28):

$$\begin{aligned} S_{\beta\alpha} &= {\rm disc} + i\int d^3\mathbf{x}\ \langle\beta_-|\left[e^{-ip_1\cdot x}\overleftrightarrow{\partial}_{x^0}\varphi_{\rm in}(x) - e^{-ip_1\cdot x}\overleftrightarrow{\partial}_{x^0}\varphi_{\rm out}(x)\right]|\alpha'_+\rangle \\ &= {\rm disc} + i\mathcal{Z}_\varphi^{-1/2}(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty})\int d^3\mathbf{x}\ \langle\beta_-|e^{-ip_1\cdot x}\overleftrightarrow{\partial}_{x^0}\varphi_H(x)|\alpha'_+\rangle , \end{aligned}$$

where in the second step we have used the asymptotic condition (13.31) with the canonical field operator $\varphi_H$. One can use as well any other renormalized canonical operator $\tilde{\varphi}_H$ or a composite operator $O$ in conjunction with the appropriate $\mathcal{Z}$ factor. The difference of limits is now traded for the $x^0$ integral:

$$S_{\beta\alpha} = {\rm disc} + i\mathcal{Z}_\varphi^{-1/2}\lim_{T_x\to\infty}\int_{-T_x}^{T_x} d^4x\ \partial_{x^0}\langle\beta_-|e^{-ip_1\cdot x}\overleftrightarrow{\partial}_{x^0}\varphi_H(x)|\alpha'_+\rangle ,$$

and, using the fact that

$$\begin{aligned} \partial_{x^0}[e^{-ip_1\cdot x}\overleftrightarrow{\partial}_{x^0}\varphi_H(x)] &= e^{-ip_1\cdot x}\partial^2_{x^0}\varphi_H(x) - [\partial^2_{x^0}e^{-ip_1\cdot x}\varphi_H(x) \\ &= e^{-ip_1\cdot x}\partial^2_{x^0}\varphi_H(x) + [(-\nabla^2_\mathbf{x} + M^2_{\rm ph})e^{-ip_1\cdot x}]\varphi_H(x) \\ &\longrightarrow e^{-ip_1\cdot x}\,(\partial^2_x + M^2_{\rm ph})\varphi_H(x) , \end{aligned}$$

where the arrow in the last step means (spatial) integration by parts, we arrive at the result

$$S_{\beta\alpha} = {\rm disc} + i\mathcal{Z}_\varphi^{-1/2}\lim_{T_x\to\infty}\int_{-T_x}^{T_x} d^4x\, e^{-ip_1\cdot x}\,(\partial^2_x + M^2_{\rm ph})\langle\beta_-|\varphi_H(x)|\alpha'_+\rangle . \qquad (13.65)$$

"Reduction" of a particle (with momentum $\mathbf{p}'_1$) from the state $\langle\beta_-|$ would proceed similarly with the result differing from (13.65) by the replacement of $e^{-ip_1\cdot x}$ by $e^{+ip'_1\cdot x}$.

In the second step we "reduce" the next particle (e.g. a one of momentum $\mathbf{p}_2$ from the *in* state) by writing the matrix element in (13.65) in the form

$$\langle\beta_-|\varphi_H(x)|\alpha'_+\rangle = \langle\beta_-|a^\dagger_{\rm out}(\mathbf{p}_2)\varphi_H(x)|\alpha''_+\rangle$$
$$-\langle\beta_-|\left[a^\dagger_{\rm out}(\mathbf{p}_2)\varphi_H(x) - \varphi_H(x)a^\dagger_{\rm in}(\mathbf{p}_2)\right]|\alpha''_+\rangle\ .$$

The first term, if nonzero, contributes to the disconnected part of the $S$ matrix and will be put into the "disc" term. In the second term we again use the formula (8.28) to express $a^\dagger$'ses in terms of the *in* and *out* field operators and use next the weak asymptotic condition (13.31) to get

$$S_{\beta\alpha} = {\rm disc}' + (i\mathcal{Z}_\varphi^{-1/2})^2 \lim_{T_x\to\infty}\int_{-T_x}^{T_x} d^4x\, e^{-ip_1\cdot x}\,(\partial^2_x + M^2_{\rm ph})$$
$$\times\int d^3\mathbf{y}\left(\lim_{y^0\to+\infty}\langle\beta_-|e^{-ip_2\cdot y}\overleftrightarrow{\partial}_{y^0}\varphi_H(y)\varphi_H(x)|\alpha''_+\rangle\right.$$
$$\left. - \lim_{y^0\to-\infty}\langle\beta_-|\varphi_H(x)e^{-ip_2\cdot y}\overleftrightarrow{\partial}_{y^0}\varphi_H(y)|\alpha''_+\rangle\right)$$

If the limits in the big bracket are taken in such a way that $T_y > T_x$, they together can be written as the integral over $dy^0$ with the help of the chronological ordered product:

$$\lim_{T_y\to\infty}\int_{-T_y}^{T_y} d^4y\,\partial_{y^0}\langle\beta_-|T\left[e^{-ip_2\cdot y}\overleftrightarrow{\partial}_{y^0}\varphi_H(y)\varphi_H(x)\right]|\alpha''_+\rangle$$
$$= \lim_{T_y\to\infty}\int_{-T_y}^{T_y} d^4y\,\partial_{y^0}\left(e^{-ip_2\cdot y}\overleftrightarrow{\partial}_{y^0}\langle\beta_-|T\left[\varphi_H(y)\varphi_H(x)\right]|\alpha''_+\rangle\right)\ .$$

Although

$$\partial_{y^0}T[\varphi_H(y)\varphi_H(x)] = \delta(y^0 - x^0)\,[\varphi_H(y),\ \varphi_H(x)] + T[\partial_{y^0}\varphi_H(y)\varphi_H(x)]\ ,$$

the derivative $\overleftrightarrow{\partial}_{y^0}$ was taken out of the chronological product because the term with $\delta(y^0 - x^0)$ vanishes by the canonical equal-time commutation relations. The rest goes as previously: $\partial^2_{y^0}$ acting on $e^{-ip_2\cdot y}$ is replaced by $-\nabla^2_{\mathbf{y}} + M^2_{\rm ph}$ and moved (by parts) onto the matrix element completing the d'Alembertian. The result is

$$S_{\beta\alpha} = {\rm disc}' + (i\mathcal{Z}_\varphi^{-1/2})^2 \lim_{T_y>T_x\to\infty}\int_{-T_x}^{T_x} d^4x\int_{-T_y}^{T_y} d^4y\, e^{-ip_1\cdot x}e^{-ip_2\cdot y}\,(\partial^2_x + M^2_{\rm ph})$$
$$(\partial^2_y + M^2_{\rm ph})\langle\beta_-|T[\varphi_H(y)\varphi_H(x)]|\alpha''_+\rangle\ . \qquad (13.66)$$

The procedure can be continued yielding for the transition amplitude the formula

$$S^{\rm con}(\mathbf{p}'_n,\dots,\mathbf{p}_r,\dots) = (i\mathcal{Z}_\varphi^{-1/2})^n\int d^4x_n\dots\int d^4x_1\, e^{ip'_n\cdot x_n}\dots e^{-ip_1\cdot x_1}$$
$$(\partial^2_{x_n} + M^2_{\rm ph})\dots(\partial^2_{x_1} + M^2_{\rm ph})\langle\Omega|T[\varphi_H(x_n)\dots\varphi_H(x_1)]|\Omega\rangle\ . \qquad (13.67)$$

This is obviously equivalent to the formula (13.39): Integrating in the exponents $e^{-ip_k\cdot x_k}$ and $e^{+ip'_j\cdot x_j}$ puts the external lines of the Fourier transform of the Green's function $\langle\Omega|T[\varphi_H(x_n)\dots\varphi_H(x_1)]|\Omega\rangle$ on-shell, while the differential operators $i(\partial^2_{x_i}+M^2_{\rm ph})$ remove the denominators of the full propagators on its external lines. Finally, the factors $\mathcal{Z}_\varphi^{-1/2}$ combine with $\mathcal{Z}_\varphi$ in the numerators of the propagators giving $\mathcal{Z}_\varphi^{+1/2}$ per each external line as in (13.39).

For charged spin 1/2 particles similar reduction formulae can be easily found using the formulae (8.108). Without going into details we present the resulting formulae

$$i\mathcal{Z}^{-1/2}\int d^4x\,\langle\Omega|T[\dots\bar\psi_H(x)\dots]|\Omega\rangle\,(i\overleftarrow{\not\partial}_x+m_{\rm ph})\,u(\mathbf{p},\sigma)\,e^{-ip\cdot x}$$
$$i\mathcal{Z}^{-1/2}\int d^4x\,\bar v(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,(i\overrightarrow{\not\partial}_x-m_{\rm ph})\,\langle\Omega|T[\dots\psi_H(x)\dots]|\Omega\rangle \qquad (13.68)$$

for a particle and an antiparticle in the *in* state, respectively, and

$$i\mathcal{Z}^{-1/2}\int d^4x\,\bar u(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,(i\overrightarrow{\not\partial}_x-m_{\rm ph})\,\langle\Omega|T[\dots\psi_H(x)\dots]|\Omega\rangle$$
$$i\mathcal{Z}^{-1/2}\int d^4x\,\langle\Omega|T[\dots\bar\psi_H(x)\dots]|\Omega\rangle\,(i\overleftarrow{\not\partial}_x+m_{\rm ph})\,v(\mathbf{p},\sigma)\,e^{+ip\cdot x} \qquad (13.69)$$

for a particle and an antiparticle in the *out* state, respectively. Again, this is equivalent to the formula (13.39): the differential operators together with the $\mathcal{Z}^{-1/2}$ factors remove the external lines propagators leaving $\mathcal{Z}^{+1/2}$ per each fermionic external line and the functions $u$ and $v$ (or $\bar u$ and $\bar v$) give the corresponding factor present in the formula (13.39). Similar steps lead to the corresponding reduction formulae for spin 1 particles etc.

Compared to the momentum space approach of subsection 13.4, the formulation in the position space is less universal as it requires using the specific equations satisfied by the $u$ and $v$ functions appropriate for a given type of particle. The advantage of this formulation is the possibility of using the equations of motion satisfied by the canonical operators $\varphi_H(x)$ or $\psi_H(x)$ in the specific field theory. For example, one can use the $\varphi^4$ theory equations of motion

$$(\partial^2_x+M^2_{\rm ph})\varphi_H(x)=-\frac{\lambda}{3!}\varphi^3_H(x)-(M^2-M^2_{\rm ph})\varphi_H(x)\ ,$$

to explicitly act with the differential operators on the field operators in (13.67). Of course, acting with the time derivatives one must then differentiate also the theta functions present in the definition of the chronological ordering.

The use of the equations of motion finds the typical application in quantum electrodynamics of spin $\frac{1}{2}$ particles where the canonical photon field operator satisfies the equation

$$\partial^2_x A^i_H(x)=eJ^i_{\rm EM}\ , \qquad (13.70)$$