Problems in Quantum Field Theory of Fundamental Interactions II.

Problem II.1
Show that the function $\Delta_+(x,m)$ defined by the integral

$$\Delta_+(x,m)=\int d\Gamma_{\mathbf{k}}\, e^{-ik\cdot x}\equiv\int\frac{d^3\mathbf{k}}{(2\pi)^3 2E(\mathbf{k},m)}\,e^{-ik_\mu x^\mu},$$

in which $E(\mathbf{k},m)=\sqrt{\mathbf{k}^2+m^2}$ is, for $x^2<0$, an even function of x^μ.
Hint: Use the Lorentz invariance of the integral defining $\Delta_+(x)\equiv\Delta_+(x,m)$.

Problem II.2
The functions $u_l(\mathbf{p},\sigma)$ and $v_l(\mathbf{p},\sigma)$ corresponding to definite spin projections on the z-axis in the rest frame of a massive particle which enter the free field (interaction picture) operators

$$\phi_l^{(+)}(x)=\int d\Gamma_{\mathbf{p}}\sum_\sigma u_l(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,a(\mathbf{p},\sigma)\,,$$
$$\phi_l^{(-)}(x)=\int d\Gamma_{\mathbf{p}}\sum_\sigma v_l(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,a^\dagger(\mathbf{p},\sigma)\,,$$

(the indices l transform according to some regular representation $D_{lk}(\Lambda)$ of the Lorentz group) and appear in Feynman rules for initial and and final state of the particle are given in the general case by the formulae

$$u_l(\mathbf{p},\sigma)=\sum_k D_{lk}(L_p)\,u_k(\mathbf{0},\sigma)\,,$$
$$v_l(\mathbf{p},\sigma)=\sum_k D_{lk}(L_p)\,v_k(\mathbf{0},\sigma)\,,$$

in which L_p is the standard Lorentz transformation. Using the results of Problem I.21 give the analogous general formulae for functions $u_l(\mathbf{p},\lambda)$ and $v_l(\mathbf{p},\lambda)$ corresponding to a massive particle of definite helicity (spin projection onto their momentum).

Problem II.3
Construct explicitly the "wave functions" $u_l(\mathbf{p},\sigma)$ and $v_l(\mathbf{p},\sigma)$ entering the (free) field operator $V^\mu(x)$ transforming as a Lorentz vector and constructed out of the creation and annihilation operators of a massive spin 1 particle with definite spin projection σ onto the z-axis (in the particle's rest frame). Construct also the functions $u_l(\mathbf{p},\lambda)$ and $v_l(\mathbf{p},\lambda)$ corresponding to a definite helicity λ of such a particle. (All these u_l and v_l functions corresponding to massive spin 1 parrticles are uually denoted ϵ^μ and $\epsilon^{\mu *}$). Show that the field operator $V^\mu(x)$ written in terms of the creation and annihilation operators corresponding to definite helicities also satisfies the relations $\mathcal{P}V^\mu(x)\mathcal{P}^{-1}=-\eta^* P^\mu_{\ \nu}V^\nu(P\cdot x)$ and $\mathcal{T}V^\mu(x)\mathcal{T}^{-1}=-\zeta^* T^\mu_{\ \nu}V^\nu(T\cdot x)$.

Problem II.4
Using the explicit zero momentum forms of the u and v functions (spinors) corresponding to massive spin 1/2 particles with the spin projection σ onto the z-axis show that

$$\begin{aligned}\bar{u}(\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= \ \ 2m\,\delta_{\sigma\sigma'}\,,\\ \bar{v}(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') &= -2m\,\delta_{\sigma\sigma'}\,,\\ \bar{u}(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') = \bar{v}(\mathbf{p},\sigma')\cdot u(\mathbf{p},\sigma') &= 0\,,\end{aligned}$$

and that

$$\begin{aligned}u^\dagger(\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= 2E_{\mathbf{p}}\,\delta_{\sigma\sigma'}\,,\\ v^\dagger(\mathbf{p},\sigma)\cdot v(\mathbf{p},\sigma') &= 2E_{\mathbf{p}}\,\delta_{\sigma\sigma'}\,,\\ u^\dagger(\mathbf{p},\sigma)\cdot v(-\mathbf{p},\sigma') = v^\dagger(-\mathbf{p},\sigma)\cdot u(\mathbf{p},\sigma') &= 0\,.\end{aligned}$$

Consider both, Dirac and chiral (Weyl) representations. Prove similar relations satisfied by the helicity spinors $u(\mathbf{p},\lambda)$ and $v(\mathbf{p},\lambda)$.

Problem II.5
Construct explicitly the spinors $u(\mathbf{p},\sigma=\pm\frac{1}{2})$, and $v(\mathbf{p},\sigma=\pm\frac{1}{2})$ using the appropriate Lorentz transformation. Give the formulae in both, Dirac and Weyl (chiral), representations of the gamma matrices. Check that up to a phase factor

$$\begin{aligned}u(\mathbf{p},\sigma) &= \frac{\not{p}+m}{\sqrt{2m(E+m)}}\,u(\mathbf{0},\sigma)\,,\\ v(\mathbf{p},\sigma) &= \frac{\not{p}-m}{\sqrt{2m(E+m)}}\,v(\mathbf{0},\sigma)\,.\end{aligned}$$

Problem II.6
Construct explicitly the spinors $u(\mathbf{p},\lambda=\pm\frac{1}{2})$, and $v(\mathbf{p},\lambda=\pm\frac{1}{2})$, corresponding to fermions of definite helicity, using the appropriate Lorentz transformation. Give the formulae in both, Dirac and Weyl (chiral), representations of the gamma matrices.

Problem II.7
Consider the field operator $\psi(x)=\kappa_+\psi^{(+)}(x)+\kappa_-\psi^{(-)}(x)$ of a spin 1/2 Majorana particle (which is its own antiparticle) with

$$\begin{aligned}\psi^{(+)}(x) &= \int d\Gamma_{\mathbf{p}}\sum_\sigma u(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,b(\mathbf{p},\sigma)\,,\\ \psi^{(-)}(x) &= \int d\Gamma_{\mathbf{p}}\sum_\sigma v(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,b^\dagger(\mathbf{p},\sigma)\,.\end{aligned}$$

Show that the local causality requires such particles to be fermions i.e. that it is the anticommutators: $\{\psi(x),\ \psi^\dagger(y)\}$, $\{\psi(x),\ \psi(y)\}$ and $\{\psi^\dagger(x),\ \psi^\dagger(y)\}$ which can be made to vanish for $(x-y)^2 < 0$, provided $|\kappa_+| = |\kappa_-|$ and $b_u = -b_v$ (recall that $b_u = \pm 1$ and $b_v = \pm 1$ are the eigenvalues of the $\beta = \gamma^0$ matrix on the spinors $u(\mathbf{0}, \sigma)$ and $v(\mathbf{0}, \sigma)$, respectively). Show that the intrinsic parity of a Majorana fermion must be $\pm i$.

Problem II.8
Show that the operators

$$\psi_\alpha(x)\,(C^{-1})_{\alpha\beta}\,\psi_\beta(x)\,, \qquad \text{and} \qquad \bar{\psi}_\alpha(x)\,C_{\alpha\beta}\,\bar{\psi}_\beta(x)\,,$$

constructed out of the Majorana spinor field operators associated with a neutral massive spin $\frac{1}{2}$ particle are Lorentz scalars.

Problem II.9 Let us introduce the Feynman's notation: $\not{a} \equiv \gamma^\mu a_\mu$, where a_μ is an arbitrary four-vector. Using the trace property tr(AB)=tr(BA) and the defining relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ show that the following relations hold:

$$\begin{aligned}
&\mathrm{tr}(\not{a}) = 0\,,\\
&\mathrm{tr}(\not{a}\,\not{b}) = 4a_\mu b^\mu\,,\\
&\mathrm{tr}(\not{a}_1\,\not{a}_2 \ldots \not{a}_{2k+1}) = 0\,,\\
&\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) = 4\,[(a\!\cdot\! b)(c\!\cdot\! d) + (a\!\cdot\! d)(c\!\cdot\! b) - (a\!\cdot\! c)(b\!\cdot\! d)]\,,\\
&\mathrm{tr}(\gamma^5) = 0\,,\\
&\mathrm{tr}(\not{a}\,\gamma^5) = 0\,,\\
&\mathrm{tr}(\not{a}\,\not{b}\,\gamma^5) = 0\,,\\
&\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\gamma^5) = 0\,,\\
&\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}\,\gamma^5) = -4i\,\epsilon^{\mu\nu\lambda\rho} a_\mu b_\nu c_\lambda d_\rho\,.
\end{aligned}$$

The conventions are: $\epsilon_{0123} = -1$ and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -(i/4!)\,\epsilon_{\mu\nu\lambda\rho}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho$.

Problem II.10
Show that

$$\begin{aligned}
&\bar{u}(\mathbf{p}_2, \sigma_2)\,[(p_2+p_1)^\mu \mathbf{P} + i\sigma^{\mu\nu}(p_2-p_1)_\nu \mathbf{P}]\,u(\mathbf{p}_1, \sigma_1)\\
&\qquad\qquad = \bar{u}(\mathbf{p}_2, \sigma_2)\,(m_2\gamma^\mu \mathbf{P} + m_1 \mathbf{P}\gamma^\mu)\,u(\mathbf{p}_1, \sigma_1)\,,
\end{aligned}$$

where $p_1^2 = p_2^2 = m_2^2$, $\sigma^{\mu\nu} = (i/2)[\gamma^\mu,\ \gamma^\nu]$ and $\mathbf{P}$ can be 1, γ^5 or one of the projectors $\mathbf{P}_L$, $\mathbf{P}_R$ defined as

$$\mathbf{P}_L = \frac{1}{2}(1-\gamma^5)\,, \qquad \mathbf{P}_R = \frac{1}{2}(1+\gamma^5)\,.$$

Use these results to obtain the standard Gordon's identity

$$\bar{u}(\mathbf{p}_2, \sigma_2)\gamma^\mu u(\mathbf{p}_1, \sigma_1) = \bar{u}(\mathbf{p}_2, \sigma_2)\left[\frac{(p_2+p_1)^\mu}{2m} + \frac{i}{2m}\,\sigma^{\mu\nu}(p_2-p_1)_\nu\right]u(\mathbf{p}_1, \sigma_1)\,,$$

in which $p_1^2 = m^2$, $p_2^2 = m^2$.

Hint: Use the fact that

$$\bar{u}_2\mathbf{A}(\not{p}_1 - m_1)\,u_1 + \bar{u}_2\,(\not{p}_2 - m_2)\mathbf{B}\,u_1 = 0 + 0 = 0\,,$$

for any two matrices **A** and **B** in the spinor space.

Problem II.11
Express $\epsilon^{\mu\nu\lambda\rho}\gamma_\rho$ through the gamma matrices (without the epsilon tensor). Find also a corresponding representation of $\epsilon^{\mu\nu\lambda\rho}\gamma_\lambda\gamma_\rho$. Use the second result to prove that

$$\sigma^{\mu\nu}\gamma^5 \otimes \sigma_{\mu\nu}\gamma^5 = \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}\,,$$
$$\sigma^{\mu\nu}\gamma^5 \otimes \sigma_{\mu\nu} = \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}\gamma^5\,,$$

i.e. that $(\sigma^{\mu\nu}\gamma^5)_{\alpha\beta}(\sigma_{\mu\nu}\gamma^5)_{\alpha'\beta'} = (\sigma^{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})_{\alpha'\beta'}$, etc.

Problem II.12
Show that in d dimensions

$$\gamma^\mu\gamma_\mu = d\,,$$
$$\gamma^\mu\ \not{a}\ \gamma_\mu = (2-d)\ \not{a}\,,$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \gamma_\mu = 4\ a_\mu b^\mu + (d-4)\ \not{a}\ \not{b},$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \not{c}\ \gamma_\mu = -2\ \not{c}\ \not{b}\ \not{a} + (4-d)\ \not{a}\ \not{b}\ \not{c},$$
$$\gamma^\mu\ \not{a}\ \not{b}\ \not{c}\ \not{d}\ \gamma_\mu = 2\ \not{d}\ \not{a}\ \not{b}\ \not{c} + 2\ \not{c}\ \not{b}\ \not{a}\ \not{d} + (d-4)\ \not{a}\ \not{b}\ \not{c}\ \not{d}\,.$$

Problem II.13 (Fierz rearrangement)
Derive the general formula allowing to decompose products of four spinors u_1, $\bar{u}_2$, u_3, $\bar{u}_4$ of the generic form

$$[\bar{u}_4\Gamma u_3]\,[\bar{u}_2\Gamma' u_1]\,,$$

in which Γ and Γ' are two arbitrary matrices, into the sum of products

$$\left[\bar{u}_4\Gamma^M u_1\right]\,\left[\bar{u}_2\Gamma^N u_3\right]\,,$$

where Γ^M form a complete basis of matrices in the spinor space. Write down explicitly rearrangements of the five basic structures

$$[\bar{u}_4 u_3]\,[\bar{u}_2 u_1]\,,\qquad [\bar{u}_4\gamma^\mu u_3]\,[\bar{u}_2\gamma_\mu u_1]\,,\qquad [\bar{u}_4\sigma^{\mu\nu}u_3]\,[\bar{u}_2\sigma_{\mu\nu}u_1]\,,$$
$$\left[\bar{u}_4\gamma^\mu\gamma^5 u_3\right]\,\left[\bar{u}_2\gamma_\mu\gamma^5 u_1\right]\,,\qquad \left[\bar{u}_4\gamma^5 u_3\right]\,\left[\bar{u}_2\gamma^5 u_1\right]\,.$$

Hint: Take for the basis Γ^M the following sixteen linearly independent matrices

$$1\,,\qquad \gamma^\mu\,,\qquad \sigma^{\lambda\rho}\,,\qquad \gamma^\nu\gamma^5\,,\qquad i\gamma^5\,.$$

Problem II.14
Using the general formulae derived in Problem II.13 write down the Fierz rearrangements of the following structures:

$$[\bar{u}_4\gamma^\mu \mathbf{P}_L u_3][\bar{u}_2\gamma_\mu \mathbf{P}_L u_1]\,, \qquad [\bar{u}_4\gamma^\mu \mathbf{P}_R u_3][\bar{u}_2\gamma_\mu \mathbf{P}_R u_1]\,, \qquad [\bar{u}_4\gamma^\mu \mathbf{P}_L u_3][\bar{u}_2\gamma_\mu \mathbf{P}_R u_1]\,,$$

and

$$[\bar{u}_4 \mathbf{P}_L u_3][\bar{u}_2 \mathbf{P}_L u_1]\,, \qquad [\bar{u}_4 \mathbf{P}_R u_3][\bar{u}_2 \mathbf{P}_R u_1]\,, \qquad [\bar{u}_4 \mathbf{P}_R u_3][\bar{u}_2 \mathbf{P}_L u_1]\,.$$

Write down also the corresponding identities for matrix elements (i.e. the identities written without spinors). Using these result express the product $[\bar{u}_4\sigma^{\mu\nu}u_3][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_L u_1]$ through $[\bar{u}_4\sigma^{\mu\nu}u_1][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_L u_3]$ and $[\bar{u}_4\mathbf{P}_L u_1][\bar{u}_2\mathbf{P}_L u_3]$ (and similarly $[\bar{u}_4\sigma^{\mu\nu}u_3][\bar{u}_2\sigma_{\mu\nu}\mathbf{P}_R u_1]$).

Problem II.15
Check that if λ_α transforms as $\lambda'_\alpha = M_\alpha{}^\beta \lambda_\beta$ under Lorentz (or, more precisely, SL(2,C)) transformations, then $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ transforms with the matrix M^{T-1}, i.e. $\lambda'^\alpha = (M^{T-1})^\alpha{}_\beta \lambda^\beta$. Similarly, show that if $\bar{\chi}'_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta}\bar{\chi}_{\dot\beta}$ then $\bar{\chi}^{\dot\alpha} = \bar{\chi}_{\dot\beta}\,\bar{\epsilon}^{\dot\beta\dot\alpha}$ transforms as $\bar{\chi}'^{\dot\alpha} = (M^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\,\bar{\chi}^{\dot\beta}$. Use the convention

$$\epsilon^{12} = -\epsilon^{21} = -1\,, \qquad \bar{\epsilon}^{\dot1\dot2} = -\bar{\epsilon}^{\dot2\dot1} = 1\,,$$
$$\epsilon_{12} = -\epsilon_{21} = -1\,, \qquad \bar{\epsilon}_{\dot1\dot2} = -\bar{\epsilon}_{\dot2\dot1} = 1\,.$$

Problem II.16
Prove by direct calculation the following identities

$$\begin{aligned}
(\sigma^\mu)_{\alpha\dot\beta}\,(\bar\sigma_\mu)^{\dot\gamma\sigma} &= 2\,\delta^\sigma_\alpha\delta^{\dot\gamma}_{\dot\beta}\,,\\
(\sigma^\mu)_{\alpha\dot\beta}\,(\sigma_\mu)_{\gamma\dot\sigma} &= -2\,\epsilon_{\alpha\gamma}\bar\epsilon_{\dot\beta\dot\sigma}\,,\\
(\sigma^\mu)_{\sigma\dot\delta} &= \bar\epsilon_{\dot\delta\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\epsilon_{\beta\sigma}\,,\\
(\bar\sigma^\mu)^{\dot\alpha\beta} &= \epsilon^{\beta\sigma}\,(\sigma^\mu)_{\sigma\dot\delta}\,\bar\epsilon^{\dot\delta\dot\alpha}\,.
\end{aligned}$$

Show then that for two anticommuting spinors (field operators or Grassmann variables)

$$\begin{aligned}
\bar\chi\bar\sigma^\mu\lambda &= -\lambda\sigma^\mu\bar\chi\,,\\
(\chi\sigma^\mu\bar\lambda)^\dagger &= (\lambda\sigma^\mu\bar\chi)\,,
\end{aligned}$$

and finally that

$$\bar\chi_i\bar\sigma^\mu T^a_{ij}\lambda_j = \lambda_i\sigma^\mu(-T^{a*})_{ij}\bar\chi_j\,,$$

where T^a_{ij} are some Hermitian matrices (e.g. generators of an internal symmetry group).

Problem II.17
Check that the matrices M and $(M^\dagger)^{-1}$

$$M \equiv \exp\left(-\frac{i}{2}\,\sigma^k(\eta^k - i\xi^k)\right),$$
$$(M^\dagger)^{-1} \equiv \exp\left(-\frac{i}{2}\,\sigma^k(\eta^k + i\xi^k)\right),$$

of the two inequivalent $SL(2,C)$ representations of the lowest dimension can be written in the following covariant forms

$$M = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\,\frac{1}{2}\sigma^{\mu\nu}_{2\times 2}\right),$$
$$(M^\dagger)^{-1} = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\,\frac{1}{2}\bar{\sigma}^{\mu\nu}_{2\times 2}\right),$$

where

$$(\sigma^{\mu\nu}_{2\times 2})_\alpha^{\ \beta} = \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta},$$
$$(\bar{\sigma}^{\mu\nu}_{2\times 2})^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}_{\ \dot{\beta}}.$$

Find how the parameters η^i and ξ^i are related to the parameters $\omega_{\mu\nu}$. Considering infinitesimal transformations, check by explicit calculation that for $\omega_{\mu\nu}$ related to the Lorentz transformation Λ by $\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}$ the following relations ("canonical" definitions of the twofold covering of the Lorentz group)

$$M(V_\mu\sigma^\mu)M^\dagger = \sigma^\mu\,V'_\mu,$$
$$(M^\dagger)^{-1}(V_\mu\bar{\sigma}^\mu)M^{-1} = \bar{\sigma}^\mu\,V'_\mu,$$

where $V'^\mu = \Lambda^\mu_{\ \nu}V^\nu$, which equivalently can be written in the form

$$\Lambda^\mu_{\ \nu}M\sigma^\nu M^\dagger = \sigma^\mu,$$
$$\Lambda^\mu_{\ \nu}(M^\dagger)^{-1}\bar{\sigma}^\nu M^{-1} = \bar{\sigma}^\mu,$$

are satisfied.[1] These relations imply that if V^μ is a Lorentz four-vector and λ and $\bar{\chi}$ transform as $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of $SL(2,C)$, the quantities

$$V^\mu(\lambda\sigma_\mu\bar{\chi}), \qquad \text{and} \qquad V^\mu(\bar{\chi}\bar{\sigma}_\mu\lambda),$$

transform as scalars.

[1] For infinitesimal parameters $\omega_{\mu\nu}$ there is no ambiguity related to the fact the both M and $-M$ satisfy the same "canonical" definition.

Problem II.18
Check the following identities

$$\mathrm{tr}(\sigma^\mu\bar\sigma^\nu\sigma^\lambda\bar\sigma^\kappa) = 2\left(g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}\right) + 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,$$
$$\mathrm{tr}(\bar\sigma^\mu\sigma^\nu\bar\sigma^\lambda\sigma^\kappa) = 2\left(g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}\right) - 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,$$

and

$$\mathrm{tr}\left(\sigma^{\mu\nu}_{2\times2}\,\sigma^{\lambda\kappa}_{2\times2}\right) = 2\left(g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda}\right) - 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,$$
$$\mathrm{tr}\left(\bar\sigma^{\mu\nu}_{2\times2}\,\bar\sigma^{\lambda\kappa}_{2\times2}\right) = 2\left(g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda}\right) + 2i\,\epsilon^{\mu\nu\lambda\kappa}\,,$$

($\sigma^{\mu\nu}_{2\times2}$ and $\bar\sigma^{\mu\nu}_{2\times2}$ are defined in Problem II.17) and

$$\sigma^\mu\bar\sigma^\kappa\sigma^\lambda + \sigma^\lambda\bar\sigma^\kappa\sigma^\mu = 2\left(g^{\mu\kappa}\sigma^\lambda + g^{\lambda\kappa}\sigma^\mu - g^{\mu\lambda}\sigma^\kappa\right),$$
$$\bar\sigma^\mu\sigma^\kappa\bar\sigma^\lambda + \bar\sigma^\lambda\sigma^\kappa\bar\sigma^\mu = 2\left(g^{\mu\kappa}\bar\sigma^\lambda + g^{\lambda\kappa}\bar\sigma^\mu - g^{\mu\lambda}\bar\sigma^\kappa\right),$$

and, finally, that

$$\sigma^\mu\bar\sigma^\kappa\sigma^\lambda - \sigma^\lambda\bar\sigma^\kappa\sigma^\mu = \ \ 2i\,\epsilon^{\mu\kappa\lambda\nu}\sigma_\nu\,,$$
$$\bar\sigma^\mu\sigma^\kappa\bar\sigma^\lambda - \bar\sigma^\lambda\sigma^\kappa\bar\sigma^\mu = -2i\,\epsilon^{\mu\kappa\lambda\nu}\bar\sigma_\nu\,.$$

(Recall that we use $\epsilon^{0123} = +1$).

Problem II.19
Decomposing four-component Grassman algebra valued (anticommuting) spinors/fermionic field operators ψ_i into two-component ones (using the appropriate representation of the gamma matrices) prove the following relations

$$\begin{aligned}\left(\bar\psi_1\gamma^\mu\mathbf{P}_L\psi_2\right)\left(\bar\psi_3\gamma_\mu\mathbf{P}_L\psi_4\right) &= \left(\bar\psi_1\gamma^\mu\mathbf{P}_L\psi_4\right)\left(\bar\psi_3\gamma_\mu\mathbf{P}_L\psi_2\right)\\ &= 2\left(\bar\psi_1\mathbf{P}_R\psi^c_3\right)\left(\bar\psi^c_2\mathbf{P}_L\psi_4\right) = 2\left(\bar\psi_3\mathbf{P}_R\psi^c_1\right)\left(\bar\psi^c_2\mathbf{P}_L\psi_4\right)\\ &= 2\left(\bar\psi_1\mathbf{P}_R\psi^c_3\right)\left(\bar\psi^c_4\mathbf{P}_L\psi_2\right) = 2\left(\bar\psi_3\mathbf{P}_R\psi^c_1\right)\left(\bar\psi^c_4\mathbf{P}_L\psi_2\right),\end{aligned}$$

a similar one with $\mathbf{P}_L \leftrightarrow \mathbf{P}_R$ and

$$\begin{aligned}\left(\bar\psi_1\gamma^\mu\mathbf{P}_L\psi_2\right)\left(\bar\psi_3\gamma_\mu\mathbf{P}_R\psi_4\right) &= -2\left(\bar\psi_1\mathbf{P}_R\psi_4\right)\left(\bar\psi_3\mathbf{P}_L\psi_2\right) = -2\left(\bar\psi_1\mathbf{P}_R\psi_4\right)\left(\bar\psi^c_2\mathbf{P}_L\psi^c_3\right)\\ &= -2\left(\bar\psi^c_4\mathbf{P}_R\psi^c_1\right)\left(\bar\psi_3\mathbf{P}_L\psi_2\right) = -2\left(\bar\psi^c_4\mathbf{P}_R\psi^c_1\right)\left(\bar\psi^c_2\mathbf{P}_L\psi^c_3\right).\end{aligned}$$

in which

$$\psi^c = C\bar\psi \equiv C\gamma^0\psi^*\,,$$

is the charge conjugated spinor. Write down explicitly the corresponding matrix identities involving the charge conjugation matrix C. Compare these formulae to the ones obtained in Problem II.14.

Prove also the following relations involving the C matrix:

$$(\gamma^\mu \mathbf{P}_L C)_{\alpha\beta} (C\gamma^\mu \mathbf{P}_L)_{\alpha'\beta'} = -2 (\mathbf{P}_R)_{\alpha\alpha'} (\mathbf{P}_L)_{\beta\beta'} \, ,$$
$$(\gamma^\mu \mathbf{P}_R C)_{\alpha\beta} (C\gamma^\mu \mathbf{P}_R)_{\alpha'\beta'} = -2 (\mathbf{P}_L)_{\alpha\alpha'} (\mathbf{P}_R)_{\beta\beta'} \, ,$$

and

$$(\gamma^\mu \mathbf{P}_L C)_{\alpha\beta} (C\gamma_\mu \mathbf{P}_R)_{\alpha'\beta'} = - (\gamma^\mu \mathbf{P}_L)_{\alpha\alpha'} (\gamma_\mu \mathbf{P}_R)_{\beta\beta'} \, ,$$
$$(\gamma^\mu \mathbf{P}_R C)_{\alpha\beta} (C\gamma_\mu \mathbf{P}_L)_{\alpha'\beta'} = - (\gamma^\mu \mathbf{P}_R)_{\alpha\alpha'} (\gamma_\mu \mathbf{P}_L)_{\beta\beta'} \, .$$

Hint: Show that if

$$\psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad \text{then} \qquad \psi^c = C\gamma^0\psi^\dagger = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} .$$

Problem II.20
Decomposing anticommuting four-component spinors ψ_i into two-component (Weyl) spinors show that

$$\left(\bar{\psi}_1 \mathbf{P}_L \psi_2\right) \left(\bar{\psi}_3 \mathbf{P}_L \psi_4\right) = -\frac{1}{2} \left(\bar{\psi}_1 \mathbf{P}_L \psi_4\right) \left(\bar{\psi}_3 \mathbf{P}_L \psi_2\right) - \frac{1}{8} \left(\bar{\psi}_1 \sigma_{\mu\nu} \mathbf{P}_L \psi_4\right) \left(\bar{\psi}_3 \sigma^{\mu\nu} \mathbf{P}_L \psi_2\right) ,$$
$$\left(\bar{\psi}_1 \mathbf{P}_R \psi_2\right) \left(\bar{\psi}_3 \mathbf{P}_R \psi_4\right) = -\frac{1}{2} \left(\bar{\psi}_1 \mathbf{P}_R \psi_4\right) \left(\bar{\psi}_3 \mathbf{P}_R \psi_2\right) - \frac{1}{8} \left(\bar{\psi}_1 \sigma_{\mu\nu} \mathbf{P}_R \psi_4\right) \left(\bar{\psi}_3 \sigma^{\mu\nu} \mathbf{P}_R \psi_2\right) ,$$

where $\sigma^{\mu\nu} \equiv (i/2)\,[\gamma^\mu,\ \gamma^\nu]$. Use the relations (see Problem II.16) $(\sigma^\mu)_{\alpha\dot{\beta}}(\sigma_\mu)_{\rho\dot{\kappa}} = -2\epsilon_{\alpha\rho}\epsilon_{\dot{\beta}\dot{\kappa}}$, $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\bar{\sigma}_\mu)^{\dot{\rho}\kappa} = -2\epsilon^{\dot{\alpha}\dot{\rho}}\epsilon^{\beta\kappa}$, etc.

Using these results prove also the following identities

$$(\mathbf{P}_L C)_{\alpha\beta} (C\mathbf{P}_L)_{\alpha'\beta'} = \frac{1}{2} (\mathbf{P}_L)_{\alpha\alpha'} (\mathbf{P}_L)_{\beta\beta'} - \frac{1}{8} (\sigma^{\mu\nu})_{\alpha\alpha'} (\sigma_{\mu\nu}\mathbf{P}_L)_{\beta\beta'} \, ,$$
$$(\mathbf{P}_R C)_{\alpha\beta} (C\mathbf{P}_R)_{\alpha'\beta'} = \frac{1}{2} (\mathbf{P}_R)_{\alpha\alpha'} (\mathbf{P}_R)_{\beta\beta'} - \frac{1}{8} (\sigma^{\mu\nu})_{\alpha\alpha'} (\sigma_{\mu\nu}\mathbf{P}_R)_{\beta\beta'} \, .$$

Hint: To derive the first two identities start with the last term on the right and use the summation rules for the sigma matrices derived in Problem II.16. Notice that for any four left-chiral Weyl spinors ψ, χ, λ and φ the following identity

$$(\psi\chi)(\lambda\varphi) + (\psi\varphi)(\lambda\chi) + (\psi\lambda)(\varphi\chi) = 0 \, .$$

must hold.

Problem II.21
Show that the operator (a "current")

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \, ,$$

is a four-vector with respect to Lorentz transformations. How does $j^\mu(x)$ transform under the parity, time reversal and charge conjugation operations, i.e. to what equal $\mathcal{P}j^\mu(x)\mathcal{P}^{-1}$,

$\mathcal{T} j^\mu(x)\mathcal{T}^{-1}$ and $\mathcal{C} j^\mu(x)\,\mathcal{C}^{-1}$ respectively? Assume that with respect to Lorentz, P, C and T transformations $\psi(x)$ behaves as the free-field operator of a spin 1/2 particle. Show also that if $\psi(x)$ is a free-field operator, $j^\mu(x)$ satisfies the continuity equation (is conserved) $\partial_\mu j^\mu = 0$. Show (formally) that in QED $j^\mu(x)$ is conserved also as a Heisenberg picture operator.

Problem II.22
Knowing that for a massless particle

$$\mathcal{P}|\mathbf{p},\lambda\rangle = \eta_\lambda\, e^{\pm i\pi\lambda}|-\mathbf{p},-\lambda\rangle\,,$$
$$\mathcal{T}|\mathbf{p},\lambda\rangle = \zeta_\lambda\, e^{\mp i\pi\lambda}|-\mathbf{p},\lambda\rangle\,,$$

(λ is the particle helicity) find how the free field operator $A^\mu(x)$ of massless spin 1 particles transforms under parity and time reversal operations. What must be the photon intrinsic parity and charge conjugation properties if the photon interaction $\mathcal{H}_{\rm int}(x) = e\,QA^\mu(x)\,j_\mu(x)$, with the current $j_\mu(x)$ which has the transformation properties established in Problem II.21, is to conserve parity and to be charge conjugation invariant?

Problem II.23
Generalize the results of Problems II.21 and II.22 to operators $A^a_\mu(x)$ of nonabelian gauge theories which couple to fermions through the interaction

$$\mathcal{H}_{\rm int} = g\,A^a_\mu\,\bar\psi_i\gamma^\mu T^a_{ij}\,\psi_j,$$

where T^a_{ij} are Hermitian generators of a compact group like $SU(N)$ or $SO(N)$. How does then transform the field strength operator

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\,,$$

in which f^{abc} are the group structure constants?

Find also the P, C and T transformation properties of the additional interaction term

$$\mathcal{H}'_{\rm int} = \kappa\,\epsilon^{\mu\nu\lambda\rho}F^b_{\mu\nu}F^b_{\lambda\rho}\,,$$

in which κ is a (real) coupling constant. Although $\mathcal{H}'_{\rm int}$ is a total four-divergence and does not affect physical predictions of Abelian gauge theories like QED, it becomes important in nonanbelian gauge theories.[2]

Problem II.24
Consider the interaction term

$$\mathcal{H}_{\rm int} = \varphi\,\bar\psi(c\,\mathbf{P}_L + c^*\mathbf{P}_R)\psi\,,$$

[2] The presence of such an interaction term in the Lagrangian of QCD would lead to violation of the CP-invariance by the strong interactions; what makes the coefficient κ in $\mathcal{H}'_{\rm int}$ smaller than $\sim 10^{-9}$ (as required by the experimental data) is the essence of the so-called *strong CP problem*.

in which φ and ψ are the field operators of a neutral spin 0 particle and of charged spin 1/2 fermion and its antiparticle, respectively. Check that $\mathcal{H}_{\rm int}$ is Hermitian. Using the transformation rules of the field operators φ, ψ and $\bar{\psi}$ under parity, time reversal and charge conjugation formulate conditions on the coefficient c which ensure that P, T, C, CP, CT, PT and CPT, respectively can be[3] good symmetries of the interaction. In particular, check that if the phase factors η, ξ and ζ are chosen so that

$$\mathcal{CPT}\mathcal{H}_{\rm int}(x)(\mathcal{CPT})^{-1} = \mathcal{H}_{\rm int}(-x)$$

(i.e. so that CPT is a good symmetry), the conditions ensuring invariance of $\mathcal{H}_{\rm int}$ under CP are equivalent to those ensuring its T-invariance.

Problem II.25
Consider the Hamiltonian term describing interactions of spin zero particles which are not their own antiparticles with fermions and their antifermions:

$$\mathcal{H}_{\rm int} = \phi\,\bar{\psi}(c_L\,\mathbf{P}_L + c_R\,\mathbf{P}_R)\psi + \text{H.c.}$$

Interaction of this form effectively model decays of neutral mesons K^0, $\overline{K}^0$ or $B^0(B^0_s)$, $\overline{B}^0(\overline{B}^0_s)$ into a pair $\ell^-\ell^+$ (see also Problem V.12). Write down the H.c. part. Formulate conditions on the coefficients c_L and c_R which ensure that P, T, C, CP, CT, PT and CPT, respectively are good symmetries of the interaction $\mathcal{H}_{\rm int}$. Check that if CPT is a good symmetry, the conditions ensuring invariance of $\mathcal{H}_{\rm int}$ under CP are equivalent to those ensuring its invariance under T.

Problem II.26
Do the same for the interaction

$$\mathcal{H}_{\rm int} = \phi\,\bar{\psi}^{(a)}_i(c^{ij}_L\,\mathbf{P}_L + c^{ij}_R\,\mathbf{P}_R)\psi^{(b)}_j + \text{H.c.}$$

in which ϕ is the operator of spin 0 particles carrying a conserved charge and $\psi^{(a)}_i$, and $\psi^{(b)}_i$ with $i = 1, \dots, N$ are field operators of $2N$ species of fermions. Notice, that $\psi^{(a)}_i$ and $\psi^{(b)}_i$ must carry different charges.

Problem II.27
Assume the interaction of a massive neutral vector boson (spin 1 particle) with two fermions in the general form

$$\mathcal{H}_{\rm int} = V_\mu\,\bar{\psi}_1\gamma^\mu\,(c_L\,\mathbf{P}_L + c_R\,\mathbf{P}_R)\,\psi_2 + \text{H.c.}$$

For $\psi_1 = \psi_2$ this describes the interactions of the neutral massive Z^0 vector boson with fermions. For $\psi_1 \neq \psi_2$ this can be considered an effective description of loop-induced decays of Z^0 into $b\bar{s}$ and $s\bar{b}$ (etc.) quark pairs. Write down the H.c. part of $\mathcal{H}_{\rm int}$.

[3]Whether they really are symmetries depends on other interaction terms involving ψ and $\bar{\psi}$ which may be present in the complete interaction $\mathcal{H}_{\rm int}$.

What are the conditions on c_L and c_R ensuring that $\mathcal{H}_{\rm int}$ respects P, C, T, CP and CPT symmetries?

Problem II.28
The same as in preceding Problems but for $V^{\mu\dagger} \neq V^\mu$ and the interaction

$$\mathcal{H}_{\rm int} = V_\mu \, \bar{\psi}_i^{(a)} \gamma^\mu \left(c_L^{ij} \, \mathbf{P}_L + c_R^{ij} \, \mathbf{P}_R \right) \psi_j^{(b)} + \text{H.c.}$$

where c_L^{ij} and c_R^{ij} with $i, j = 1, \dots, N$, are arbitrary complex matrices, and V_μ carry a conserved quantum number. Notice that similarly as in Problem II.26 $\psi_i^{(a)}$ and $\psi_i^{(b)}$ must carry different quantum numbers. How does the condition for CP violation looks like, if one of these matrices vanishes? What if, as in the interactions of the $W^\pm$ vector bosons, the nonzero coupling (c_L^{ij} in the $W^\pm$ case) is proportional to a unitary matrix?

Problem II.29
Consider the interaction

$$\mathcal{H}_{\rm int} = \bar{\psi}_i \sigma^{\mu\nu} \left(c_L^{ij} \, \mathbf{P}_L + c_R^{ij} \, \mathbf{P}_R \right) \psi_j \, f_{\mu\nu} \, ,$$

in which ψ_i are N charged (Dirac) fermion spinor fields and $f^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor. Loop-induced interaction of this type describes e.g. the celebrated decay of the b-quark into the s-quark and the photon (see Problems V.32 and V.33). What condition imposes on the matrices c_L^{ij} and c_R^{ij} hermiticity of the interaction? Write down the conditions imposed on this interaction by P-, C-, T-, CP- and CPT-invariance. Show that CPT-invariance makes the conditions for CP- and T-invariance equivalent to each other.

Problem II.30
Consider the interaction

$$\mathcal{H}_{\rm int} = \partial^\mu \phi \, \bar{\psi}_1 \gamma_\mu \left(c_L \, \mathbf{P}_L + c_R \, \mathbf{P}_R \right) \psi_2 + \text{H.c.}$$

generalizing the effective description of the interactions of charged $\pi^\pm$ pions with the $\ell\nu$ pairs (see Problem V.13). Write down the H.c. part explicitly. Investigate conservation of P, C, T, etc. by this interaction.

Problem II.31
Obtain the explicit form of the Feynman propagator $iS^{\rm F}_{\alpha\beta}(x-y)$ of a massive spin 1/2 particle directly from the defining formula

$$iS^{\rm F}_{\alpha\beta}(x-y) = \theta(x^0 - y^0) \left\{ \psi_\alpha^{(+)}(x), \; \bar{\psi}_\beta^{(-)}(y) \right\} - \theta(y^0 - x^0) \left\{ \bar{\psi}_\beta^{(+)}(y), \; \psi_\alpha^{(-)}(x) \right\} .$$

Problem II.32
Write down the most general, consistent with the Poincaré symmetry, form of the matrix element of an operator $J^\mu(x)$ (transforming as a four-vector under proper changes of the

Lorentz frame) between two one-particle states of spinless and between two states of spin zero and spin $\frac{1}{2}$ massive particles. Investigate the constraints imposed on the independent formfactors by hermiticity of the operator $J^\mu(x)$ and/or by the P- and T-invariance of the underlying interaction assuming that the operator $J^\mu(x)$ has well defined transformation properties under P and T. In particular, show that if the intrinsic parities of the two particles are equal, the matrix element of an axial vector operator $A^\mu(x)$ (i.e. the operator transforming under parity according to the rule $\mathcal{P}\,A^\mu(x)\,\mathcal{P}^{-1} = -P^\mu{}_\nu A^\nu(P\cdot x)$) must vanish if the particles are spinless but does not need to vanish if thy are spin $\frac{1}{2}$ ones.

Problem II.33

Using the results of Problem II.32 discuss in detail matrix elements of the electromagnetic current $J^\mu_{\rm EM}(x)$ between two one-particle states of the same particle taking into account conservation of $J^\mu_{\rm EM}(x)$. Consider the cases of $J^\mu_{\rm EM}(x)$ in a theory which (like QED+QCD) respects parity and time reversal and in a theory which (like the Standard Theory) is not invariant with respect to P and T (and, hence, CP). Using the formula

$$\langle \mathbf{p}_2, \sigma_2|\, e\int d^3\mathbf{x}\, J^\mu_{\rm EM}(x) A_\mu(\mathbf{x})|\mathbf{p}_1, \sigma_1\rangle = \Delta E\, \langle \mathbf{p}_2, \sigma_2|\mathbf{p}_1, \sigma_1\rangle\,,$$

($e > 0$ is the elementary charge), which should be understood in the limit $\mathbf{p}_2 \to \mathbf{p}_1$, for the correction ΔE to the energy of the particle in an applied static and uniform[4] external (c-number) electromagnetic field $A_\mu(\mathbf{x})$, relate the values of the formfactors at zero momentum transfer, $q^\mu = (p_2 - p_1)^\mu \to 0$, to the static (i.e. for $\mathbf{p}_1 \to 0$) characteristics of the particle like charge, magnetic moment, etc. In particular show that if P- and T- (i.e. CP-) invariance of the underlying dynamics is not assumed, two additional formfactors are allowed of which one corresponds to a nonzero electric dipole moment of the particle and the other one to its anapole moment.[5]

Problem II.34

Find the most general possible form of the matrix element of the electromagnetic current operator $J^\mu_{\rm EM}(x)$ between two one-particle states of the same spin $\frac{1}{2}$ Majorana particle which is its own CPT conjugate.[6]

---

[4] The applied electric and magnetic fields must be static and uniform in order that the one-particle states $|\mathbf{p}, \sigma\rangle$ remain eigenstates (with shifted energy) of the full Hamiltonian. The formula is then just the ordinary formula of quantum mechanics (applied here to non-normalizable states $|\mathbf{p}, \sigma\rangle$) for the first order correction induced by the perturbation $H_{\rm pert} = e\int d^3\mathbf{x}\, J^\mu_{\rm EM}(0, \mathbf{x}) A_\mu(\mathbf{x})$.

[5] There is an important conceptual difference between theories (like pure QCD) with respect to which the current $J^\mu_{\rm EM}$ is an "external conserved current" (a current of a global symmetry of such a theory) and theories like QED or the Standard Theory (of which QED is a part) in which to the canonical symmetry current $J^\mu_{\rm EM}$ couples a dynamical photon field. In the first class of theories the current operator does not need any renormalization and the proposed reasoning is directly applicable. In the second class of theories there are in fact two different conserved current operators: the true global symmetry current indeed does not renormalize, but the canonical one, $J^\mu_{\rm EM}$, does and mixes under renormalization with the operator $\partial_\nu f^{\nu\mu}$. This renormalization is in this proposed problem to some extent trivial and can be removed by including it in the renormalization of the electric charge which creates the external field A^μ. The proposed reasoning becomes then justified as in the first class of theories.

[6] A C-invariant theory admits existence of essentially neutral particles which are their own conjugates

Problem II.35
Consider the second quantized formulation of the theory of relativistic spin $\frac{1}{2}$ particles (and their antiparticles) of mass m and electric charge Q coupled to an external static electromagnetic field via

$$\hat{H}_{\rm pert} = e\int d^3\mathbf{x}\, J^{\mu}_{\rm EM}(\mathbf{x})A_{\mu}(\mathbf{x})\,,$$

but not interacting with one another. Expanding in powers of $1/c$ the matrix element between the one-particle states of the Hamiltonian of this system and treating it as the matrix element of a nonrelativistic one-particle Hamiltonian, find relativistic corrections to the ordinary quantum mechanical Hamiltonian of a spin $\frac{1}{2}$ particle moving in an external electromagnetic field. In particular obtain in this way the well-known Darwin term and the spin-orbit interaction term (as well as the first correction arising from the expansion of the relativistic kinetic energy) which usually are derived with the help of the Foldy-Wouthuysen transformation applied to the Dirac equation describing electron in an external (static) electromagnetic field.

Problem II.36
Using the relativistic corrections to the (nonrelativistic) Pauli Hamiltonian of a spin $\frac{1}{2}$ particle in an external electromagnetic field derived in Problem II.35, compute the first order relativistic corrections to the spectrum of the Hydrogen-like atom.
Hint: Derive first the Kramers recurrence relation

$$2E_n(s+1)\langle\hat{r}^s\rangle + Z|Q|e^2(2s+1)\langle\hat{r}^{s-1}\rangle + \frac{\hbar^2}{m}\left[\frac{s^2-1}{4} - l\,(l+1)\right] s\,\langle\hat{r}^{s-2}\rangle = 0\,,$$

relating expectation values of the operator r^s in the unperturbed energy eigenstates $|n,l,m_l\rangle$ of the Hydrogen-like atom Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + \frac{ZQe^2}{\hat{r}} \equiv \frac{\hat{\mathbf{P}}^2}{2m} - \frac{Z|Q|e^2}{\hat{r}}\,,$$

(Q must be negative for bound states to exist). This recurrence relation has a "hole" which does not allow for immediate calculation of $\langle\hat{r}^s\rangle$ for $s\leq -2$. Find a quick method of computing $\langle\hat{r}^{-2}\rangle$ and use it in the Kramers relation to obtain $\langle\hat{r}^{-3}\rangle$.

Problem II.37
Find the explicit form of the function $\Delta_+(x;m)$ for $m\neq 0$. Using this result find the $x^2\sim 0$ behaviour of the propagator $i\Delta_F(x) = \langle\Omega_0|T[\varphi_I(x)\varphi_I(y)]|\Omega_0\rangle$ of a free spinless particle and of the commutator $[\varphi_I(x),\ \varphi_I(0)]$ of the interaction picture (free) scalar field operators.
Hint: In the Fourier integral representation of $\Delta_+(x;m)$ substitute $|\mathbf{k}| = \mathrm{sh}\,\theta$ and consult Ryzhik-Gradstein for integral and power series representations of the Bessel functions (reproduced here in the Appendix).

---

under the C-operation. True electroweak interactions are not C-invariant but since CPT is always a good symmetry of relativistically invariant quantum field theories, each particle state must have its CPT partner.

Appendix. Ryzhik & Gradstein formulae for the Bessel functions

8.421. For $x > 0$, $-1 < \mathrm{Re}\, z < 1$

$$1. \qquad H_\nu^{(1)}(x) = \frac{e^{-\frac{i}{2}\nu\pi}}{i\pi} \int_{-\infty}^{+\infty} dt\, e^{-\nu t + ix\,\mathrm{ch} t}\,,$$

$$2. \qquad H_\nu^{(2)}(x) = -\frac{e^{\frac{i}{2}\nu\pi}}{i\pi} \int_{-\infty}^{+\infty} dt\, e^{-\nu t - ix\,\mathrm{ch} t}\,,$$

8.405.

$$1. \qquad H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)\,,$$

$$2. \qquad H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z)\,.$$

8.432. For $x > 0$, $-1 < \mathrm{Re}\, z < 1$

$$4. \qquad K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}} \int_0^\infty dt\, \mathrm{ch}\, \nu t\, \cos(x\, \mathrm{sh}\, t)\,.$$

8.402. For $|\arg z| < \pi$

$$J_\nu(z) = \sum_{k=0}^\infty (-1)^k \frac{z^{\nu+2k}}{2^{\nu+2k}\, k!\, \Gamma(1+\nu+k)}\,.$$

8.403. For $|\arg z| < \pi$

$$2. \qquad \pi N_n(z) = 2\left(\ln\frac{z}{2}\right) J_n(z) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - \sum_{k=0}^\infty (-1)^k \frac{1}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n} [\psi(k+1) + \psi(k+n+1)]\,.$$

8.446. For ν integer, $\nu = n$,

$$K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - (-1)^n \sum_{k=0}^\infty \frac{1}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n} \left[\ln\frac{z}{2} - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(k+n+1)\right].$$