BESSEL POTENTIALS AND GREEN FUNCTIONS ON PSEUDO-EUCLIDEAN SPACES

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We review properties of Bessel potentials, that is, inverse Fourier transforms of (regularizations of) $(m^2+p^2)^{-\frac{\mu}{2}}$ on a pseudo-Euclidean space with signature $(q, d-q)$. We are mostly interested in the Lorentzian signature $(1, d-1)$, and the case $\mu = 2$, related to the Klein–Gordon equation $(-\Box + m^2)f = 0$. We analyze properties of various "propagators", which play an important role in quantum field theory, such as the retarded/advanced propagators or Feynman/anti-Feynman propagators. We consistently use hypergeometric functions instead of Bessel functions, which makes most formulae much more transparent. We pay attention to distributional properties of various Bessel potentials. We include in our analysis the "tachyonic case", corresponding to the "wrong" sign in the Klein–Gordon equation.

Keywords: Bessel potential, Riesz potential, Green function, Klein–Gordon equation, pseudo-Euclidean spaces, Minkowski space.

1. Introduction

Let us start with the Bessel potentials on the Euclidean space $\mathbb{R}^d$. Let $\operatorname{Re}\mu > 0$ and $m \geq 0$. If $m = 0$ we will usually additionally assume that $d > \operatorname{Re}\mu$. Consider the function

$$G_{\mu,m}(x) = \int \frac{e^{ipx}}{(m^2 + p^2)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{1.1}$$

on the Euclidean space $\mathbb{R}^d$. Note that $G_{\mu,m}(x-y)$ can be interpreted as the integral kernel of the operator $(m^2 - \Delta)^{-\frac{\mu}{2}}$.

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We have

$$G_{\mu,m}(x) = m^{d-\mu} G_{\mu,1}(mx), \tag{1.2}$$

so the case $m > 0$ reduces to $m = 1$. $G_{\mu,1}(x)$ can be expressed in terms of the *Macdonald function*, one of solutions of the *modified Bessel equation*. Therefore, $G_{\mu,1}(x)$ is often called the *Bessel potential* of order μ. The function $G_{\mu,0}(x)$ is called the *Riesz potential* of order μ.

It is remarkable that the theory of Bessel potentials is very similar for all $\mu > 0$. However, the case $\mu = 2$ is probably the most important. In this case we will usually omit μ from the notation, setting $G_m(x) := G_{2,m}(x)$, and obtaining the *Green function* of the inhomogeneous *Helmholtz equation*

$$(-\Delta + m^2) g(x) = f(x). \tag{1.3}$$

In other words,

$$(-\Delta + m^2) G_m(x) = \delta(x). \tag{1.4}$$

Note that in dimension $d = 3$ we have

$$G_m(x) = \frac{e^{-m|x|}}{4\pi|x|}.$$

Thus for $m > 0$ it coincides with the *Yukawa potential* and for $m = 0$ with the *Coulomb potential*.

Suppose now $\mathbb{R}^{q,d-q}$ is the *pseudo-Euclidean space of signature* $(q, d-q)$. In other words, as a set it is $\mathbb{R}^d$ with the scalar product for $x, y \in \mathbb{R}^{q,p}$ given by

$$xy = -x_1 y_1 \cdots - x_q y_q + x_{q+1} y_{q+1} + \cdots + x_d y_d. \tag{1.5}$$

The definition (1.1) is usually no longer correct for $m^2 \in \mathbb{R}$, since $\frac{1}{(m^2+p^2)^{\frac{\mu}{2}}}$ may fail to be locally integrable, and hence may not define a tempered distribution. It still works for complex nonreal m^2. A possible pair of generalizations of (1.1) to m^2 real is the pair of functions, which correspond to the limits from above and below:

$$G^{\mathrm{F}}_{\mu,m}(x) = \int \frac{e^{ipx}}{(m^2 + p^2 - i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d}, \tag{1.6}$$

$$G^{\overline{\mathrm{F}}}_{\mu,m}(x) = \int \frac{e^{ipx}}{(m^2 + p^2 + i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d}. \tag{1.7}$$

Formulae (1.6) and (1.7) have an obvious interpretation as boundary values of integral kernels of appropriate functions of the pseudoLaplacian

$$\Box := -\partial_1^2 \cdots - \partial_q^2 + \partial_{q+1}^2 + \partial_d^2. \tag{1.8}$$

Again, the case $m > 0$ reduces to $m = 1$. $G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x)$ can be expressed by Macdonald and Hankel functions. (The Hankel functions are special functions solving the standard Bessel equation.)

The symbols F and $\overline{\mathrm{F}}$ are motivated by the special case of Green functions in the Lorentzian case. $G_{2,m}^{\mathrm{F}/\overline{\mathrm{F}}}(x)$ coincide then with the *Feynman*, resp. the *anti-Feynman propagators*, which play an important role in quantum field theory, as we explain below.

In our paper we will discuss all signatures, including the *Euclidean* $(0, d)$ and *anti-Euclidean* $(d, 0)$. However, we are mostly interested in the *Lorentzian signature*. The Lorentzian signature comes in two varieties: "mostly pluses" $(1, d-1)$ and "mostly minuses" $(d-1, 1)$. We will treat the former as the standard one.

The Lorentzian case is especially interesting and rich. This is related to the fact that the *Minkowski space* $\mathbb{R}^{1,d-1}$ can be equipped with a causal structure and the set $p^2 + m^2 = 0$ has two connected components. Therefore, besides $G_{\mu,m}^{\mathrm{F}/\overline{\mathrm{F}}}$, we can introduce the distributions

$$G_{\mu,m}^{\vee}(x) = \int \frac{e^{ipx}}{(m^2 + p^2 - i0\mathrm{sgn}\, p^0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d}, \tag{1.9}$$

$$G_{\mu,m}^{\wedge}(x) = \int \frac{e^{ipx}}{(m^2 + p^2 + i0\mathrm{sgn}\, p^0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d}, \tag{1.10}$$

which are invariant wrt orthochronous Lorentz transformations. Remarkably, $G_{\mu,m}^{\vee/\wedge}$ is supported in the forward, resp. backward cone. Therefore, $G_{\mu,m}^{\vee}$ is called the *forward (or retarded)*, and $G_{\mu,m}^{\wedge}$ the *backward (or advanced) Bessel potential*.

In the Lorentzian case, the pseudo-Laplacian is usually called the *d'Alembertian*

$$\Box := -\partial_0^2 + \partial_1^2 + \cdots + \partial_{d-1}^2, \tag{1.11}$$

and $-\Box + m^2$ is called the *Klein–Gordon operator*. By a *Green function* of the (inhomogeneous) *Klein–Gordon equation*

$$(-\Box + m^2) f(x) = g(x). \tag{1.12}$$

we will mean a distribution $G^{\bullet}(x)$ satisfying

$$(-\Box + m^2) G^{\bullet}(x) = \delta(x). \tag{1.13}$$

The Klein–Gordon equation possesses many Green functions. Among them, we have the Feynman and anti-Feynman Green functions given by the formulae (1.6) and (1.7) with $\mu = 2$. Another distinguished pair consists of the retarded (or forward) Green function and the advanced (or backward) Green function, defined by demanding that their support is contained in the forward, resp. backward cone. For $m^2 \geq 0$ the retarded Green function is given by (1.9) and the advanced Green function by (1.10) with $\mu = 2$.

The Feynman, anti-Feynman, forward, and backward Green functions of the Klein–Gordon equation have important applications in physics, especially in classical and quantum field theory. The forward and backward Green functions can be used to express the Cauchy problem. The Feynman, resp. anti-Feynman Green functions express the time-ordered, resp. anti-time-ordered vacuum expectation values of fields in quantum field theory. Importantly, they satisfy the identity

$$G_m^{\mathrm{F}} + G_m^{\overline{\mathrm{F}}} = G_m^{\vee} + G_m^{\wedge}. \tag{1.14}$$

In our paper, we also consider the Lorentzian case with the “wrong sign of m^2”. This case corresponds to the *tachyonic* Klein–Gordon equation

$$(-\Box - m^2)f(x) = g(x). \tag{1.15}$$

Remarkably, all four basic Green functions, Feynman G_m^{F}, anti-Feynman $G_m^{\overline{\mathrm{F}}}$, forward $G_m^{\vee}$, and backward $G_m^{\wedge}$, can be defined in the tachyonic case. For the Feynman and anti-Feynman Green functions we can still use the formulae (1.6) and (1.7), where m^2 is replaced with $-m^2$. Their interpretation in terms of the vacuum expectation values is however lost, since the tachyonic theory has no vacuum state. (In particular, in the tachyonic case we do not have a counterpart of the positive/negative frequency Green functions (5.40)). The forward and backward Green functions are defined by their support properties. For them we cannot use the formulae (1.9) and (1.10). In fact, the set $p^2 - m^2 = 0$ is now connected, and cutting it with sgn p^0 is no longer invariant. Nevertheless, one can use the analytic continuation in m to uniquely define Green functions with correct support properties also in the tachyonic case. We point out that the identity (1.14) is no longer true in the tachyonic case.

The difference of two Green functions is a solution of the homogeneous Helmholtz/Klein–Gordon equation. Certain distinguished solutions are important for physics applications. In the Lorentzian case, we have the Pauli–Jordan propagator; for $m^2 \geq 0$ also the positive frequency and the negative frequency two-point functions. We illustrate applications of distinguished solutions to the Helmholtz/Klein–Gordon equation by computing averages of plane waves over the sphere (in the Euclidean case), as well as over the hyperbolic and de Sitter space (in the Lorentzian case).

Let us say a few words about the history of Bessel potentials. The name *Bessel potentials* was introduced in the 60s by Aronszajn and Smith, who studied them in the Euclidean case in [1]. Around the same time, they were also investigated by Calderon [2]. Bessel potentials are frequently viewed in the literature as smoothed versions of Riesz potentials (see, for example, [3] where they are defined using the integral formula (2.5)). They are often used to define Bessel potential spaces that generalize standard Sobolev spaces (see [4]), and the idea to use Bessel kernels is due to Deny [5]. For a comprehensive treatment of (Euclidean) Bessel potentials, we refer the reader to [1], where many properties of Bessel potentials are exhaustively studied.

The Lorentzian versions of Bessel potentials, typically in dimension 1+3, often appear in the literature on quantum field theory. They are ingredients of formulae for scattering amplitudes based on Feynman diagrams and on the Epstein–Glaser approach [6, 7]. The famous textbooks by Björken–Drell [8] and by Bogoliubov–Shirkov [9] contain appendices devoted to distinguished Green functions and solutions of the Klein–Gordon equation in the physical dimension 1+3. They carry various names. For instance, often the term *Green function* is replaced by *propagator*, etc.

Formulae for Bessel potentials in various signatures are known and are available in collections of integrals [10, 11]. In Chapter III.2 of [12] one can find Fourier transforms of powers of quadratic forms with any signature, including the formula (4.5) of the general case studied in this paper. Although there exists a large literature about Bessel potentials, our presentation contains several new points, which we have not seen in the literature and believe are important.

The first new point involves the special functions that we use. Various kinds of the Bessel equation can be reduced to equation

$$(z\partial_z^2 + (\alpha + 1)\partial_z - 1)v(z) = 0, \tag{1.16}$$

which can be called the ${}_0F_1$ *hypergeometric equation*. Eq. (1.16) has two singular points: 0 and ∞. The singularity at 0 is regular (Fuchsian), and the solution obtained by the well-known Frobenius method is the ${}_0F_1$ *hypergeometric function*, which we denote F_α. We usually prefer its *Olver normalized* version $\mathbf{F}_\alpha := \frac{F_\alpha}{\Gamma(\alpha+1)}$, closely related to the Bessel function, both standard and modified.

Another standard solution of the ${}_0F_1$ equation, corresponding to the irregular singularity at ∞, is the function that we denote U_α, This function is perhaps less known. Up to a coefficient, it coincides with the *Meijer G-function* $G^{2,0}_{0,2}(-;0,-\alpha;z)$. The function U_α is closely related to the Macdonald and Hankel functions.

In our paper, we treat $\mathbf{F}_\alpha$ and U_α functions as basic elements of our description of Bessel potentials. In our opinion, they are much more convenient for this purpose, rather than functions from the Bessel family, as it is done in the conventional treatment of this topic. The corresponding formulae are simpler and more transparent. This is especially visible when we consider non-Euclidean signatures, where the formulae involve analytic continuation across two branches and an irregular distribution at the junction of these branches. The $\mathbf{F}_\alpha$ and U_α functions are also convenient to see the transition from the Minkowski space to the de Sitter and the universal cover of the Anti-de Sitter space, as discussed in [13]. In fact, on the Minkowski space retarded/advanced and Feynman/anti-Feynman Bessel potentials are expressed in terms of $\mathbf{F}_\alpha$ and U_α, and on the de Sitter and Anti-de Sitter space we need closely related *Gegenbauer functions* instead.

We also believe that there are some important novel features in our presentation of the Lorentzian case, which is tailored to the needs of quantum field theory. In our opinion, it is quite remarkable how rich the theory of Bessel potentials is in the Lorentzian signature. We have four distinct Lorentz invariant Green functions of the Klein–Gordon equation, with important applications in physics. If we also include a few useful distinguished solutions to the Klein–Gordon equation (such as the Pauli–Jordan propagator, positive and negative frequency solution), then we obtain a whole menagerie of functions.

In our discussion, we cover not only the massive and massless case, but also the tachyonic case. This case is quite curious, even though usually ignored in the physics literature. We also discuss identity (1.14), true for $m^2 \geq 0$, but wrong in the tachyonic case. Remarkably, this identity sometimes, but not always, generalizes to curved spacetimes, as analyzed recently in [13].

In our treatment, we pay special attention to the distributional character of Bessel potentials. This is unproblematic in the Euclidean signature, where Bessel potentials are given by (locally) integrable functions. This is not the case in non-Euclidean signatures. In particular, it is interesting to look at the functions $\mathbf{F}_\alpha$ and U_α as defining distributions on the real line. With this interpretation in mind, well-known identities have to be reformulated, see e.g. (2.47).

Finally, let us mention that there exists a large literature about Green functions of the Klein–Gordon equation on curved spacetimes. In the generic context their explicit expression is not possible, and often instead of exact Green functions one restricts oneself to *parametrices*, that is inverses modulo smoothing terms. The existence of exactly four parametrices that generalize $G^{\mathrm{F}/\overline{\mathrm{F}}}$ and $G^{\vee/\wedge}$ is the result of a famous paper by Duistermaat and Hörmander [14]. It is also remarkable that expansions similar to (5.34)–(5.37) describe singular parts of these parametrices also in curved spacetimes, where they can be derived from the Hadamard recursion relations (see Chapter 4 of [15] or Chapter 2 of [16].) The universality of these singular parts is an important idea in quantum field theory on curved spacetimes [7].

2. Special functions related to the ${}_0F_1$ equation

2.1. The ${}_0F_1$ equation

Our presentation of Bessel potentials will use extensively ${}_0F_1$ hypergeometric functions, closely related to functions from the Bessel family. Surprisingly, they are seldom used and discussed in the literature. Therefore, we devote this section to a concise exposition of their properties, mostly following [17] and [18]. In particular, we will treat these functions as distributions on the real line, as explained in Section 2.5, which leads to useful distributional identities which we have not seen in the literature.

Let $c \in \mathbb{C}$. The ${}_0F_1$ *equation* is

$$(z\partial_z^2 + c\partial_z - 1)v(z) = 0. \tag{2.1}$$

If $c \neq 0, -1, -2, \ldots$, then the only solution of the ${}_0F_1$ equation equal to 1 at $z = 0$ is called the ${}_0F_1$ *hypergeometric function*:

$$F(c; z) := \sum_{j=0}^{\infty} \frac{1}{(c)_j} \frac{z^j}{j!},$$

where $(c)_j$ denotes the Pochhammer symbol:

$$\begin{aligned}
&(a)_0 = 1, \\
&(a)_n := a(a+1)\ldots(a+n-1), && n = 1, 2, \ldots \\
&(a)_n := \frac{1}{(a-n)\ldots(a-1)}, && n = \ldots, -2, -1.
\end{aligned}$$

$F(c;z)$ is defined for $c \neq 0,-1,-2,\dots$. Sometimes it is more convenient to consider the function

$$\mathbf{F}(c;z) := \frac{F(c;z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{1}{\Gamma(c+j)} \frac{z^j}{j!}$$

defined for all c. For all parameters, we have an integral representation called the *Schläfli formula*

$$\frac{1}{2\pi i} \int\limits_{]-\infty,0^+,-\infty[} e^t e^{\frac{z}{t}} t^{-c} dt = \mathbf{F}(c,z), \qquad \operatorname{Re} z > 0,$$

where the contour $]-\infty,0^+,-\infty[$ starts at $-\infty$, goes around 0 counterclockwise and returns to $-\infty$.

Instead of c it is often more natural to use $\alpha := c-1$. Thus, we denote

$$F_\alpha(z) := F(\alpha+1;z), \quad \mathbf{F}_\alpha(z) := \mathbf{F}(\alpha+1;z). \tag{2.2}$$

The following function is also a solution of the ${}_0F_1$ Eq. (1.16),

$$U_\alpha(z) := e^{-2\sqrt{z}} z^{-\frac{\alpha}{2}-\frac{1}{4}} {}_2F_0\left(\frac{1}{2}+\alpha, \frac{1}{2}-\alpha; -; -\frac{1}{4\sqrt{z}}\right),$$

where we used the ${}_2F_0$ function, see e.g. [17, 18]. U_α is a multivalued function. When talking about multivalued functions, we will usually consider their *principal branches* on the domain $\mathbb{C} \setminus]-\infty,0]$.

The function U_α rarely appears in the literature, except as a special case of Meijer's function, see (2.32) below. Typically, it is represented through Macdonald or Hankel functions, which we describe further in Eqs. (2.35), (2.37), and (2.36). In our opinion, however, the function U_α is often more convenient than Macdonald or Hankel functions.

$U_\alpha(z)$ has a symmetry

$$U_\alpha(z) = z^{-\alpha} U_{-\alpha}(z). \tag{2.3}$$

Alternatively, the function U_α can be defined by the integral representations valid for all α,

$$\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} e^{-\frac{z}{t}} t^{-\alpha-1} dt = U_\alpha(z), \qquad \operatorname{Re} z > 0. \tag{2.4}$$

For further reference, it is convenient to rewrite (2.4) as follows: For $\operatorname{Re}(m) > 0$, we have

$$\int_0^\infty e^{-tm^2-\frac{x^2}{4t}} t^{-\alpha-1} dt = \sqrt{\pi} m^{2\alpha} U_\alpha\left(\frac{m^2x^2}{4}\right). \tag{2.5}$$

For $\operatorname{Re}(m) \geq 0$ (2.5) is still true in the sense of oscillatory integrals. By substituting $x^2 \mapsto e^{\pm i\frac{\pi}{2}} x^2$, $m^2 \mapsto e^{\pm i\frac{\pi}{2}} m^2$, into (2.5) we obtain a pair of identities valid in terms of oscillatory integrals for $m > 0$,

$$\int_0^\infty e^{\mp itm^2 \mp \frac{x^2}{4t}} t^{-\alpha-1} dt = e^{i\frac{\pi\alpha}{2}} \sqrt{\pi} m^{2\alpha} U_\alpha\left(e^{\pm i\pi}\frac{m^2x^2}{4}\right). \tag{2.6}$$

As $|z| \to \infty$ and $|\arg z| < 2\pi - \epsilon$, $\epsilon > 0$, we have

$$U_\alpha(z) \sim \exp(-2z^{\frac{1}{2}}) z^{-\frac{\alpha}{2}-\frac{1}{4}}. \tag{2.7}$$

U_α is the unique solution of (1.16) with this property. (Note that the validity of (2.7) extends beyond $|\arg z| < \pi$, that is, beyond the principal sheet of the Riemann surface.)

We can express U_α in terms of the solutions of with a simple behaviour at zero

$$U_\alpha(z) = \frac{\sqrt{\pi}}{\sin \pi(-\alpha)} \mathbf{F}_\alpha(z) + \frac{\sqrt{\pi}}{\sin \pi\alpha} z^{-\alpha} \mathbf{F}_{-\alpha}(z). \tag{2.8}$$

Alternatively, we can use the U_α function and its analytic continuation around 0 in the clockwise or anti-clockwise direction as the basis of solutions

$$\mathbf{F}_\alpha(z) = \frac{\mp i}{2\sqrt{\pi}} (e^{\mp i\pi\alpha} U_\alpha(z) - e^{\pm i\pi\alpha} U_\alpha(e^{\pm i2\pi} z)). \tag{2.9}$$

Here is a version of (2.9) adapted to some applications:

$$\mathbf{F}_\alpha(-z) = \frac{i}{2\sqrt{\pi}} \left(e^{i\pi\alpha} U_\alpha(e^{i\pi} z) - e^{-i\pi\alpha} U_\alpha(e^{-i\pi} z)\right), \tag{2.10}$$

$$z^{-\alpha} \mathbf{F}_{-\alpha}(-z) = \frac{i}{2\sqrt{\pi}} \left(U_\alpha(e^{i\pi} z) - U_\alpha(e^{-i\pi} z)\right). \tag{2.11}$$

We have the recurrence relations

$$\partial_z \mathbf{F}_\alpha(z) = \mathbf{F}_{\alpha+1}(z), \tag{2.12}$$

$$(z\partial_z + \alpha)\, \mathbf{F}_\alpha(z) = \mathbf{F}_{\alpha-1}(z); \tag{2.13}$$

$$\partial_z U_\alpha(z) = -U_{\alpha+1}(z), \tag{2.14}$$

$$(z\partial_z + \alpha)\, U_\alpha(z) = -U_{\alpha-1}(z). \tag{2.15}$$

$\alpha = m \in \mathbb{Z}$ is the degenerate case of the ${}_0F_1$ equation at 0. We have then

$$\mathbf{F}_m(z) = \sum_{n=\max(0,-m)} \frac{1}{n!(m+n)!} z^n.$$

This easily implies the identity

$$\mathbf{F}_m(z) = z^{-m} \mathbf{F}_{-m}(z). \tag{2.16}$$

In the degenerate case $U_\alpha(z)$ needs to be reexpressed using the de l'Hospital formula

$$U_m(z) = \frac{(-1)^{m+1}}{\sqrt{\pi}}\Bigg(\sum_{k=1}^{m}\frac{(-1)^{k-1}(k-1)!}{(m-k)!}z^{-k} + \sum_{j=0}^{\infty}\frac{\ln(z)-\psi(j+m+1)-\psi(j+1)}{j!(m+j)!}z^j\Bigg). \tag{2.17}$$

In the degenerate case, the integral representation simplifies yielding the so-called *Bessel integral representation*. Besides, we have a generating function

$$\frac{1}{2\pi i}\int_{[0^+]} e^{t+\frac{z}{t}}t^{-m-1}dt = \mathbf{F}_m(z) = z^{-m}\mathbf{F}_{-m}(z),$$

$$e^t e^{\frac{z}{t}} = \sum_{m\in\mathbb{Z}} t^m\mathbf{F}_m(z).$$

Above, $[0^+]$ denotes the contour encircling 0 in the counterclockwise direction.

In the half-integer case, we can express the ${}_0F_1$ function in terms of elementary functions. Indeed,

$$F_{-\frac{1}{2}}(z) = \cosh 2\sqrt{z},\ U_{-\frac{1}{2}}(z) = \exp(-2\sqrt{z}), \tag{2.18}$$

$$F_{\frac{1}{2}}(z) = \frac{\sinh 2\sqrt{z}}{2\sqrt{z}},\ U_{\frac{1}{2}}(z) = \frac{\exp(-2\sqrt{z})}{\sqrt{z}}, \tag{2.19}$$

and by the recurrence relations, we have for $k \in \mathbb{N}$

$$F_{-\frac{1}{2}-k}(z) = z^{k+\frac{1}{2}}\partial_z^k\left(\frac{\cosh(2\sqrt{z})}{\sqrt{z}}\right), \tag{2.20}$$

$$F_{\frac{1}{2}+k}(z) = \partial_z^k\left(\frac{\sinh(2\sqrt{z})}{2\sqrt{z}}\right), \tag{2.21}$$

$$U_{-\frac{1}{2}-k}(z) = (-1)^k z^{k+\frac{1}{2}}\partial_z^k\left(\frac{\exp(-2\sqrt{z})}{\sqrt{z}}\right), \tag{2.22}$$

$$U_{\frac{1}{2}+k}(z) = (-1)^k\partial_z^k\left(\frac{\exp(-2\sqrt{z})}{\sqrt{z}}\right). \tag{2.23}$$

2.2. Relationship to confluent functions

Recall that the confluent equation is

$$(w\partial_w^2 + (c-w)\partial_w - c)f(w) = 0. \tag{2.24}$$

Its standard solutions are

Kummer's confluent function $\quad {}_1F_1(a;c;w) := \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n n!}w^n,$

and Tricomi's confluent function $\quad U(a;c;w) := z^{-a}{}_2F_0(a, 1+a-c; -; -w^{-1}).$

The ${}_0F_1$ equation can be reduced to a special class of the confluent equation by the so-called *Kummer's 2nd transformation*

$$z\partial_z^2 + (\alpha+1)\partial_z - 1 \tag{2.25}$$

$$= \frac{4}{w}e^{-w/2}\left(w\partial_w^2 + (2\alpha+1-w)\partial_w - \alpha - \frac{1}{2}\right)e^{w/2}, \tag{2.26}$$

where $w = \pm 4\sqrt{z}$, $z = \frac{1}{16}w^2$. F_α and U_α can be expressed in terms of Kummer's and Tricomi's confluent function as follows:

$$F_\alpha(z) = e^{\mp 2\sqrt{z}}{}_1F_1\left(\alpha + \frac{1}{2}, 2\alpha+1, \pm 4\sqrt{z}\right), \tag{2.27}$$

$$U_\alpha(z) = \frac{e^{-2\sqrt{z}}}{2^{2\alpha+1}}U\left(\alpha + \frac{1}{2}, 2\alpha+1, 4\sqrt{z}\right). \tag{2.28}$$

2.3. Relationship to Meijer G-functions

Solutions of hypergeometric equations ${}_pF_q$ can be expressed in terms of Meijer G-functions [19]. In particular, the ${}_0F_1$ equation can be solved by two distinguished functions

$$G_{0,2}^{1,0}\left(\begin{matrix} \\ 0, -\alpha\end{matrix}\middle| -z\right) := \frac{1}{2\pi i}\int_{L_1}\frac{\Gamma(-s)e^{i\pi s}}{\Gamma(\alpha+1+s)}z^s\,ds, \tag{2.29}$$

$$G_{0,2}^{2,0}\left(\begin{matrix} \\ 0, -\alpha\end{matrix}\middle| z\right) := \frac{1}{2\pi i}\int_{L_2}\Gamma(-s)\Gamma(-\alpha-s)z^s\,ds. \tag{2.30}$$

Here, the contour L_1 goes from $+\infty$ to $+\infty$ and encircles $\mathbb{N}_0$, and the contour L_2 also goes from $+\infty$ to $+\infty$ and encircles $\mathbb{N}_0 \cup (\mathbb{N}_0 - \alpha)$, both counterclockwise. Computing the residues and using the connection formula (2.8) we obtain

$$\mathbf{F}_\alpha(z) = G_{0,2}^{1,0}\left(\begin{matrix} \\ 0, -\alpha\end{matrix}\middle| -z\right), \tag{2.31}$$

$$U_\alpha(z) = \frac{1}{\sqrt{\pi}}G_{0,2}^{2,0}\left(\begin{matrix} \\ 0, -\alpha\end{matrix}\middle| z\right). \tag{2.32}$$

2.4. Relationship to Bessel functions

In the literature, the ${}_0F_1$ equation is seldom used. Much more frequent is the *modified Bessel equation*, which is equivalent to the ${}_0F_1$ equation. It is given by the operator

$$z^{\frac{\alpha}{2}}\big(z\partial_z^2 + (\alpha+1)\partial_z - 1\big)z^{-\frac{\alpha}{2}} = \partial_w^2 + \frac{1}{w}\partial_w - 1 - \frac{\alpha^2}{w^2},$$

where $z = \frac{w^2}{4}$, $w = \pm 2\sqrt{z}$.

Even more frequent is the (standard) *Bessel equation* given by

$$-z^{\frac{\alpha}{2}}\left(z\partial_z^2 + (\alpha+1)\partial_z - 1\right)z^{-\frac{\alpha}{2}} = \partial_u^2 + \frac{1}{u}\partial_u + 1 - \frac{\alpha^2}{u^2},$$

where $z = -\frac{u^2}{4}$, $u = \pm 2i\sqrt{z}$. Clearly, we can pass from the modified Bessel to the Bessel equation by $w = \pm iu$.

The function $\mathbf{F}_\alpha$ is also seldom used. Instead, one uses the *modified Bessel function* and, even more frequently, the *Bessel function*:

$$I_\alpha(w) = \left(\frac{w}{2}\right)^\alpha \mathbf{F}_\alpha\left(\frac{w^2}{4}\right), \tag{2.33}$$

$$J_\alpha(w) = \left(\frac{w}{2}\right)^\alpha \mathbf{F}_\alpha\left(-\frac{w^2}{4}\right). \tag{2.34}$$

They solve the modified Bessel, resp. the Bessel equation.

Instead of the U_α function one uses the *Macdonald function*, solving the modified Bessel equation

$$K_\alpha(w) = \frac{\sqrt{\pi}}{2}\left(\frac{w}{2}\right)^\alpha U_\alpha\left(\frac{w^2}{4}\right), \tag{2.35}$$

and the Hankel functions of the 1st and 2nd kind, solving the Bessel equation:

$$H_\alpha^{(1)}(w) = H_\alpha^+(w) = \frac{-i}{\sqrt{\pi}}\left(\frac{e^{-i\pi}w}{2}\right)^\alpha U_\alpha\left(e^{-i\pi}\frac{w^2}{4}\right), \tag{2.36}$$

$$H_\alpha^{(2)}(w) = H_\alpha^-(w) = \frac{i}{\sqrt{\pi}}\left(\frac{e^{i\pi}w}{2}\right)^\alpha U_\alpha\left(e^{i\pi}\frac{w^2}{4}\right). \tag{2.37}$$

Here are the relations between various functions from the Bessel family:

$$H_\alpha^\pm(z) = \frac{2}{\pi}e^{\mp i\frac{\pi}{2}(\alpha+1)}K_\alpha(\mp iz), \tag{2.38}$$

$$H_{-\alpha}^\pm(z) = e^{\pm\alpha\pi i}H_\alpha^\pm(z), \tag{2.39}$$

$$J_\alpha(z) = \frac{1}{2}\left(H_\alpha^+(z) + H_\alpha^-(z)\right), \tag{2.40}$$

$$I_\alpha(z) = \frac{1}{\pi}\left(\mp iK_\alpha(e^{\mp i\pi}z) \pm ie^{i\pi m}K_\alpha(z)\right). \tag{2.41}$$

2.5. $\mathbf{F}_\alpha$ and U_α functions as distributions

The function $U_\alpha(z)$ (and many others that we consider in this paper) are multivalued analytic functions defined on the Riemann surface of the logarithm. It has its *principal branch* on $\mathbb{C}\setminus]-\infty,0]$. For its analytic continuation around 0 we will often use the self-explanatory notation $U_\alpha(e^{i\phi}z)$, where $z \in \mathbb{C}\setminus]-\infty,0]$ and $\phi \in \mathbb{R}$.

We will often consider $U_\alpha(w)$ on the real line. For $w > 0$ this is unambiguous. For $w < 0$ one needs to add $\pm i0$ indicating whether we are infinitesimally above or below the real line. At $w = 0$ this function has a singularity, which may require a more careful treatment in terms of distributions (see Appendix A.2 for notation about some common distributions).

Thus we introduce the distribution on the real line

$$U_\alpha(w \pm i0) := \lim_{\epsilon \searrow 0} U_\alpha(w \pm i\epsilon), \tag{2.42}$$

where the right-hand side should be understood as the limit in the distributional sense. Note that for $w \neq 0$ these distributions are regular (in the sense of Appendix A.2) and given by analytic functions:

$$U_\alpha(w \pm i0) = U_\alpha(w), \quad w > 0; \tag{2.43}$$

$$U_\alpha(w \pm i0) = U_\alpha\big(e^{\pm i\pi}(-w)\big), \quad w < 0. \tag{2.44}$$

At $w = 0$ these distributions are irregular if $\operatorname{Re}\alpha \geq 1$. We can then write $U_\alpha(w \pm i0)$ as the sum of an irregular and regular part as follows:

$$U_\alpha(w \pm i0) = U_\alpha^{\text{sing}}(w \pm i0) + U_\alpha^{\text{reg}}(w), \tag{2.45}$$

$$U_\alpha^{\text{sing}}(w \pm i0) := \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \operatorname{Re}\alpha \rfloor - 1} \frac{(-1)^j \Gamma(\alpha - j)}{j!} (w \pm i0)^{j-\alpha}. \tag{2.46}$$

This easily follows from (2.8) and (2.17).

Recall that for $\alpha \notin \mathbb{N}$ the symbol $w_-^{-\alpha}$ defined in (A.9) denotes the standard regularization of $|w|^{-\alpha}\theta(-w)$. The identity (2.11) for $w \in \mathbb{R}\setminus\{0\}$ can be rewritten as

$$w_-^{-\alpha}\mathbf{F}_{-\alpha}(w) := \frac{i}{2\sqrt{\pi}} \left(U_\alpha(w + i0) - U_\alpha(w - i0)\right). \tag{2.47}$$

(Note that both sides of (2.47) are zero for $w > 0$.) It is easy to see that for $\alpha \notin \mathbb{N}$ (2.47) is a correct distributional identity, where the l.h.s. is the product of the distribution $w_-^{-\alpha}$ and of the smooth function $\mathbf{F}_{-\alpha}(w)$, whereas the r.h.s. is a linear combination of distributions defined in (2.42). (2.47) can be decomposed into a singular and regular part as follows,

$$w_-^{-\alpha}\mathbf{F}_{-\alpha}(w) = \sum_{j=0}^{\lfloor \operatorname{Re}\alpha \rfloor - 1} \frac{w_-^{-\alpha+j}(-1)^j}{\Gamma(-\alpha + j + 1)j!} + \sum_{j=\lfloor \operatorname{Re}\alpha \rfloor}^{\infty} \frac{w_-^{-\alpha+j}(-1)^j}{\Gamma(-\alpha + j + 1)j!} \tag{2.48}$$

The r.h.s. of (2.47) is well defined also for $\alpha \in \mathbb{N}$. We will *define* for such α the symbol on the l.h.s. of (2.47) by the r.h.s. Using (A.14) for $\alpha \in \mathbb{N}$ we can thus write

$$w_-^{-\alpha}\mathbf{F}_{-\alpha}(w) = (-1)^{\alpha+1} \sum_{j=0}^{\alpha-1} \frac{(-1)^j \delta^{(\alpha-1-j)}(w)}{j!} + (-1)^\alpha \mathbf{F}_\alpha(w)\theta(-w). \tag{2.49}$$

(Compare with (2.16), where you do not see the distributions supported at zero).

Of course, in the context described in this subsection, the distribution $U_\alpha(w \pm i0)$ defined as in (2.42) can be also expressed in terms of K_α and $H_\alpha^\pm$, where we would have to treat $\sqrt{w}$, resp. $\sqrt{-w}$ with $w \in \mathbb{R}$ as their arguments. It is then important to indicate precisely how the analytic continuation of the square root is performed—whether we bypass the branch point at zero from above or from below, adding $\pm i0$ to the variable:

$$K_\alpha\big(\sqrt{w \mp i0}\big) := \begin{cases} K_\alpha\big(\sqrt{w}\big), & w > 0, \\ K_\alpha\big(\mp i\sqrt{-w}\big) = \pm i\frac{\pi}{2} e^{\pm i\pi\alpha} H_\alpha^\pm\big(\sqrt{-w}\big), & w < 0; \end{cases} \tag{2.50a}$$

$$H_\alpha^\pm\big(\sqrt{-w \pm i0}\big) := \begin{cases} H_\alpha^\pm\big(\pm i\sqrt{w}\big) = \mp i\frac{2}{\pi} e^{\mp i\pi\alpha} K_\alpha\big(\sqrt{w}\big), & w > 0, \\ H_\alpha^\pm\big(\sqrt{-w}\big), & w < 0. \end{cases} \tag{2.50b}$$

We believe, however, that it is more convenient in such situations to use the function U_α. Indeed, we have

$$U_\alpha\left(\frac{w \mp i0}{4}\right) = \begin{cases} \frac{2^{\alpha+1}}{\sqrt{\pi}} (w \mp i0)^{-\frac{\alpha}{2}} K_\alpha\big(\sqrt{w \mp i0}\big), \\ \pm i 2^\alpha \sqrt{\pi} (w \mp i0)^{-\frac{\alpha}{2}} H_\alpha^\pm\big(\sqrt{-w \pm i0}\big). \end{cases} \tag{2.51}$$

3. Euclidean and anti-Euclidean signature

This section is devoted to Bessel potentials on the Euclidean space $\mathbb{R}^d$. $|x| := \sqrt{x^2}$ will denote the Euclidean norm of $x \in \mathbb{R}^d$.

In this section, we will provide various expressions both in terms of the Bessel family functions $I_\alpha, J_\alpha, K_\alpha, H_\alpha^\pm$, as well as in terms of the hypergeometric functions F_α, U_α.

3.1. General exponents—Euclidean case

Consider first the Euclidean signature. For $m > 0$ and $\operatorname{Re}\mu > 0$ the function $\frac{1}{(p^2+m^2)^{\frac{\mu}{2}}}$ defines a tempered distribution, hence one can compute its Fourier transform.

THEOREM 1. *Let* $m > 0$.

$$G_{\mu,m}(x) = \int \frac{e^{ipx}}{(p^2 + m^2)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{3.1}$$

$$= \frac{2}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{|x|}{2m}\right)^{\frac{\mu-d}{2}} K_{\frac{d-\mu}{2}}(m|x|) \tag{3.2}$$

$$= \frac{\sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} U_{\frac{d-\mu}{2}}\left(\frac{m^2 x^2}{4}\right). \tag{3.3}$$

Proof: By (A.1),

$$\frac{1}{(2\pi)^d}\int\frac{e^{ipx}dp}{(m^2+p^2)^{\frac{\mu}{2}}} \tag{3.4}$$

$$=\frac{1}{(2\pi)^d\Gamma(\frac{\mu}{2})}\int_0^\infty ds\int dps^{\frac{\mu}{2}-1}e^{-(m^2+p^2)s}e^{ipx} \tag{3.5}$$

$$=\frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{\mu}{2})}\int_0^\infty dss^{\frac{\mu}{2}-\frac{d}{2}-1}e^{-m^2s-\frac{x^2}{4s}} \tag{3.6}$$

Then we use (2.5). □

Note that the integrand of (3.1) is integrable for $\mathrm{Re}\,\mu > d$. Therefore, $G_{\mu,m}$ is bounded for such μ. For instance,

$$G_{\mu,m}(0)=\frac{1}{(2\pi)^d}\int\frac{dp}{(p^2+m^2)^{\frac{\mu}{2}}}=\frac{m^{d-\mu}\Gamma(\frac{\mu-d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{\mu}{2})},\quad \mathrm{Re}\,\mu > d. \tag{3.7}$$

3.2. General exponents—massless case

For $0<\mathrm{Re}\,\mu<d$ the following function is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and is bounded at infinity, hence it defines a regular distribution in $\mathcal{S}'(\mathbb{R}^d)$,

$$G_{\mu,0}(x):=\int\frac{e^{ipx}}{|p|^\mu}\frac{dp}{(2\pi)^d} \tag{3.8}$$

$$=\frac{\Gamma(\frac{d-\mu}{2})}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}}\left(\frac{|x|}{2}\right)^{\mu-d}. \tag{3.9}$$

It is called the *Riesz potential*, and it is the massless limit of Bessel potentials.

THEOREM 2. *Let* $0<\mathrm{Re}\,\mu<d$. *Then*

$$G_{\mu,0}(x)=\lim_{m\to0}G_{\mu,m}(x) \tag{3.10}$$

in the sense of $\mathcal{S}'(\mathbb{R}^d)$.

Proof: One can prove this fact in the position space, see Subsection 4.2, where we give a proof in the case of a general signature. Instead, in this section we describe a proof based on the momentum space.

For $0<\mu<d$, $|p|^{-\mu}$ is a regular distribution. By using the Dominated Convergence Theorem we see that the pointwise limit

$$\lim_{m\to0}(p^2+m^2)^{-\frac{\mu}{2}}=|p|^{-\mu} \tag{3.11}$$

is a limit in the sense of $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transformation is a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$. Therefore, for considered μ, (3.10) is true. □

3.3. General exponent—anti-Euclidean case

Suppose now the scalar product is negative definite. For $m^2 > 0$, the function $\frac{1}{(-p^2+m^2)^{\frac{\mu}{2}}}$ does not define uniquely a distribution, therefore one cannot compute its Fourier transform. However, if $m^2 \in \mathbb{C} \setminus [0, \infty[$, then $\frac{1}{(-p^2+m^2)^{\frac{\mu}{2}}}$ is a tempered distribution, and one can take its limit from above or below in the distributional sense,

$$\frac{1}{(-p^2 + m^2 \pm i0)^{\frac{\mu}{2}}} := \lim_{\epsilon \searrow 0} \frac{1}{(-p^2 + m^2 \pm i\epsilon)^{\frac{\mu}{2}}}. \tag{3.12}$$

Thus we obtain two kinds of Bessel potentials in the anti-Euclidean case.

THEOREM 3.

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) = \int \frac{e^{ipx}}{(-p^2 + m^2 \mp i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{3.13}$$

$$= \frac{\mp i(\pm i)^d \pi}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{|x|}{2m}\right)^{\frac{\mu-d}{2}} H^{\mp}_{\frac{\mu-d}{2}}(m|x|) \tag{3.14}$$

$$= \frac{\mp i e^{\pm i \frac{\pi\mu}{2}} \pi}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{|x|}{2m}\right)^{\frac{\mu-d}{2}} H^{\mp}_{\frac{d-\mu}{2}}(m|x|). \tag{3.15}$$

$$= \frac{e^{\pm i\pi\frac{d}{2}} \sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} U_{\frac{d-\mu}{2}}\left(\frac{e^{\pm i\pi} m^2 x^2}{4}\right). \tag{3.16}$$

Proof: Using (A.2) and then (2.5) we obtain (3.16). □

Note that the Euclidean Bessel potential $G_{\mu,m}$ is well defined not only for $m \geq 0$, but also for $\mathrm{Re}\,(m) > 0$, which guarantees $m^2 \in \mathbb{C} \setminus]-\infty, 0]$. Taking the limit at the imaginary line we can express the anti-Euclidean Bessel potential in terms of the Euclidean one,

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) = e^{\mp i\pi\frac{\mu}{2}} G_{\mu,\pm im}(x). \tag{3.17}$$

3.4. Green functions of the Helmholtz equation

Bessel potentials with $\mu = 2$ are Green functions of the Helmholtz equation

$$(-E - \Delta) f(x) = g(x). \tag{3.18}$$

More precisely, the Green function for $-E = m^2$ is

$$G_m(x) := \int \frac{e^{ipx}}{(p^2+m^2)} \frac{dp}{(2\pi)^d} \tag{3.19}$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{|x|}{m}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(m|x|) \tag{3.20}$$

$$= \frac{\sqrt{\pi} m^{d-2}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2x^2}{4}\right), \tag{3.21}$$

and for $-E = -m^2$ we have two distinguished Green functions:

$$G_{\mp im}(x) = \int \frac{e^{ixp}}{(p^2-m^2\mp i0)} \frac{dp}{(2\pi)^d} \tag{3.22}$$

$$= \pm\frac{i}{4}\left(\frac{m}{2\pi|x|}\right)^{\frac{d}{2}-1} H^{\pm}_{\frac{d}{2}-1}(m|x|) \tag{3.23}$$

$$= -(-i)^d \frac{\sqrt{\pi} m^{d-2}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(-\frac{m^2(x^2\pm i0)}{4}\right). \tag{3.24}$$

$G_{\mp im}(x)$ coincide with the case $\mu = 2$ of the anti-Euclidean Bessel potential (3.13) multiplied by -1.

3.5. Averages of plane waves on sphere

Consider the sphere in $\mathbb{R}^d$ of radius m, denoted $\mathbb{S}_m^{d-1} = \mathbb{S}_m$. Let $d\Omega_m$ be the natural measure on $\mathbb{S}_m$. As an application of Bessel potentials, we will compute the Fourier transform of the measure on $\mathbb{S}_m$.

THEOREM 4.

$$\int_{\mathbb{S}_m} e^{ipx} d\Omega_m(p) = 2m^{d-1}\pi^{\frac{d}{2}} \mathbf{F}_{\frac{d}{2}-1}\left(-\frac{m^2x^2}{4}\right) \tag{3.25}$$

$$= m^{d-1}(2\pi)^{\frac{d}{2}}(m|x|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(m|x|). \tag{3.26}$$

Proof: By the Sochocki–Plemejl formula we have

$$\delta(|p|-m) = 2m\delta(p^2-m^2) = \frac{2m}{2\pi i}\left(\frac{1}{p^2-m^2-i0} - \frac{1}{p^2-m^2+i0}\right). \tag{3.27}$$

Therefore,

$$\int_{\mathbb{S}_m} e^{ipx}\Omega_m(p) = \int e^{ipx}\delta(|p|-m)dp \tag{3.28}$$

$$= \frac{2m}{2\pi i}\int e^{ipx}\left(\frac{1}{p^2-m^2-i0} - \frac{1}{p^2-m^2+i0}\right)dp \tag{3.29}$$

$$= \frac{m(2\pi)^d}{\pi i}(G_{-im}(x) - G_{im}(x)) \tag{3.30}$$

$$= m^{d-1}\pi^{\frac{d-1}{2}}\left((-i)^{d-1}U_{\frac{d}{2}-1}\left(\frac{e^{-i\pi}m^2x^2}{4}\right) - i^{d-1}U_{\frac{d}{2}-1}\left(\frac{e^{i\pi}m^2x^2}{4}\right)\right) \tag{3.31}$$

$$= 2m^{d-1}\pi^{\frac{d}{2}}\mathbf{F}_{\frac{d}{2}-1}\left(-\frac{m^2x^2}{4}\right), \tag{3.32}$$

where at the end we used (2.10). □

Consider a radial function $\mathbb{R}^d \ni p \mapsto f(|p|)$. Its Fourier transform is also radial. (3.25) yields the identity

$$\int f(|p|)e^{-ipx}dp = 2\pi^{\frac{d}{2}}\int_0^\infty f(k)\mathbf{F}_{\frac{d}{2}-1}\left(-\frac{k^2x^2}{4}\right)k^{d-1}dk \tag{3.33}$$

$$= (2\pi)^{\frac{d}{2}}\int_0^\infty f(k)J_{\frac{d}{2}-1}(k|x|)(k|x|)^{-\frac{d}{2}+1}k^{d-1}dk, \tag{3.34}$$

where $k = |p|$ has the meaning of the length of p.

Using $\mathbf{F}_{-\frac{1}{2}}(-z) = \frac{\cos 2\sqrt{z}}{\sqrt{\pi}}$ and $\mathbf{F}_{\frac{1}{2}}(-z) = \frac{\sin 2\sqrt{z}}{\sqrt{\pi z}}$ we obtain the low dimensional cases of (3.33):

$$\int f(|p|)e^{-ipx}dp = 2\int_0^\infty f(k)\cos(k|x|)dk, \qquad d = 1; \tag{3.35}$$

$$= 2\pi\int_0^\infty f(k)\mathbf{F}_0\left(-\frac{k^2x^2}{4}\right)dk = 2\pi\int_0^\infty f(k)kJ_0(k|x|)dk, \qquad d = 2; \tag{3.36}$$

$$= 4\pi\int_0^\infty f(k)k^2\frac{\sin(k|x|)}{k|x|}dk, \qquad d = 3. \tag{3.37}$$

3.6. Integral representations of the U_α function

As an illustration of the usefulness of (3.33), we will derive a certain integral representation of U_α.

Applying (3.33) to (3.3) we obtain

$$2\int_0^\infty \frac{k^{d-1}dk}{(k^2+1)^{\frac{\mu}{2}}}\mathbf{F}_{\frac{d}{2}-1}\left(-\frac{r^2k^2}{4}\right) = \frac{\sqrt{\pi}}{\Gamma(\frac{\mu}{2})}U_{\frac{d-\mu}{2}}\left(\frac{r^2}{4}\right). \tag{3.38}$$

Specifying $d = 1$ and $d = 3$ we obtain

$$2\int_0^\infty \frac{\cos(kr)}{(k^2+1)^{\frac{\mu}{2}}}dk = \frac{\sqrt{\pi}}{\Gamma(\frac{\mu}{2})}U_{\frac{1-\mu}{2}}\left(\frac{r^2}{4}\right), \tag{3.39}$$

$$4\int_0^\infty \frac{k\sin(kr)}{(k^2+1)^{\frac{\mu}{2}}r}dk = \frac{\sqrt{\pi}}{\Gamma(\frac{\mu}{2})}U_{\frac{3-\mu}{2}}\left(\frac{r^2}{4}\right). \tag{3.40}$$

(3.40) could be also deduced from (3.39) by differentiating wrt r and using the recurrence relation (2.14). Setting $\alpha = \frac{\mu-1}{2}$ in (3.39), we obtain the Poisson representation of the U_α function,

$$U_\alpha\left(\frac{r^2}{4}\right) = \frac{\Gamma(\frac{1}{2}-\alpha)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ikr}(k^2+1)^{\alpha-\frac{1}{2}} dk, \qquad \alpha < 0. \tag{3.41}$$

4. General signature

4.1. Positive mass

Consider now a pseudo-Euclidean space of general signature $\mathbb{R}^{q,d-q}$. $\frac{1}{(p^2+m^2)^{\frac{\mu}{2}}}$ no longer defines a tempered distribution in the general signature. Just as in the anti-Euclidean case, there are two natural regularizations of this function,

$$\frac{1}{(p^2+m^2\pm i0)^{\frac{\mu}{2}}} := \lim_{\epsilon\searrow 0} \frac{1}{(p^2+m^2\pm i\epsilon)^{\frac{\mu}{2}}}. \tag{4.1}$$

They lead to two kinds of the Bessel potential.

THEOREM 5. *Let* $m > 0$ *(or more generally* $\mathrm{Re}\,(m) > 0$*). Then*

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) = \int \frac{e^{ipx}}{(m^2+p^2\mp i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{4.2}$$

$$= \frac{2(\pm i)^q}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{\sqrt{x^2\pm i0}}{2m}\right)^{\frac{\mu-d}{2}} K_{\frac{d-\mu}{2}}\left(\sqrt{m^2(x^2\pm i0)}\right) \tag{4.3}$$

$$= \mp \frac{\pi i(\pm i)^q}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{\sqrt{x^2\pm i0}}{2m}\right)^{\frac{\mu-d}{2}} H^{\mp}_{\frac{\mu-d}{2}}\left(\sqrt{m^2(-x^2\mp i0)}\right) \tag{4.4}$$

$$= \frac{(\pm i)^q \sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} U_{\frac{d-\mu}{2}}\left(\frac{m^2(x^2\pm i0)}{4}\right). \tag{4.5}$$

REMARK 1. In (4.3) and (4.4) we use the notation explained in (2.50a) and (2.50b). Note that (4.3) works best for $x^2 > 0$, because then we can ignore $\pm i0$. Likewise, (4.4) is best suited for $x^2 < 0$, because then we can ignore $\mp i0$.

Anyway, in our opinion the expression in terms of U_α, (4.5), is preferable.

Proof of Theorem 5. Using (A.2) and (A.4) we obtain

$$\frac{1}{(2\pi)^d} \int \frac{e^{ipx} dp}{(m^2+p^2\mp i0)^{\frac{\mu}{2}}} = \frac{e^{\pm i\frac{\pi\mu}{4}}}{(2\pi)^d \Gamma(\frac{\mu}{2})} \int_0^\infty dt \int dp e^{\mp it(m^2+p^2)} t^{\frac{\mu}{2}-1} e^{ipx}$$

$$= \frac{(\pm i)^q e^{\pm i\frac{\pi}{2}(\frac{\mu-d}{2})} \pi^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{\mu}{2})} \int_0^\infty dt e^{\mp i(tm^2-\frac{x^2}{4t})} t^{\frac{\mu-d}{2}-1}. \tag{4.6}$$

Then we apply (2.6). □

4.2. Zero mass

For $0 < \operatorname{Re}\mu < d$ let us introduce two distributions in $\mathcal{S}'$

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,0}(x) := \int \frac{e^{ipx}}{(p^2 \mp i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{4.7}$$

$$= \frac{(\pm i)^q \Gamma(\frac{d-\mu}{2})}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{x^2 \pm i0}{4}\right)^{\frac{\mu-d}{2}}. \tag{4.8}$$

They will be called *Feynman/anti-Feynman Riesz potentials*. They are massless limits of the corresponding Bessel potentials.

Theorem 6. *For $0 < \operatorname{Re}\mu < d$ we have*

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,0}(x) = \lim_{m\searrow 0} G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) \tag{4.9}$$

in the sense of $\mathcal{S}'$.

Proof: Surprisingly, a momentum space proof, from the Euclidean case, seems to be difficult to generalize to the non-Euclidean case. Instead, we will present a proof in the position space.

Using the decomposition (2.45) of the function U_α, we can write

$$G_{\mu,m}(x) = \frac{(\pm i)^q}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\sum_{j=0}^{\lfloor \operatorname{Re}\frac{d-\mu}{2} \rfloor -1} \frac{(-1)^j m^j \Gamma(\frac{d-\mu}{2} - j)}{j!} \left(\frac{(x^2 \pm i0)}{4}\right)^{j-\frac{d-\mu}{2}} \right. \tag{4.10}$$

$$\left. + m^{d-\mu} U^{\mathrm{reg}}_{\frac{d-\mu}{2}}\left(\frac{m^2(x^2 \pm i0)}{4}\right)\right). \tag{4.11}$$

The line (4.10) obviously converges to (4.8). By (2.7), $U^{\mathrm{reg}}_{\frac{d-\mu}{2}}$ is a continuous function of a polynomial growth at infinity. Therefore, the second line (4.11) converges to zero in $\mathcal{S}'$. □

Note that as a consequence of the above theorem and of the continuity of the Fourier transformation on $\mathcal{S}'(\mathbb{R}^d)$ we can infer that

$$\lim_{m\searrow 0} \frac{1}{(p^2 + m^2 \mp i0)^{\frac{\mu}{2}}} = \frac{1}{(p^2 \mp i0)^{\frac{\mu}{2}}} \tag{4.12}$$

in the sense of $\mathcal{S}'$.

4.3. Scaling degree of distributions

Let us start by defining the action of a dilation by λ on a distribution $T(x)$ as $T_\lambda(x) = T(\lambda x)$, by which we mean the dual action to the dilation on test functions

$$\langle T_\lambda | f \rangle = \int T(\lambda x) f(x) dx = \lambda^{-d} \int T(x) f(\lambda^{-1} x) dx. \tag{4.13}$$

Given a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$, we define its scaling degree $\mathrm{sd}(T)$ as

$$\mathrm{sd}(T) = \inf \left\{ \omega : \lim_{\lambda \searrow 0} \lambda^{\omega} T_\lambda = 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^d) \right\}. \tag{4.14}$$

The scaling degree of a distribution is often used in mathematical analysis of quantum field theory [6, 7].

Let us compute the scaling degree of Bessel potentials.

Theorem 7.

$$\mathrm{sd} G_{m,\mu}^{\mathrm{F}/\overline{\mathrm{F}}} = \begin{cases} d-\mu, & 0 < \mu \le d, \\ 0, & d \le \mu. \end{cases} \tag{4.15}$$

Proof: For $0 < \mu < d$, the Riesz potentials $G_{\mu,0}^{\mathrm{F}/\overline{\mathrm{F}}}$ defined in (4.8) are homogeneous,

$$G_{\mu,0}^{\mathrm{F}/\overline{\mathrm{F}}}(\lambda x) = \lambda^{\mu-d} G_{\mu,0}^{\mathrm{F}/\overline{\mathrm{F}}}(x). \tag{4.16}$$

So $\mathrm{sd} G_{\mu,0}^{\mathrm{F}/\overline{\mathrm{F}}} = d - \mu$.

By the definition of the Bessel potential, the mass dependence is (1.2),

$$G_{\mu,m}(\lambda x) = \lambda^{\mu-d} G_{\mu,\lambda m}(x), \tag{4.17}$$

so, according to Theorem 6,

$$\lim_{\lambda \searrow 0} \lambda^{d-\mu} G_{\mu,m}(\lambda x) = \lim_{\lambda \searrow 0} G_{\mu,\lambda m}(x) = G_{\mu,0}(x), \tag{4.18}$$

which shows that $\mathrm{sd} G_{\mu,m}^{\mathrm{F}/\overline{\mathrm{F}}} = d - \mu$ for any mass m and $0 < \mu < d$.

For $d < \mu$, $G_m^{F/\overline{\mathrm{F}}}$ is a continuous bounded function, so its scaling degree is 0. For $d = \mu$, we have

$$G_{d,m}(x) = \frac{(\pm i)^q \sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}} U_0\left(\frac{m^2(x^2 \pm i0)}{4} \right). \tag{4.19}$$

Now, we can use the bound (2.7) and the expansion (2.17),

$$|U_0(z \pm i0)| \le C|z|^{-\frac{1}{4}}, \quad z \in \mathbb{R}, \quad |z| > 1, \tag{4.20}$$

$$U_0(z \pm i0) = \ln(z \pm i0)\mathbf{F}_0(z) + H(z), \tag{4.21}$$

where H is an entire function, just as $\mathbf{F}_0$. Using this we easily show that for $\omega > 0$,

$$\lambda^{\omega} G_{d,m}(\lambda x) \to 0 \tag{4.22}$$

in the sense of $\mathcal{S}'$. □

5. The Minkowski signature

The Lorentzian signature is especially important, both because of its physical relevance and rich mathematical properties. The spaces $\mathbb{R}^{1,d-1}$ and $\mathbb{R}^{d-1,1}$ are two kinds of a Minkowski space, that is, a pseudo-Euclidean space with a Lorentzian signature. We will treat $\mathbb{R}^{1,d-1}$ as the standard form of a Minkowski space. x^0 will denote the first coordinate of $\mathbb{R}^{1,d-1}$, which we assume to be timelike (having a negative coefficient in the scalar product). The remaining, spacelike coordinates will be denoted $\vec{x}$, so that $x = (x^0, \vec{x})$. In other words,

$$x^2 = -(x^0)^2 + \vec{x}^2 = -(x^0)^2 + (x^1)^2 + \cdots + (x^{d-1})^2. \tag{5.1}$$

The future and the past light cone will be denoted

$$J^\vee := \{x \in \mathbb{R}^{1,d-1} \quad : \quad x^2 \leq 0, \quad x^0 \geq 0\},$$
$$J^\wedge := \{x \in \mathbb{R}^{1,d-1} \quad : \quad x^2 \leq 0, \quad x^0 \leq 0\}.$$

In this section, we will only use the hypergeometric functions $\mathbf{F}_\alpha, U_\alpha$.

5.1. General exponent

Let $m > 0$. The set $m^2 + p^2$ consists of two connected components: the future and the past mass hyperboloid. Therefore, the following four regularizations of $\frac{1}{(m^2+p^2)^{\frac{\mu}{2}}}$ are tempered distributions invariant wrt the orthochronous Lorentz group,

$$\frac{1}{(m^2 + p^2 \pm i0)^{\frac{\mu}{2}}}, \quad \frac{1}{(m^2 + p^2 \pm i0\mathrm{sgn}\ p^0)^{\frac{\mu}{2}}}. \tag{5.2}$$

Their inverse Fourier transforms define four kinds of Bessel potentials:

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) := \int \frac{e^{ipx}}{(m^2 + p^2 \mp i0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d} \tag{5.3}$$

$$G^{\vee/\wedge}_{\mu,m}(x) := \int \frac{e^{ipx}}{(m^2 + p^2 \mp i0\mathrm{sgn}\ p^0)^{\frac{\mu}{2}}} \frac{dp}{(2\pi)^d}. \tag{5.4}$$

By the following well-known argument, found e.g. in various standard textbooks on quantum field theory, we can show that $G^{\vee/\wedge}_{\mu,m}$ have causal supports.

Theorem 8. $\mathrm{supp}\, G^{\vee/\wedge}_{\mu,m} \subset J^{\vee/\wedge}$.

Proof: For definiteness, consider (5.4) with the minus sign. In order to prove that its support is contained in $J^\vee$, by the Lorentz invariance it suffices to prove that it is zero for $x^0 < 0$. We write

$$\int \frac{e^{ipx}dp}{(p^2 + m^2 - i0\mathrm{sgn}\ p^0)^{\frac{\mu}{2}}} = \int \frac{e^{-ip^0x^0 + i\vec{p}\vec{x}}dp^0 d\vec{p}}{\left(\vec{p}^2 + m^2 - (p^0 + i0)^2\right)^{\frac{\mu}{2}}}.$$

Next, we continuously deform the contour of integration, replacing $p^0 + i0$ by $p^0 + iR$, where $R \in [0, \infty[$. We do not cross any singularities of the integrand and note that $e^{-ix^0(p^0+iR)}$ goes to zero (remember that $x^0 < 0$). □

THEOREM 9. *We have the identity*

$$G^{\mathrm{F}}_{\mu,m}(x) + G^{\overline{\mathrm{F}}}_{\mu,m}(x) = G^{\vee}_{\mu,m}(x) + G^{\wedge}_{\mu,m}(x). \tag{5.5}$$

Here are the expressions for the Bessel potentials in the position space:

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,m}(x) = \frac{\pm i\sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} U_{\frac{d-\mu}{2}}\left(\frac{m^2(x^2 \pm i0)}{4}\right), \tag{5.6}$$

$$G^{\vee/\wedge}_{\mu,m}(x) = \theta(\pm x^0)\frac{2\pi}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}}\left(\tfrac{x^2}{4}\right)_-^{\frac{\mu-d}{2}} \mathbf{F}_{\frac{\mu-d}{2}}\left(\frac{m^2x^2}{4}\right). \tag{5.7}$$

where in (5.7) *we used the notation introduced in* (2.47).

Formula (5.7) *involves the multiplication of a distribution by a discontinuous function, which in general is not well defined. At the end of this subsection we explain how this formula can be correctly interpreted.*

Proof: The identity (5.5) follows immediately from the defining formulae, that is from (5.3) and (5.4).

(5.6) is a special case of (4.5). Using (5.6) and (5.7) we obtain a simple expression for the sum of two Bessel potentials,

$$G^{\vee}_{\mu,m}(x) + G^{\wedge}_{\mu,m}(x) = \frac{-i\sqrt{\pi} m^{d-\mu}}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}}\left(U_{\frac{d-\mu}{2}}\left(\frac{m^2x^2 - i0}{4}\right) - U_{\frac{d-\mu}{2}}\left(\frac{m^2x^2 + i0}{4}\right)\right) \tag{5.8}$$

$$= \frac{2\pi}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}}\left(\tfrac{x^2}{4}\right)_-^{\frac{\mu-d}{2}} \mathbf{F}_{\frac{\mu-d}{2}}\left(\frac{m^2x^2}{4}\right), \tag{5.9}$$

where again we used the notation introduced in (2.47). (5.9) is clearly supported in $J^\wedge \cup J^\vee$. By Theorem 8, we know that $G^{\vee/\wedge}_{\mu,m}$ are supported in $J^{\vee/\wedge}$. Thus to find expressions for $G^{\vee/\wedge}_{\mu,m}$ we need to "split the distribution" (5.9) into two terms, one supported in $J^\vee$ and the other in $J^\wedge$.

Using Proposition 1 to justify the multiplication of a distribution (5.9) by the (discontinuous) function $\theta(\pm x^0)$, we can define

$$\tilde{G}^{\vee/\wedge}_{\mu,m}(x) = \theta(\pm x^0)\frac{2\pi}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}}\left(\tfrac{x^2}{4}\right)_-^{\frac{\mu-d}{2}} \mathbf{F}_{\frac{\mu-d}{2}}\left(\frac{m^2x^2}{4}\right). \tag{5.10}$$

Clearly, $\tilde{G}^{\vee/\wedge}_{\mu,m}$ are supported in $J^{\vee/\wedge}$. Besides,

$$G^{\vee}_{\mu,m}(x) + G^{\wedge}_{\mu,m}(x) = \tilde{G}^{\vee}_{\mu,m}(x) + \tilde{G}^{\wedge}_{\mu,m}(x). \tag{5.11}$$

But $J^\vee \cap J^\wedge = \{0\}$. Therefore, $G^{\vee/\wedge}_{\mu,m} - \tilde{G}^{\vee/\wedge}_{\mu,m}$ is a distribution supported in $\{0\}$, that is, a linear combination of $\delta^{(\alpha)}(x)$,

$$B^{\vee/\wedge}_{\mu,m} := G^{\vee/\wedge}_{\mu,m} - \tilde{G}^{\vee/\wedge}_{\mu,m} = \sum_{|\alpha|<n} c^{\vee/\wedge}_{\alpha,m}\delta^{(\alpha)}(x). \tag{5.12}$$

Eq. (5.11) implies $B^{\vee}_{\mu,m}(x) = -B^{\wedge}_{\mu,m}(x)$. The symmetry in $x \mapsto -x, \vee/\wedge \mapsto \wedge/\vee$ of (5.4) and (5.7) allows us to write

$$G^{\vee}_{\mu,m}(x) = G^{\wedge}_{\mu,m}(-x), \qquad \tilde{G}^{\vee}_{\mu,m}(x) = \tilde{G}^{\wedge}_{\mu,m}(-x),$$

and therefore

$$B^{\vee/\wedge}_{\mu,m}(x) = B^{\wedge/\vee}_{\mu,m}(-x) = -B^{\vee/\wedge}_{\mu,m}(-x).$$

Its action on a test function $\phi \in \mathcal{S}(\mathbb{R}^{1,d-1})$ is

$$\langle B^{\vee/\wedge}_{\mu,m}, \phi\rangle = \sum_{|\alpha|<n} (-1)^{|\alpha|} c^{\vee/\wedge}_{\alpha,m} (\partial^{\alpha}_{x}\phi)(0) = \sum_{|\alpha|<n} c^{\vee/\wedge}_{\alpha,m} (\partial^{\alpha}_{-x}\phi)(0), \tag{5.13}$$

so $c^{\vee/\wedge}_{\alpha,m} = 0$ for even $|\alpha|$. $G^{\vee/\wedge}_{\mu,m}$ and $\tilde{G}^{\vee/\wedge}_{\mu,m}$ are invariant with respect to the action of the proper Lorentz group. The same must apply to their difference $B^{\vee/\wedge}_{\mu,m}$. Derivatives evaluated at 0 transform as vectors under the action of the Lorentz group. However, $\langle B^{\vee/\wedge}_{\mu,m}, \phi\rangle$ is a sum of terms with only odd number of indices, so it cannot be invariant under the action of the Lorentz group unless $B^{\vee/\wedge}_{\mu,m} = 0$. We conclude that $G^{\vee/\wedge}_{\mu,m} = \tilde{G}^{\vee/\wedge}_{\mu,m}$. □

$G^{\mathrm{F}}_{\mu,m}$ will be called the *Feynman Bessel potential* and $G^{\overline{\mathrm{F}}}_{\mu,m}$ the *anti-Feynman Bessel potential*. These names are somewhat artificial in the context of a general μ. Their justification comes from the case $\mu = 2$, where these Bessel potentials coincide with the Feynman and anti-Feynman propagator known from quantum field theory.

The distribution $G^{\vee}_{\mu,m}$ will be called the *forward* or *retarded Bessel potential*, and $G^{\wedge}_{\mu,m}$ the *backward* or *advanced Bessel potential*.

For $0 < \operatorname{Re}\mu < d$ we also have the massless Riesz potentials:

$$G^{\mathrm{F}/\overline{\mathrm{F}}}_{\mu,0}(x) = \frac{\pm i\Gamma(\frac{d-\mu}{2})}{\Gamma(\frac{\mu}{2})(4\pi)^{\frac{d}{2}}} \left(\frac{x^2 \pm i0}{4}\right)^{\frac{\mu-d}{2}}, \tag{5.14}$$

$$G^{\vee/\wedge}_{\mu,0}(x) = \theta(\pm x^0) \frac{2\pi}{\Gamma(\frac{\mu}{2})\Gamma(\frac{\mu-d+2}{2})(4\pi)^{\frac{d}{2}}} \left(\tfrac{x^2}{4}\right)_{-}^{\frac{\mu-d}{2}}. \tag{5.15}$$

As we mentioned above, the formula (5.7) for the advanced and retarded Bessel potential involves a product of two distributions, and therefore it needs a justfication. We will explain two approaches how to interpret this formula.

The first approach is quite elementary. It uses the identification $\mathbb{R}^{1,d-1} \simeq \mathbb{R}\times\mathbb{R}^{d-1}$, with the first variable denoted x^0 or t. For the remaining variables $\vec{x}$ we will later use spherical coordinates (r, Ω) with $r = |\vec{x}|$. For $n, m \in \mathbb{N}_0$ and $\chi \in \mathcal{S}(\mathbb{R}^{1,d-1})$ let us introduce the semi-norms, which involve only the variables $\vec{x} \in \mathbb{R}^{d-1}$,

$$\|\chi(t,\cdot)\|_{n,m} = \sup_{\vec{x}\in\mathbb{R}^{d-1}, |\alpha|=n, |\beta|=m} = |\vec{x}^{\alpha}(\partial^{\beta}_{\vec{x}}\chi)(t,\vec{x})|.$$

Proposition 1. *Let* $\operatorname{Re}\nu < d$. *Then there exist* c_k, $k = 0, \dots, \lfloor \frac{\operatorname{Re}\nu}{2} \rfloor$, *such that for any* $\phi \in \mathcal{S}(\mathbb{R}^{1,d-1})$,

$$\left| \int \left(x^2\right)_-^{-\frac{\nu}{2}} \phi(x) dx \right| \le \sum_{k=0}^{\lfloor \frac{\operatorname{Re}\nu}{2} \rfloor} \int c_k |t|^{d-\operatorname{Re}\nu+k-1} \|\phi(t,\cdot)\|_{0,k} dt, \tag{5.16}$$

where the coefficients $|t|^{d-\operatorname{Re}\nu+k-1}$ *are locally integrable and polynomially bounded at infinity. Therefore, if* $f \in L^\infty(\mathbb{R})$, *then* $f(x^0)\left(x^2\right)_-^{-\frac{\nu}{2}}$ *defines a tempered distribution on* $\mathbb{R}^d$.

Proof: The action of $\left(x^2\right)_-^{-\frac{\nu}{2}}$ on a test function $\phi \in \mathcal{S}(\mathbb{R}^d)$ is

$$\int \left(x^2\right)_-^{-\frac{\nu}{2}} \phi(x) dx = \int_{-\infty}^{\infty} dt \int_0^{|t|} dr \int_{\mathbb{S}^{d-2}} d\Omega (r^2 - t^2)_-^{-\frac{\nu}{2}} \phi(t,r,\Omega) r^{d-2}.$$

For simplicity let us consider only $t > 0$. We can expand ϕ around $r = t$,

$$\phi(t,r,\Omega) = \sum_{k=0}^{m} \frac{(r-t)^k}{k!} \phi^{(k)}(t,t,\Omega) + (r-t)^{m+1} \psi(t,r,\Omega),$$

with $m = \lfloor \frac{\operatorname{Re}\nu}{2} \rfloor - 1$, where $\phi^{(k)}$ denote derivatives with respect to the r variable. Note that $|\psi(t,r,\Omega)| \le (m+1)! \|\phi(t,\cdot)\|_{0,m+1}$. Let

$$a_{m+1} := \int_0^{\infty} dt \int_0^t dr \int_{\mathbb{S}^{d-2}} d\Omega (t-r)^{\beta} (r+t)^{-\frac{\nu}{2}} \psi(t,r,\Omega) r^{d-2},$$

with $\beta = \lfloor \frac{\operatorname{Re}\nu}{2} \rfloor - \frac{\nu}{2}$, $\quad -1 < \operatorname{Re}\beta \le 0$, be the integral of the locally integrable function. We see that it is well defined and

$$\begin{aligned}
|a_{m+1}| &\le \int_0^{\infty} dt \int_0^t dr \int_{\mathbb{S}^{d-2}} d\Omega (t-r)^{\operatorname{Re}\beta} (t+r)^{-\operatorname{Re}\frac{\nu}{2}} |\psi(t,r,\Omega)| r^{d-2} \\
&\le (m{+}1)! \int_0^{\infty} dt \|\phi(t,\cdot)\|_{0,m+1} t^{\lfloor \frac{\operatorname{Re}\nu}{2} \rfloor + d - \operatorname{Re}\nu - 1} \int_0^1 dr' (1{-}r')^{\beta} (1{+}r')^{-\frac{\operatorname{Re}\nu}{2}} r'^{d-2} \int_{\mathbb{S}^{d-2}} d\Omega \\
&=: C(d,\nu,m+1) \int_0^{\infty} dt t^{\lfloor \frac{\nu}{2} \rfloor + d - \operatorname{Re}\nu - 1} \|\phi(t,\cdot)\|_{0,m+1}.
\end{aligned}$$

Next, we look at each term of the expansion of $\phi(t,r,\Omega)$ in k,

$$\begin{aligned}
a_k &= \int_0^{\infty} dt \int_0^t dr \int_{\mathbb{S}^{d-2}} d\Omega (r^2 - t^2)_-^{-\frac{\nu}{2}} \frac{(r-t)^k}{k!} \phi^{(k)}(t,t,\Omega) r^{d-2} \\
&= \frac{(-1)^k}{k!} \int_0^{\infty} dt \int_0^t dr (r-t)_-^{-\frac{\nu}{2}+k} (t+r)^{-\frac{\nu}{2}} r^{d-2} \int_{\mathbb{S}^{d-2}} d\Omega \phi^{(k)}(t,t,\Omega).
\end{aligned}$$

Here, $(t-r)_-^{-\frac{\nu}{2}+k}$ is the (irregular) distribution, defined by (A.12). It yields a finite expression

$$\int_0^t dr(r-t)_-^{-\frac{\nu}{2}+k}(t+r)^{-\frac{\nu}{2}}r^{d-2} = t^{d-\nu+k-1}\int_0^1 dr'(r'-1)_-^{-\frac{\nu}{2}+k}(1+r')^{-\frac{\nu}{2}}r^{d-2}$$
$$=: t^{d-\nu+k-1}\tilde{C}(d,\nu,k).$$

Because $d-\operatorname{Re}\nu+k-1 \geq d-\operatorname{Re}\nu-2 > -1$, dependence on t is locally integrable and bounded by a polynomial. For $k=0,1,\dots,m+1$ we can write

$$|a_k| \leq C(d,\nu,k)\int_0^\infty dt\ \ t^{d-\operatorname{Re}\nu+k-1}\|\phi(t,\cdot)\|_{0,k}.$$

For fixed d,ν, we have the inequality (5.16) showing that homogeneous distributions are tempered distribution. □

Now we have $d-\mu < d$, and therefore Proposition 1 shows that we can multiply the distribution $\left(\frac{x^2}{4}\right)_-^{\frac{\mu-d}{2}}$ by the discontinuous but bounded function $\theta(\pm x^0)$. The resulting distribution is then multiplied by the smooth function $\mathbf{F}_{\frac{\mu-d}{2}}\left(\frac{m^2x^2}{4}\right)$, obtaining the right-hand side of (5.7).

An alternative way to define the product in (5.7) is based on the concept of the *wave front set* [20]. Here are the wave front sets of the distributions contained in (5.7):

$$\mathrm{WF}\,(\theta(t)) = \left\{((0,\vec{x}),(\tau,0)) \ \ : \ \ \vec{x}\in\mathbb{R}^{d-1}, \tau\neq 0\right\},$$
$$\mathrm{WF}\left((x^2)_-^{-\frac{\nu}{2}}\right) = \left\{((t,\vec{x}),(-\lambda t,\lambda\vec{x})) \ \ : \ \ t^2-\vec{x}^2=0, (t,\vec{x})\neq 0, \lambda\neq 0\right\}$$
$$\cup\left\{((0,0),(\tau,\vec{k})) \ \ : \ \ \tau^2-\vec{k}^2=0, (\tau,\vec{k})\neq 0\right\},$$

where $(\tau,\vec{k})$ denotes the dual variable to $(t,\vec{x})$. The fiberwise sum of wavefront sets $\mathrm{WF}\,(\theta(t)) + \mathrm{WF}\left((x^2)_-^{-\frac{\nu}{2}}\right)$ does not contain an element of the form $((t,\vec{x}),(0,0))$. Therefore, by Hörmander's criterion [20, p. 267], the product of these two distributions is well defined.

5.2. Green functions of the Klein–Gordon equation

Consider the *Klein–Gordon equation*

$$(-E-\Box)f(x) = g(x), \tag{5.17}$$

where E is a parameter, usually real. We will consider 3 cases:

$$\text{massive case:} \quad -E = m^2, \tag{5.18}$$
$$\text{massless case:} \quad -E = 0, \tag{5.19}$$
$$\text{tachyonic case:} \quad -E = -m^2. \tag{5.20}$$

The massive and massless cases are quite similar and they often appear in physical applications. They are often discussed in detail in the literature. The tachyonic case is more exotic and less known, but also interesting.

The Klein–Gordon equation possesses several useful Green functions, that is, distributions satisfying

$$(-E - \Box)G^{\bullet}(x) = \delta(x). \tag{5.21}$$

One can try to define Green functions of the Klein–Gordon equation the Fourier transformation. Unfortunately, for $E \in \mathbb{R}$, $\frac{1}{(-E+p^2)}$ is not a well-defined distribution because of zeros of its denominator. One way to regularize it is to add $\pm i0$, which leads to the so-called Feynman and anti-Feynman Green function

$$G_m^{\mathrm{F}/\overline{\mathrm{F}}}(x) = \int \frac{e^{ipx}}{(-E + p^2 \mp i0)} \frac{dp}{(2\pi)^d}. \tag{5.22}$$

As follows from a general theory of hyperbolic equations, the Klein–Gordon equation (5.17) possesses also another important pair of Green functions: the retarded (or forward) Green function $G^{\vee}$ and the advanced (backward) Green function $G^{\wedge}$. They are uniquely defined by the conditions

$$\operatorname{supp} G^{\vee/\wedge} \subset J^{\pm}. \tag{5.23}$$

Note that the above definition provides $G^{\vee/\wedge}$ for all $E \in \mathbb{R}$. In the case $-E \geq 0$, with $-E = m^2$ they coincide with $G_m^{\vee/\wedge}$ defined already with the help of Fourier transformation. In the tachyonic case they will be denoted $G_{im}^{\vee/\wedge} = G_{-im}^{\vee/\wedge}$ and they need a separate discussion, see Subsection 5.5

We will also consider certain distinguished solutions of the (homogeneous) Klein–Gordon equation, that is functions G° satisfying

$$(-E - \Box)G^{\circ}(x) = 0. \tag{5.24}$$

One can look for them with the ansatz

$$G^{\circ}(x) = \int e^{ipx} g^{\circ}(p)\delta(-E + p^2)\frac{dp}{(2\pi)^{d-1}}, \tag{5.25}$$

where g° is a distribution on $p^2 - E = 0$. Above, for $E \in \mathbb{R}$, we use the notation

$$\delta(p^2 - E)dp = \frac{\delta\big(p^0 - \sqrt{\vec{p}^2 - E}\big)}{2\sqrt{\vec{p}^2 - E}} d\vec{p} + \frac{\delta\big(p^0 + \sqrt{\vec{p}^2 - E}\big)}{2\sqrt{\vec{p}^2 - E}} d\vec{p}, \tag{5.26}$$

where for $\vec{p}^2 < E$, (5.26) = 0.

Below we consider separately the massive, massless and tachyonic cases of the Klein–Gordon equation. In all three cases, we will be able to define $G^{\overline{\mathrm{F}}/\mathrm{F}}$ and $G^{\vee/\wedge}$.

5.3. Massive Klein–Gordon equation

Let us consider $-E = m^2$, that is the massive Klein–Gordon equation. The corresponding Green functions satisfy

$$(m^2 - \Box)G_m^{\bullet}(x) = \delta(x). \tag{5.27}$$

Specifying Theorem 9 to $\mu = 2$, we obtain the following expressions for the Feynman and anti-Feynman Green functions.

THEOREM 10.

$$G_m^{\mathrm{F}/\overline{\mathrm{F}}}(x) = \int \frac{e^{ipx}}{(m^2 + p^2 \mp i0)} \frac{dp}{(2\pi)^d} \tag{5.28}$$

$$= \frac{\pm i\sqrt{\pi} m^{d-2}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2(x^2 \pm i0)}{4}\right). \tag{5.29}$$

The retarded and advanced Green functions of the Klein–Gordon equation are obtained by specifying Theorem 9 to $\mu = 2$. In the following theorem, we also identify their regular and singular part.

THEOREM 11.

$$G_m^{\vee/\wedge}(x) = \int \frac{e^{ipx}}{m^2 + p^2 \mp i0\mathrm{sgn}\ p^0} \frac{dp}{(2\pi)^d} \tag{5.30}$$

$$= \theta(\pm x^0) \frac{-i\sqrt{\pi} m^{d-2}}{(4\pi)^{\frac{d}{2}}} \left(U_{\frac{d}{2}-1}\left(\frac{m^2x^2 - i0}{4}\right) - U_{\frac{d}{2}-1}\left(\frac{m^2x^2 + i0}{4}\right)\right) \tag{5.31}$$

$$= \theta(\pm x^0) \frac{2\pi m^{d-2}}{(4\pi)^{\frac{d}{2}}} \left(\tfrac{x^2}{4}\right)_-^{1-\frac{d}{2}} \mathbf{F}_{1-\frac{d}{2}}\left(\frac{m^2x^2}{4}\right). \tag{5.32}$$

We can decompose $G^{\vee/\wedge}$ into a singular and regular part,

$$G_m^{\vee/\wedge}(x) = G_{m,\mathrm{sing}}^{\vee/\wedge}(x) + G_{m,\mathrm{reg}}^{\vee/\wedge}(x). \tag{5.33}$$

For d odd this decomposition can be chosen as

$$G_{m,\mathrm{sing}}^{\vee/\wedge}(x) = \frac{\theta(\pm x^0)}{2\pi^{\frac{d}{2}-1}} \sum_{j=0}^{\frac{d-5}{2}} \frac{(-1)^j}{j!\Gamma(2 - \frac{d}{2} + j)} \left(\frac{m^2}{4}\right)^j (x^2)_-^{1-\frac{d}{2}+j}, \tag{5.34}$$

$$G_{m,\mathrm{reg}}^{\vee/\wedge}(x) = \frac{\theta(\pm x^0)}{2\pi^{\frac{d}{2}-1}} \sum_{j=\frac{d-3}{2}}^{\infty} \frac{(-1)^j}{j!\Gamma(2 - \frac{d}{2} + j)} \left(\frac{m^2}{4}\right)^j (-x^2)^{1-\frac{d}{2}+j} \theta(-x^2). \tag{5.35}$$

For d even:

$$G_{m,\mathrm{sing}}^{\vee/\wedge}(x) = \theta(\pm x^0) \frac{1}{2\pi^{\frac{d}{2}-1}} \sum_{j=0}^{\frac{d}{2}-2} \frac{(-1)^{j+1}}{(\frac{d}{2} - 2 - j)!} \left(\frac{m^2}{4}\right)^{\frac{d}{2}-2-j} \delta^{(j)}(x^2), \tag{5.36}$$

$$G_{m,\mathrm{reg}}^{\vee/\wedge}(x) = \theta(\pm x^0) \frac{2\pi m^{d-2}}{(4\pi)^{\frac{d}{2}}} \mathbf{F}_{\frac{d}{2}-1}\left(\frac{m^2x^2}{4}\right) \theta(-x^2). \tag{5.37}$$

Proof: The formula for the Green function of the Klein–Gordon equation is given by Eq. (5.32), which was computed earlier for a general μ (5.7). The decomposition into (5.34) and (5.35) is due to (2.48). For even d, the decomposition can be rewritten using (2.49). □

Introduce the following distinguished solutions of the Klein–Gordon equation $-\Box + m^2$:

$$G_m^{\mathrm{PJ}}(x) := \frac{i}{(2\pi)^d} \int e^{ix\cdot p} \mathrm{sgn}\ (p^0)\delta(p^2+m^2)dp \tag{5.38}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d\vec{p}}{\sqrt{\vec{p}^2+m^2}} e^{i\vec{x}\vec{p}} \sin\left(x^0\sqrt{\vec{p}^2+m^2}\right), \tag{5.39}$$

$$G_m^{(\pm)}(x) := \frac{1}{(2\pi)^d} \int e^{ix\cdot p} \theta(\pm p^0)\delta(p^2+m^2)dp \tag{5.40}$$

$$= \frac{1}{(2\pi)^{d-1}} \int \frac{d\vec{p}}{2\sqrt{\vec{p}^2+m^2}} e^{\mp i x^0 \sqrt{\vec{p}^2+m^2} + i\vec{x}\vec{p}}. \tag{5.41}$$

Following [13], we will call distinguished Green functions and solutions jointly *propagators*. G_m^{PJ} is supported in $J^\vee \cup J^\wedge$. Here are the expressions for these solutions in terms of positions:

$$G_m^{\mathrm{PJ}}(x) = \mathrm{sgn}\ (x^0) \frac{2\pi}{(4\pi)^{\frac{d}{2}}} \left(\frac{x^2}{4}\right)_-^{\frac{2-d}{2}} \mathbf{F}_{\frac{2-d}{2}}\left(\frac{m^2x^2}{4}\right), \tag{5.42}$$

$$G_m^{(\pm)}(x) = \frac{\sqrt{\pi} m^{d-\mu}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d-2}{2}}\left(\frac{m^2x^2 \pm i\mathrm{sgn}\ x^0 0}{4}\right). \tag{5.43}$$

Note the identities satisfied by the propagators:

$$G_m^\vee - G_m^\wedge = G_m^{\mathrm{PJ}} \tag{5.44a}$$

$$= iG_m^{(+)} - iG_m^{(-)}, \tag{5.44b}$$

$$G_m^{\mathrm{F}} - G_m^{\overline{\mathrm{F}}} = iG_m^{(+)} + iG_m^{(-)}, \tag{5.44c}$$

$$G_m^{\mathrm{F}} + G_m^{\overline{\mathrm{F}}} = G_m^\vee + G_m^\wedge, \tag{5.44d}$$

$$G_m^{\mathrm{F}} = iG_m^{(+)} + G_m^\wedge = iG_m^{(-)} + G_m^\vee, \tag{5.44e}$$

$$G_m^{\overline{\mathrm{F}}} = -iG_m^{(+)} + G_m^\vee = -iG_m^{(-)} + G_m^\wedge. \tag{5.44f}$$

To prove these identities we use repeatedly

$$\theta(\pm p^0) 2\pi i \delta(p^2+m^2) = \theta(\pm p^0)\left(\frac{1}{p^2+m^2-i0} - \frac{1}{p^2+m^2+i0}\right), \tag{5.45}$$

5.4. Massless Klein–Gordon equation

The massless case is quite similar to the massive one: we need only to set $m = 0$ in the previous subsection. In particular, all identities (5.44) are satisfied. There are a few simplifications. Only the most singular part of the massive propagator remains in the massless case. This is the special case of Riesz potentials, massless limit of Bessel potentials, that we studied in Section 3.2.

THEOREM 12.

$$G_0^{\mathrm{F}/\overline{\mathrm{F}}}(x) = \pm \frac{i\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\big(x^2 \pm i0\big)^{1-\frac{d}{2}}, \tag{5.46}$$

$$G_0^{\vee/\wedge}(x) = \theta(\pm x^0)\frac{1}{2\pi^{\frac{d}{2}-1}\Gamma(2-\frac{d}{2})}(x^2)_-^{1-\frac{d}{2}}, \tag{5.47}$$

$$G_0^{\mathrm{PJ}}(x) = \mathrm{sgn}\,(x^0)\frac{1}{2\pi^{\frac{d}{2}-1}\Gamma(2-\frac{d}{2})}(x^2)_-^{1-\frac{d}{2}}, \tag{5.48}$$

$$G_0^{(\pm)}(x) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\big(x^2 \pm i0\mathrm{sgn}\,(x^0)\big)^{1-\frac{d}{2}}. \tag{5.49}$$

For d odd (5.47) *and* (5.48) *can be rewritten as*

$$G_0^{\vee/\wedge}(x) = \theta(\pm x^0)\frac{(-1)^{\frac{d}{2}-2}}{2\pi^{\frac{d}{2}-1}}\delta^{(\frac{d}{2}-2)}(x^2), \tag{5.50}$$

$$G_0^{\mathrm{PJ}}(x) = \mathrm{sgn}\,(x^0)\frac{(-1)^{\frac{d}{2}-2}}{2\pi^{\frac{d}{2}-1}}\delta^{(\frac{d}{2}-2)}(x^2). \tag{5.51}$$

Note that using (A.12) we can write identity (5.44d) as

$$G_0^{\mathrm{F}}(x) + G_0^{\overline{\mathrm{F}}}(x) = G_0^{\vee} + G_0^{\wedge} = \frac{1}{2\pi^{\frac{d}{2}-1}}\rho_-^{\frac{d}{2}-1}(x), \tag{5.52}$$

which agrees with the fact that massless retarded/advanced Green functions, also known as Riesz distributions (see [16]), are expressed by homogeneous distributions supported on $J^{\vee/\wedge}$.

5.5. Tachyonic Klein–Gordon equation

Let us now consider the *tachyonic Klein–Gordon equation*, which means, with $E = m^2$. Its Green functions satisfy

$$(-m^2 - \square)G^{\bullet}(x) = \delta(x). \tag{5.53}$$

Usually, tachyonic quantum fields are considered to be unphysical [21]. Nevertheless, every now and then there are attempts to analyze them in the physics literature, see [22], and more recently [23].

We have a minor notational problem how to indicate that we replaced m^2 with $-m^2$. Naively, one would think it should be indicated by both $+im$ and $-im$ instead of m. However, this would suggest the analytic continuation $e^{i\phi}$, $\pm\phi \in [0, \pi]$, which is not always appropriate. This problem appears in the case of the Feynman propagator: we will write G^{F}_{im}, but not G^{F}_{-im}. Similarly, for the anti-Feynman propagator we will write $G^{\overline{\mathrm{F}}}_{-im}$, but not $G^{\overline{\mathrm{F}}}_{im}$. In the case of retarded/advanced propagators, this problem will be absent, since the analytic continuation can be performed in m^2: thus $G^{\wedge/\vee}_{im} = G^{\wedge/\vee}_{-im}$.

We define the Feynman and anti-Feynman Green functions by adding $\mp i0$ to the denominator $-m^2 + p^2$ in the momentum representation. In the following theorem, we compute their form in position variables.

THEOREM 13.

$$G^{\mathrm{F}}_{im}(x)/G^{\overline{\mathrm{F}}}_{-im}(x) = \int \frac{e^{ipx}}{(-m^2 + p^2 \mp i0)} \frac{dp}{(2\pi)^d} \tag{5.54}$$

$$= \frac{\sqrt{\pi} m^{d-2} (\mp i)^{d+1}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2 \mp i0)}{4}\right). \tag{5.55}$$

In particular, for $x^2 > 0$ we have

$$G^{\mathrm{F}}_{im}(x)/G^{\overline{\mathrm{F}}}_{-im}(x) = \frac{\pm i\sqrt{\pi} m^{d-2} e^{\mp i\pi(\frac{d}{2}-1)}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2 \mp i0)}{4}\right)$$

and for $x^2 < 0$

$$G^{\mathrm{F}}_{im}(x)/G^{\overline{\mathrm{F}}}_{-im}(x) = \frac{\pm i\sqrt{\pi} m^{d-2} e^{\mp i\pi(\frac{d}{2}-1)}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2|x^2|}{4}\right).$$

Proof: Let us start from the usual (positive mass) Feynman propagator, defined in (5.28) and (5.29). Then we continue analytically $G^{\mathrm{F}}_m(x)$ and $G^{\overline{\mathrm{F}}}_m$, replacing m with $me^{i\phi}$, where $\phi \in [0, \frac{\pi}{2}]$ in the former and $\phi \in [-\frac{\pi}{2}, 0]$ in the latter case. (Note that during the analytic continuation the denominator has to have a constant sign of its imaginary part, that is, $\pm\mathrm{Im}(m^2 e^{2i\phi} + i0) > 0$.) The analytic continuation yields

$$G^{\mathrm{F}}_{im}(x)/G^{\overline{\mathrm{F}}}_{-im}(x) = \frac{\pm i\sqrt{\pi} m^{d-2} e^{\pm i\pi(\frac{d}{2}-1)}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{e^{\pm i\pi} m^2(x^2 \pm i0)}{4}\right) \tag{5.56}$$

$$= \frac{\pm i\sqrt{\pi} m^{d-2} e^{\pm i\pi(\frac{d}{2}-1)}}{(4\pi)^{\frac{d}{2}}} U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2 \mp i0)}{4}\right), \tag{5.57}$$

which coincides with (5.55). □

Unfortunately, the tachyonic Feynman and anti-Feynman propagator do not have the usual physical interpretation, as the vacuum expectation value of the time-ordered, resp. anti-time-ordered product of fields. In fact, for tachyons the vacuum is ill defined. Nevertheless, some authors, e.g. [22], try to use the above Feynman propagator to define interacting tachyonic quantum field theory.

Retarded and advanced tachyonic Green functions $G_{im}^{\vee/\wedge}$ are not tempered distributions on $\mathbb{R}^{1,d-1}$, and therefore they cannot be expressed in terms of the Fourier transformation in all variables, as in the massive and massless cases (5.30). However, they are well defined, and in the following theorem we give three equivalent formulae for these propagators.

THEOREM 14. *The forward and backward propagators in the tachyonic case are given by*

$$G_{im}^{\vee/\wedge}(x) = G_{-im}^{\vee/\wedge}(x)$$

$$= \theta(-x^2)\theta(\pm x^0)\frac{\sqrt{\pi}m^{d-2}i^{d+1}}{(4\pi)^{\frac{d}{2}}}\left(U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2)}{4}\right) - U_{\frac{d}{2}-1}\left(\frac{e^{i2\pi}m^2(-x^2)}{4}\right)\right) \tag{5.58}$$

$$= \theta(-x^2)\theta(\pm x^0)\frac{\sqrt{\pi}m^{d-2}(-i)^{d+1}}{(4\pi)^{\frac{d}{2}}}\left(U_{\frac{d}{2}-1}\left(\frac{e^{-i2\pi}m^2(-x^2)}{4}\right) - U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2)}{4}\right)\right) \tag{5.59}$$

$$= \theta(\pm x^0)\frac{2\pi}{(4\pi)^{\frac{d}{2}}}\left(\frac{x^2}{4}\right)_-^{1-\frac{d}{2}}\mathbf{F}_{1-\frac{d}{2}}\left(\frac{m^2|x^2|}{4}\right), \tag{5.60}$$

They are supported in $J^\vee$, resp. $J^\wedge$. We can decompose $G_{im}^{\vee/\wedge}$ into a singular and regular part,

$$G_{im}^{\vee/\wedge}(x) = G_{im,\mathrm{sing}}^{\vee/\wedge}(x) + G_{im,reg}^{\vee/\wedge}(x). \tag{5.61}$$

For d odd this decomposition is almost the same as (5.34)*,* (5.35) *but without the factor* $(-1)^j$*:*

$$G_{im,\mathrm{sing}}^{\vee/\wedge}(x) = \frac{\theta(\pm x^0)}{2\pi^{\frac{d}{2}-1}}\sum_{j=0}^{\frac{d-5}{2}}\frac{1}{j!\Gamma(2-\frac{d}{2}+j)}\left(\frac{m^2}{4}\right)^j(x^2)_-^{1-\frac{d}{2}+j}, \tag{5.62}$$

$$G_{im,\mathrm{reg}}^{\vee/\wedge}(x) = \frac{\theta(\pm x^0)}{2\pi^{\frac{d}{2}-1}}\sum_{j=\frac{d-3}{2}}^{\infty}\frac{1}{j!\Gamma(2-\frac{d}{2}+j)}\left(\frac{m^2}{4}\right)^j(-x^2)^{1-\frac{d}{2}+j}\theta(-x^2). \tag{5.63}$$

For d even the decomposition is similar as in (5.36) *and* (5.37)*:*

$$G_{im,\mathrm{sing}}^{\vee/\wedge}(x) = \theta(\pm x^0)\frac{1}{2\pi^{\frac{d}{2}-1}}\sum_{j=0}^{\frac{d}{2}-2}\frac{1}{(\frac{d}{2}-2-j)!}\left(\frac{m^2}{4}\right)^{\frac{d}{2}-2-j}\delta^{(j)}(x^2), \tag{5.64}$$

$$G_{im,\mathrm{reg}}^{\vee/\wedge}(x) = \theta(\pm x^0)\frac{2\pi m^{d-2}}{(4\pi)^{\frac{d}{2}}}\mathbf{F}_{\frac{d}{2}-1}\left(\frac{m^2|x^2|}{4}\right)\theta(-x^2). \tag{5.65}$$

Proof: Our starting point is the formula (5.31) for the forward and backward propagator $G_m^{\vee/\wedge}(x)$. They are analytic in m. Therefore, we can apply the analytic continuation $m \mapsto e^{i\frac{\pi}{2}}m$,

$$G^{\vee/\wedge}_{e^{i\frac{\pi}{2}}m}(x) = \theta(\pm x^0)\frac{-i\sqrt{\pi}m^{d-2}e^{i\pi(\frac{d}{2}-1)}}{(4\pi)^{\frac{d}{2}}}\Bigg(U_{\frac{d}{2}-1}\Big(\frac{e^{i\pi}(m^2x^2-i0)}{4}\Big)$$
$$-U_{\frac{d}{2}-1}\Big(\frac{e^{i\pi}(m^2x^2+i0)}{4}\Big)\Bigg). \tag{5.66}$$

This yields (5.58). Alternatively, we can apply the analytic continuation $m \mapsto e^{-i\frac{\pi}{2}}m$, which yields (5.59). □

Let us compute the sum of the tachyonic Feynman and anti-Feynman propagators:

$$G^{\mathrm{F}}_{im}(x) + G^{\overline{\mathrm{F}}}_{-im}(x) = \begin{cases} \dfrac{2\pi m^{d-2}}{(4\pi)^{\frac{d}{2}}}\mathbf{F}_{\frac{d}{2}-1}\Big(-\dfrac{m^2x^2}{4}\Big), & x^2 > 0; \\ \dfrac{4\pi}{(4\pi)^{\frac{d}{2}}}\Big(\dfrac{-x^2}{4}\Big)^{1-\frac{d}{2}}\mathbf{F}_{1-\frac{d}{2}}\Big(-\dfrac{m^2x^2}{4}\Big), & x^2 < 0. \end{cases} \tag{5.67}$$

Thus $G^{\mathrm{F}}_{im}(x) + G^{\overline{\mathrm{F}}}_{-im}(x)$ does not have a causal support, and consequently,

$$G^{\mathrm{F}}_{im}(x) + G^{\overline{\mathrm{F}}}_{-im}(x) \neq G^{\vee}_{im}(x) + G^{\wedge}_{im}(x). \tag{5.68}$$

The equality in (5.68) holds only for $x^2 < 0$.

Note that because of (5.68) we could not deduce the formulae of the forward and backward propagators from the Feynman and anti-Feynman propagators, and we had to apply a separate argument based on analytic continuation.

In the tachyonic case, we do not have the solutions $G^{(\pm)}_{im}$. However, we can define the Pauli–Jordan propagator

$$G^{\mathrm{PJ}}_{im}(x) = G^{\mathrm{PJ}}_{-im} = \frac{1}{(2\pi)^{d-1}}\int d\vec{p}\, e^{i\vec{x}\vec{p}}\frac{\sin\big(x^0\sqrt{\vec{p}^2-m^2}\big)}{\sqrt{\vec{p}^2-m^2}} \tag{5.69}$$

$$= \operatorname{sgn}(x^0)\frac{2\pi}{(4\pi)^{\frac{d}{2}}}\Big(\frac{x^2}{4}\Big)_-^{1-\frac{d}{2}}\mathbf{F}_{1-\frac{d}{2}}\Big(\frac{m^2|x^2|}{4}\Big). \tag{5.70}$$

Note that G^{PJ}_{im} cannot be written in the form (5.25).

Among the identities (5.44) only (5.44a) is still true.

5.6. Averages of plane waves on the hyperbolic plane

The Minkowski space possesses two kinds of hyperboloids. The two-sheeted hyperboloid consists of two connected components isomorphic to the hyperbolic space. In this subsection, we compute the Fourier transform of the natural measure on one of the sheets of the two-sheeted hyperboloids, similarly as in Theorem 4.

Consider the *future/past hyperboloid* in the d-dimensional Minkowski space, denoted $\mathbb{H}_{\pm,m} = \mathbb{H}^{d-1}_{\pm,m}$, consisting of points p such that $p^2 + m^2 = 0$ and $\pm p^0 > 0$. Let

$d\Omega_m$ denote the standard measure on $\mathbb{H}_{\pm,m}$. We will see that up to a coefficient its Fourier transform is essentially the "positive frequency solution of the Klein–Gordon equation."

THEOREM 15.

$$\int_{\mathbb{H}_{\pm,m}} e^{ipx} d\Omega_m(p) = m^{d-1}\pi^{\frac{d-1}{2}} U_{\frac{d}{2}-1}\left(\frac{m^2(x^2 \pm i\mathrm{sgn}\, x^0 0)}{4}\right). \tag{5.71}$$

Proof: This average, up to a coefficient, coincides with $G_m^{(\pm)}$ defined in (5.40), which we have already computed,

$$\int_{\mathbb{H}_{\pm,m}} e^{ipx} d\Omega_m(p) = 2m \int e^{ipx}\theta(\pm p^0)\delta(p^2+m^2)dp \tag{5.72}$$

$$= (2\pi)^d m G_m^{(\pm)}(x). \tag{5.73}$$

Therefore, it is enough to use the formula (5.43). □

5.7. Averages of plane waves on the de Sitter space

The one-sheeted hyperboloid in the physics literature is usually called the *de Sitter space*. It will be denoted $\mathrm{dS}_m = \mathrm{dS}_m^{d-1}$. It consists of points p such that $p^2 = m^2$. Let $d\Omega_m$ denote the standard measure on dS_m. We will compute the Fourier transform of the measure on dS_m.

THEOREM 16.

$$\int_{\mathrm{dS}_m} e^{ipx} d\Omega_m(p)$$

$$= m^{d-1}\pi^{\frac{d-1}{2}}\left(i^d U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2+i0)}{4}\right) + (-i)^d U_{\frac{d}{2}-1}\left(\frac{m^2(-x^2-i0)}{4}\right)\right) \tag{5.74}$$

$$= \begin{cases} (-1)^{\frac{d}{2}} 2m^{d-1}\pi^{\frac{d-1}{2}}\left(U_{\frac{d}{2}-1}\left(\frac{-m^2x^2 \pm i0}{4}\right) \pm \sqrt{\pi} i \left(\frac{-x^2}{4}\right)_-^{1-\frac{d}{2}} \mathbf{F}_{1-\frac{d}{2}}\left(-\frac{m^2x^2}{4}\right)\right), & \frac{d}{2} \in \mathbb{N}, \\ 2i^{d-1} m^{d-1}\pi^{\frac{d}{2}} \left(\frac{-x^2}{4}\right)_-^{1-\frac{d}{2}} \mathbf{F}_{1-\frac{d}{2}}\left(-\frac{m^2x^2}{4}\right), & \frac{d}{2} \notin \mathbb{N}. \end{cases} \tag{5.75}$$

Proof:

$$\int_{\mathrm{dS}_m} e^{ipx} d\Omega_m(p) = 2m \int e^{ipx}\delta(p^2 - m^2)dp \tag{5.76}$$

$$= \frac{m}{\pi i}\int e^{ipx}\left(\frac{1}{p^2-m^2-i0} - \frac{1}{p^2-m^2+i0}\right)dp \tag{5.77}$$

$$= \frac{m(2\pi)^d}{\pi i}\left(G_{im}^{\mathrm{F}}(x) - G_{-im}^{\overline{\mathrm{F}}}(x)\right). \tag{5.78}$$

Then we can use the result for the tachyonic Feynman and anti-Feynman propagator (5.55). □

One can see that the singular part is different in even- and odd-dimensional cases.

A. Appendix

A.1. Some identities

The following identities for $A > 0$ follow from the 2nd Euler integral:

$$\frac{1}{A^{\frac{\mu}{2}}} = \frac{1}{\Gamma(\frac{\mu}{2})}\int_0^\infty e^{-sA}s^{\frac{\mu}{2}-1}ds, \tag{A.1}$$

$$\frac{1}{(A \pm i0)^{\frac{\mu}{2}}} = \frac{e^{\mp i\frac{\pi\mu}{4}}}{\Gamma(\frac{\mu}{2})}\int_0^\infty e^{\pm itA}t^{\frac{\mu}{2}-1}dt. \tag{A.2}$$

We will also need the Fourier transform of the Gaussian function on the Euclidean space $\mathbb{R}^d$, and of the Fresnel function on the pseudo-Euclidean space $\mathbb{R}^{q,d-q}$ (with q minuses):

$$\int dpe^{-sp^2}e^{ipx} = \left(\frac{\pi}{s}\right)^{\frac{d}{2}}e^{-\frac{x^2}{4s}}, \tag{A.3}$$

$$\int dpe^{\pm itp^2}e^{ipx} = (\mp i)^q\left(\frac{\pi}{t}\right)^{\frac{d}{2}}e^{\pm i\frac{\pi}{4}d}e^{\mp i\frac{x^2}{4t}}. \tag{A.4}$$

A.2. Distributions

In this paper, we often use the language of distributions on $\mathbb{R}^d$. We say that a distribution T is *regular* if there exists a locally integrable function f such that for a test function Φ,

$$T(\Phi) = \int f(x)\Phi(x)dx. \tag{A.5}$$

We will use the integral notation also for irregular distributions, e.g.

$$\int \delta^{(n)}(x)\Phi(x)dx = (-1)^n\Phi^{(n)}(0). \tag{A.6}$$

Let us now consider some special distributions on $\mathbb{R}$. For any $\lambda \in \mathbb{C}$,

$$(\pm ix + 0)^\lambda = e^{\pm i\lambda\frac{\pi}{2}}(x \mp i0)^\lambda := \lim_{\epsilon\searrow 0}(\pm ix + \epsilon)^\lambda$$

is a tempered distribution. If $\operatorname{Re}\lambda > -1$, then it is regular and given by the locally integrable function

$$e^{\pm i\operatorname{sgn}(x)\frac{\pi}{2}\lambda}|x|^\lambda. \tag{A.7}$$

The functions

$$x_\pm^\lambda := |x|^\lambda \theta(\pm x) \tag{A.8}$$

define regular distributions only for $\operatorname{Re}\lambda > -1$. We can extend them to $\lambda \in \mathbb{C}$ except for $\lambda = -1, -2, \ldots$ by putting

$$x_\pm^\lambda := \frac{1}{2i \sin \pi\lambda}\big(-e^{-i\frac{\pi}{2}\lambda}(\mp ix + 0)^\lambda + e^{i\frac{\pi}{2}\lambda}(\pm ix + 0)^\lambda\big). \tag{A.9}$$

For $\lambda > -1$, (A.9) are regular and coincide with $\theta(\pm x)|x|^\lambda$. We have

$$x_\pm^{\lambda+1} = |x| \cdot x_\pm^\lambda. \tag{A.10}$$

Instead of $x_\pm^\lambda$, it is often more convenient to consider

$$\rho_\pm^\lambda(x) := \frac{x_\pm^\lambda}{\Gamma(\lambda+1)} \tag{A.11}$$

$$= \frac{\Gamma(-\lambda)}{2\pi i}\big(e^{-i\frac{\pi}{2}\lambda}(\mp ix + 0)^\lambda - e^{i\frac{\pi}{2}\lambda}(\pm ix + 0)^\lambda\big). \tag{A.12}$$

Note that using (A.11) and (A.12) we have defined $\rho_\pm^\lambda$ for all $\lambda \in \mathbb{C}$. We have

$$\partial_x \rho_\pm^\lambda(x) = \pm\rho_\pm^{\lambda-1}(x).$$

At integers we have

$$\rho_\pm^n(x) = \frac{x_\pm^n}{n!}, \qquad n = 0, 1, \ldots, \tag{A.13}$$

$$\rho_\pm^{-n-1}(x) = (\pm 1)^n \delta^n(x), \qquad n = 0, 1, \ldots. \tag{A.14}$$

Clearly, for $\operatorname{Re}(\lambda) \leq -1$ the distributions $\rho_\pm^\lambda$ are irregular.

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REFERENCES

[1] N. Aronszajn, K.T. Smith: Theory of Bessel potentials. I, *Ann. Inst. Fourier* **11**, 385–475 (1961).
[2] A.P. Calderon: Lebesgue spaces of differentiable functions and distributions, *Proc. Symp. Pure Math.* **4**, 33–49 (1961).
[3] E. Stein: *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton 1970.
[4] D.R. Adams and L.I. Hedberg: *Function Spaces and Potential Theory*, Springer-Verlag Berlin, Heidelberg 1996.
[5] J. Deny: Les potentiels d'energie finie, *Acta Math.* **82**, 107–183 (1950).

[6] O. Steinmann: *Perturbation Expansions in Axiomatic Field Theory*, Lecture Notes in Physics 11, Springer Berlin, Heidelberg 1971.
[7] R. Brunetti and K. Fredenhagen: Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds, *Commun. Math. Phys.* **208**, 623–661 (2000).
[8] J.D. Björken and S.D. Drell: *Relativistic Quantum Fields*, McGraw-Hill, New York 1965.
[9] N.N. Bogoliubov and D.V. Shirkov: *Introduction to the Theory of Quantized Fields*, 3rd ed., J. Wiley, 1980.
[10] Y.A. Brychkov and A.P. Prudnikov: *Integralnyje preobrazovania obobshchennych funkcji*, Nauka, 1977 (in Russian).
[11] I.S. Gradshteyn and I.M. Ryzhik: *Table of Integrals, Series, and Products*, 7th ed., Academic Press, 2007.
[12] I.M. Gelfand and G.E. Shilov: *Generalized Functions, Volume 1: Properties and Operations*, AMS Chelsea Publishing, 1967.
[13] J. Dereziński and C. Gaß: Propagators in curved spacetimes from operator theory, arXiv:2409.03279.
[14] J. Duistermaat and L. Hörmander: Fourier integral operators. II, *Acta Mathematica* **128**, 183–269 (1972).
[15] F. G. Friedlander: *The Wave Equation on a Curved Space-Time*, Cambridge University Press, 1975.
[16] C. Bär, N. Ginoux and F. Pfaeffle: *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2007.
[17] J. Dereziński: Hypergeometric type functions and their symmetries, *Ann. Henri Poincaré* **15**, 1569–1653 (2014).
[18] J. Dereziński: Group-theoretical origin of symmetries of hypergeometric class equations and functions, in: G. Filipuk, A. Lastra, S. Michalik, Y. Takei and H. Żołądek (eds.), *Complex Differential and Difference Equations, Proceedings of the school and conference held at Bedlewo, Poland, September 2–15, 2018*, De Gruyter Proc. Math., Berlin 2020, 3–27.
[19] G. Fikioris: *Mellin-Transform Method for Integral Evaluation*, Springer Cham, 2007.
[20] L. Hörmander: *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer Berlin, Heidelberg 2003.
[21] K. Jodłowski: Covariant quantum field theory of tachyons is unphysical, arXiv:2406.14225.
[22] J. Dhar and E. C. G. Sudarshan: Quantum field theory of interacting tachyons, *Phys. Rev.* **174**, 1808–1815 (1968).
[23] J. Paczos, K. Dębski, S. Cedrowski, S. Charzyński, K. Turzyński, A. Ekert, and A. Dragan: Covariant quantum field theory of tachyons, *Phys. Rev. D* **110**, 015006 (2024).