A representation theory approach to Lie–Hamilton systems based on $\mathfrak{sp}(4,\mathbb{R})$ and applications

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Outline

Lie systems

A *Lie system*[1] on a manifold M is a first-order system of ODE's in normal form on M which admits a superposition rule

[1] J. de Lucas and C. Sardón. *A Guide to Lie Systems with Compatible Geometric Structures.* (Singapore: World Scientific) 2020.

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A *Lie system*[1] on a manifold M is a first-order system of ODE's in normal form on M which admits a superposition rule, i.e., a t-independent map $\Phi : M^k \times M \to M$ such that

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(k)}(t); \lambda)$$

is the general solution of the system, where $\lambda \in M$ is a point related to the initial conditions.

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Some examples

(1) N-dimensional t-dependent frequency Smorodinsky–Winternitz oscillator.

(2) Riccati equations and Kummer–Schwartz equations.

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(2) Riccati equations and Kummer–Schwartz equations.

(3) New!: t-dependent electromagnetic fields and t-dependent coupled oscillators.

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Characterizing Lie systems

Theorem (Lie–Scheffers, 1893)

A first-order system of ordinary differential equations $\mathbf{X}$ *on a n-dimensional manifold* M *of the form*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \mathbf{X}_i(t,x), \qquad 1 \leq i \leq n$$

admits a superposition rule if and only if

(i) *Exist* $\beta^i \in C^\infty(\mathbb{R})$, $1 \leq i \leq \ell$,

$$\mathbf{X}(t,x) = \beta^i(t)\mathbf{X}_i(x),$$

(ii) *The real Lie algebra (Vessiot–Guldberg Lie algebra)*

$$V^X := \mathbb{R}\langle \mathbf{X}_i : 1 \leq i \leq \ell \rangle$$

is ℓ*-dimensional.*

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Some remarkable facts about LH systems

(1) LH systems on $\mathbb{R}^2$ have been classified under local diffeomorphisms[ab].

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(1) LH systems on $\mathbb{R}^2$ have been classified under local diffeomorphisms[a][b].

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(3) Quantum group theory can be applied to $\mathcal{H}_\omega$, yielding to the so-called *Poisson–Hopf deformations of LH systems*.

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[a] A. Ballesteros, A. Blasco, F. J. Herranz, J. de Lucas and C. Sardón. Lie–Hamilton systems on the plane: Properties, classification and applications. *J. Diff. Equ.* **258** (2015) 2873–2907.

[b] A. González-López, N. Kamran and P. J. Olver. Lie Algebras of Vector Fields in the Real Plane. *Proc. London Math. Soc.* **64** (1992) 339–368.

[c] A. Ballesteros, J. F. Cariñena, F. J. Herranz, J. de Lucas and C. Sardón. From constants of motion to superposition rules for Lie–Hamilton systems. *J. Phys. A: Math. Theor.* **46** (2013) 285203.

Outline

Lie systems from representations[2] [3]

- $\mathfrak{g}$ r-dimensional real Lie algebra with generators $X_1, \ldots, X_r$.

[2]R. Campoamor-Stursberg. Reduction by invariants and projection of linear representations of Lie algebras applied to the construction of nonlinear realizations. *J. Math. Phys.* **59** (2018) 033502.

[3] R. Campoamor-Stursberg. Invariant functions of vector field realizations of Lie algebras and some applications to representation theory and dynamical systems. *J. Phys.: Conf. Ser.* **1071** (2018) 012005.

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- (Linear) realization $\Phi_\Gamma : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ spanned by

$$\mathbf{X}_\alpha := \Phi_\Gamma(X_\alpha) = x^i \Gamma(X_\alpha)_i^j \frac{\partial}{\partial x^j}, \qquad 1 \leq \alpha \leq r.$$

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- Lie system $\mathbf{X}$ on $\mathbb{R}^n$ given by

$$\mathbf{X} := \sum_{\alpha=1}^{r} b_\alpha(t) \mathbf{X}_\alpha.$$

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With VG Lie algebra $V^X = \mathbb{R}\langle \mathbf{X}_1, \ldots, \mathbf{X}_r \rangle \simeq \mathfrak{g}$.

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The Lorentz Lie algebra $\mathfrak{so}(1,3)$

- Real form of $\mathfrak{sl}(2,\mathbb{C})$: $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})$.

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- Real form of $\mathfrak{sl}(2,\mathbb{C})$: $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})$.
- Spanned by J, P_i, K_i, H $(1 \leq i \leq 2)$, with commutation relations

$$
\begin{array}{llll}
[J, P_i] = \epsilon_{ij} P_j, & [J, K_i] = \epsilon_{ij} K_j, & [J, H] = 0, & [P_1, P_2] = -J, \\
[K_1, K_2] = -J, & [P_i, K_j] = -\delta_{ij} H, & [H, P_i] = -K_i, & [H, K_i] = -P_i.
\end{array}
$$

- Faithful and irreducible representation $\Gamma : \mathfrak{so}(1,3) \to \mathfrak{gl}(4,\mathbb{R})$

$$
A_\Gamma := \frac{1}{2} \begin{pmatrix} K_1 & P_2 & -H - P_1 & -J + K_2 \\ -P_2 & K_1 & -J + K_2 & H + P_1 \\ -H + P_1 & J + K_2 & -K_1 & P_2 \\ J + K_2 & H - P_1 & -P_2 & -K_1 \end{pmatrix}
$$

- Associated realization $\Phi_\Gamma : \mathfrak{so}(1,3) \to \mathfrak{gl}(4,\mathbb{R})$

$$\mathbf{X}_1 := \Phi_\Gamma(J) = \frac{1}{2}\left(p_2\frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_2}\right),$$

$$\mathbf{X}_2 := \Phi_\Gamma(P_1) = \frac{1}{2}\left(p_1\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial p_2}\right),$$

$$\mathbf{X}_3 := \Phi_\Gamma(P_2) = \frac{1}{2}\left(-q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}\right),$$

$$\mathbf{X}_4 := \Phi_\Gamma(H) = \frac{1}{2}\left(-p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial p_2}\right),$$

$$\mathbf{X}_5 := \Phi_\Gamma(K_1) = \frac{1}{2}\left(q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2} - p_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial p_2}\right),$$

$$\mathbf{X}_6 := \Phi_\Gamma(K_2) = \frac{1}{2}\left(p_2\frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial p_2}\right).$$

- Induced Lie system $\mathbf{X} := \sum_{\alpha=1}^{6} b_\alpha(t)\mathbf{X}_\alpha$

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = \frac{1}{2}\left(-b_3(t)q_2 + b_5(t)q_1 + (b_2(t) - b_4(t))p_1 + (b_1(t) + b_6(t))p_2\right),$$

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = \frac{1}{2}\left(b_3(t)q_1 + b_5(t)q_2 + (b_1(t) + b_6(t))p_1 + (b_4(t) - b_2(t))p_2\right),$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{1}{2}\left(-(b_2(t) + b_4(t))q_1 + (b_6(t) - b_1(t))q_2 - b_5(t)p_1 - b_3(t)p_2\right),$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \frac{1}{2}\left((b_6(t) - b_1(t))q_1 + (b_2(t) + b_4(t))q_2 + b_3(t)p_1 - b_5(t)p_2\right).$$

- Induced Lie system $\mathbf{X} := \sum_{\alpha=1}^{6} b_\alpha(t)\mathbf{X}_\alpha$

$$
\begin{aligned}
\frac{\mathrm{d}q_1}{\mathrm{d}t} &= \frac{1}{2}\left(-b_3(t)q_2 + b_5(t)q_1 + (b_2(t) - b_4(t))p_1 + (b_1(t) + b_6(t))p_2\right),\\
\frac{\mathrm{d}q_2}{\mathrm{d}t} &= \frac{1}{2}\left(b_3(t)q_1 + b_5(t)q_2 + (b_1(t) + b_6(t))p_1 + (b_4(t) - b_2(t))p_2\right),\\
\frac{\mathrm{d}p_1}{\mathrm{d}t} &= \frac{1}{2}\left(-(b_2(t) + b_4(t))q_1 + (b_6(t) - b_1(t))q_2 - b_5(t)p_1 - b_3(t)p_2\right),\\
\frac{\mathrm{d}p_2}{\mathrm{d}t} &= \frac{1}{2}\left((b_6(t) - b_1(t))q_1 + (b_2(t) + b_4(t))q_2 + b_3(t)p_1 - b_5(t)p_2\right).
\end{aligned}
$$

- $\omega = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$ canonical symplectic form on $T^*\mathbb{R}^2 \rightsquigarrow \mathcal{L}_{\mathbf{X}_\alpha}\omega = 0 \rightsquigarrow \mathbf{X}$ is an LH system.

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- Hamiltonian functions ($\iota_{\mathbf{X}_\alpha}\omega = \mathrm{d}h_\alpha$)

$$h_1 = \frac{1}{2}(q_1q_2 + p_1p_2),\ h_2 = \frac{1}{4}\left(q_1^2 - q_2^2 + p_1^2 - p_2^2\right),\ h_3 = \frac{1}{2}(q_1p_2 - q_2p_1),$$
$$h_4 = \frac{1}{4}\left(q_1^2 - q_2^2 - p_1^2 + p_2^2\right),\ h_5 = \frac{1}{2}(q_1p_1 + q_2p_2),\ h_6 = \frac{1}{2}(-q_1q_2 + p_1p_2)$$

span an LH algebra $\mathcal{H}_\omega \simeq \mathfrak{so}(1,3)$ with respect to $\{\cdot,\cdot\}_\omega$.

- Choose $a_\alpha \in C^\infty(\mathbb{R})$ such that

$$a_1 := b_1 + b_6,\ a_2 := b_2 - b_4,\ a_3 := b_3,\ a_4 := b_2 + b_4,\ a_5 := b_5,\ a_6 := b_1 - b_6.$$

yielding to the change of basis

$$h_1' := h_1 + h_6 = p_1 p_2, \qquad h_2' := h_2 - h_4 = \frac{1}{2}\left(p_1^2 - p_2^2\right),$$

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$$h'_1 := h_1 + h_6 = p_1 p_2, \qquad h'_2 := h_2 - h_4 = \frac{1}{2}\left(p_1^2 - p_2^2\right),$$
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on the LH algebra $\mathcal{H}_\omega \simeq \mathfrak{so}(1,3)$.

- Take $a_\alpha \equiv 0$ for $\alpha \neq 2,3,4$, so h'_2, h'_3, h'_4 span $\mathcal{H}_\omega \simeq \mathfrak{so}(1,3)$ and

$$h' := \frac{a_2(t)}{2}\left(p_1^2 - p_2^2\right) + \frac{a_3(t)}{2}(q_1 p_2 - q_2 p_1) + \frac{a_4(t)}{2}\left(q_1^2 - q_2^2\right).$$

is the t-dependent Hamiltonian of correspondent the system.

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- h_2' is the Lorentzian kinetic energy, h_3' is the angular momentum and h_4' is the potential of the isotropic oscillator on the Minkowskian plane $\mathbf{M}^{1+1} \equiv (\mathbb{R}^2, \mathrm{d}s^2 = \mathrm{d}q_1^2 - \mathrm{d}q_2^2)$.

h' describes an LH system on $T^*\mathbf{M}^{1+1}$, which is a coupling between two 1D LH systems on $T^*\mathbb{R}$ through the angular momentum term:

$$h' = \left(\frac{a_2(t)}{2}p_1^2 + \frac{a_4(t)}{2}q_1^2\right) - \left(\frac{a_2(t)}{2}p_2^2 + \frac{a_4(t)}{2}q_2^2\right) + \frac{a_3(t)}{2}(q_1p_2 - q_2p_1).$$

Example: a t-dependent Bateman Hamiltonian

$$a_2(t) = \frac{1}{m(t)}, \; a_3(t) = \frac{\gamma(t)}{m(t)}, \; a_4(t) = m(t)\Omega^2(t),$$

where

$$\Omega(t) := \sqrt{\frac{1}{m(t)}\left(k(t) - \frac{\gamma^2(t)}{4m(t)}\right)}$$

and $m(t), k(t), \gamma(t) > 0, \; k(t) > \frac{\gamma^2(t)}{4m(t)}$.

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$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = \frac{1}{2m(t)}\left(-\gamma(t)q_2 + p_1\right) \qquad \frac{\mathrm{d}q_2}{\mathrm{d}t} = \frac{1}{2m(t)}\left(\gamma(t)q_1 - p_2\right),$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{1}{2m(t)}\left(-m^2(t)\Omega^2(t)q_1 - \gamma(t)p_2\right), \quad \frac{\mathrm{d}p_2}{\mathrm{d}t} = \frac{1}{2m(t)}\left(m^2(t)\Omega^2(t)q_2 + \gamma(t)p_1\right).$$

is a coupling of two 1D t-dependent harmonic oscillators.

Example: a coupled Caldirola–Kanai Hamiltonian

$$m(t), k(t), \gamma(t) > 0, \qquad \lambda(t) := \frac{\gamma(t)}{m(t)}, \qquad \Omega(t) := \sqrt{\frac{k(t)}{m(t)}}$$

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$$a_2(t) = \frac{1}{m(t)} \mathrm{e}^{-2\int_0^t \lambda(s)\,\mathrm{d}s}, \qquad a_4(t) = m(t)\Omega^2(t)\mathrm{e}^{2\int_0^t \lambda(s)\,\mathrm{d}s},$$

$$\begin{aligned}
\frac{\mathrm{d}q_1}{\mathrm{d}t} &= \frac{1}{2}\left(-a_3(t)q_2 + \frac{1}{m(t)}\mathrm{e}^{-2\int_0^t \lambda(s)\,\mathrm{d}s} p_1\right), \\
\frac{\mathrm{d}q_2}{\mathrm{d}t} &= \frac{1}{2}\left(a_3(t)q_1 - \frac{1}{m(t)}\mathrm{e}^{-2\int_0^t \lambda(s)\,\mathrm{d}s} p_2\right), \\
\frac{\mathrm{d}p_1}{\mathrm{d}t} &= \frac{1}{2}\left(-m(t)\Omega^2(t)\mathrm{e}^{2\int_0^t \lambda(s)\,\mathrm{d}s} q_1 - a_3(t)p_2\right), \\
\frac{\mathrm{d}p_2}{\mathrm{d}t} &= \frac{1}{2}\left(m(t)\Omega^2(t)\mathrm{e}^{2\int_0^t \lambda(s)\,\mathrm{d}s} q_2 + a_3(t)p_1\right).
\end{aligned}$$

is a coupling of two 1D Caldirola–Kanai Hamiltonians.

The symplectic Lie algebra $\mathfrak{sp}(4,\mathbb{R})$

- Creation and annihilation operators a_i and $a_i^\dagger$ $(1 \leq i \leq 2)$:

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.$$

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- Label the basis as

$$X_{i,j} := a_i^\dagger a_j, \qquad X_{-i,j} := a_i^\dagger a_j^\dagger, \qquad X_{i,-j} := a_i a_j,$$

with

$$X_{i,j} + \varepsilon_i \varepsilon_j X_{-j,-i} = 0, \qquad \varepsilon_i := \mathrm{sgn}(i), \qquad \varepsilon_j := \mathrm{sgn}(j), \qquad -2 \leq i, j \leq 2.$$

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- Commutation relations:

$$[X_{i,j}, X_{k,\ell}] = \delta_{jk} X_{i,\ell} - \delta_{i\ell} X_{k,j} + \varepsilon_i \varepsilon_j \delta_{j,-\ell} X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k} X_{-j,\ell}, \quad -2 \leq i, j, k, \ell \leq 2.$$

- Fundamental representation $\Gamma_{\omega_1} : \mathfrak{sp}(4,\mathbb{R}) \to \mathfrak{gl}(4,\mathbb{R})$

$$A_{\Gamma_{\omega_1}} := \begin{pmatrix} X_{1,1} & X_{1,2} & -X_{-1,1} & -X_{-1,2} \\ X_{2,1} & X_{2,2} & -X_{-1,2} & -X_{-2,2} \\ X_{1,-1} & X_{1,-2} & -X_{1,1} & -X_{2,1} \\ X_{1,-2} & X_{2,-2} & -X_{1,2} & -X_{2,2} \end{pmatrix}.$$

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- Associated realization $\Phi_{\Gamma_{\omega_1}} : \mathfrak{sp}(4,\mathbb{R}) \to \mathfrak{X}(T^*\mathbb{R}^2)$

$$\mathbf{X}_1 := \Phi_{\Gamma_{\omega_1}}(X_{1,1}) = q_1\frac{\partial}{\partial q_1} - p_1\frac{\partial}{\partial p_1}, \qquad \mathbf{X}_2 := \Phi_{\Gamma_{\omega_1}}(X_{1,2}) = q_1\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial p_1},$$

$$\mathbf{X}_3 := \Phi_{\Gamma_{\omega_1}}(X_{2,1}) = q_2\frac{\partial}{\partial q_1} - p_1\frac{\partial}{\partial p_2}, \qquad \mathbf{X}_4 := \Phi_{\Gamma_{\omega_1}}(X_{2,2}) = q_2\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial p_2},$$

$$\mathbf{X}_5 := \Phi_{\Gamma_{\omega_1}}(X_{-1,1}) = -q_1\frac{\partial}{\partial p_1}, \qquad \mathbf{X}_6 := \Phi_{\Gamma_{\omega_1}}(X_{-1,2}) = -q_2\frac{\partial}{\partial p_1} - q_1\frac{\partial}{\partial p_2},$$

$$\mathbf{X}_7 := \Phi_{\Gamma_{\omega_1}}(X_{-2,2}) = -q_2\frac{\partial}{\partial p_2}, \qquad \mathbf{X}_8 := \Phi_{\Gamma_{\omega_1}}(X_{1,-1}) = p_1\frac{\partial}{\partial q_1},$$

$$\mathbf{X}_9 := \Phi_{\Gamma_{\omega_1}}(X_{1,-2}) = p_2\frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial q_2} \qquad \mathbf{X}_{10} := \Phi_{\Gamma_{\omega_1}}(X_{2,-2}) = p_2\frac{\partial}{\partial q_2}.$$

- Induced Lie system $\mathbf{X} = \sum_{\alpha=1}^{10} b_\alpha(t)\mathbf{X}_\alpha$

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = b_1(t)q_1 - b_3(t)q_2 + b_8(t)p_1 + b_9(t)p_2,$$
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$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -b_1(t)p_1 - b_2(t)p_2 - b_5(t)q_1 - b_6(t)q_2,$$
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\frac{\mathrm{d}p_1}{\mathrm{d}t} &= -b_1(t)p_1 - b_2(t)p_2 - b_5(t)q_1 - b_6(t)q_2, \\
\frac{\mathrm{d}p_2}{\mathrm{d}t} &= -b_3(t)p_1 - b_4(t)p_2 - b_6(t)q_1 - b_7(t)q_2.
\end{aligned}
$$

- $\omega = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$ canonical symplectic form on $T^*\mathbb{R}^2 \rightsquigarrow \mathcal{L}_{\mathbf{X}_\alpha}\omega = 0 \rightsquigarrow \mathbf{X}$ is an LH system.

- Hamiltonian functions ($\iota_{\mathbf{X}_\alpha}\omega = \mathrm{d}h_\alpha$)

$$h_1 = q_1 p_1, \quad h_2 = q_1 p_2, \quad h_3 = q_2 p_1, \quad h_4 = q_2 p_2, \quad h_5 = \frac{1}{2} q_1^2,$$
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  span an LH algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4, \mathbb{R})$.
- h_2, h_3, h_5, h_7, h_8 and h_{10} are a set of generators

$$h = b_2(t) q_1 p_2 + b_3(t) q_2 p_1 + \frac{b_5(t)}{2} q_1^2 + \frac{b_7(t)}{2} q_2^2 + \frac{b_8(t)}{2} p_1^2 + \frac{b_{10}(t)}{2} p_2^2.$$

Example: a t-dependent electromagnetic field

- Take $m_i, e_i, \gamma \in C^\infty(\mathbb{R})$ $(1 \leq i \leq 2)$ such that $m_1(t), m_2(t) > 0$ and $\ddot{\gamma}(t) \neq 0$.

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$$\phi := \phi_1 + \phi_2, \qquad \phi_1 := \frac{1}{2}q_1^2, \qquad \phi_2 := \frac{1}{2}q_2^2.$$

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- Choose

$$b_2(t) = -\frac{\gamma(t)e_2(t)}{2m_2(t)}, \qquad b_3(t) = \frac{\gamma(t)e_1(t)}{2m_1(t)}, \; b_5(t) = e_1(t) + \frac{\gamma^2(t)e_2^2(t)}{4m_1^2(t)},$$

$$b_7(t) = e_2(t) + \frac{\gamma^2(t)e_1^2(t)}{4m_2^2(t)}, \; b_8(t) = \frac{1}{m_1(t)}, \qquad b_{10}(t) = \frac{1}{m_2(t)}.$$

- t-dependent Hamiltonian

$$h^{\mathrm{E}} = \frac{1}{2m_1(t)}\left(p_1 - e_1(t)A_1\right)^2 + e_1(t)\phi_1 + \frac{1}{2m_2(t)}\left(p_2 - e_2(t)A_2\right)^2 + e_2(t)\phi_2.$$

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- t-dependent magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, \gamma(t)).$$

- t-dependent electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{2}\left(-2q_1 - q_1\dot{\gamma}(t), -2q_2 + q_2\dot{\gamma}(t), 0\right).$$

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The coupled systems obtained from $\mathfrak{so}(1,3)$ can be generalized to $\mathfrak{sp}(4,\mathbb{R})$-LH systems consisting on different coupled oscillators in a natural way.

Outline

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- Reduction by scaling symmetries. Contact Lie systems on three-dimensional Riemannian and Lorentzian spaces of constant curvature (Cayley–Klein approach).

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- Construction within $\mathfrak{sp}(2N, \mathbb{R})$.
- Poisson–Hopf deformations of the $\mathfrak{sp}(4, \mathbb{R})$-LH systems.

Outline

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Thank you for your attention