Integrable and chaotic generalizations of the rolling of a convex body without slipping on a plane

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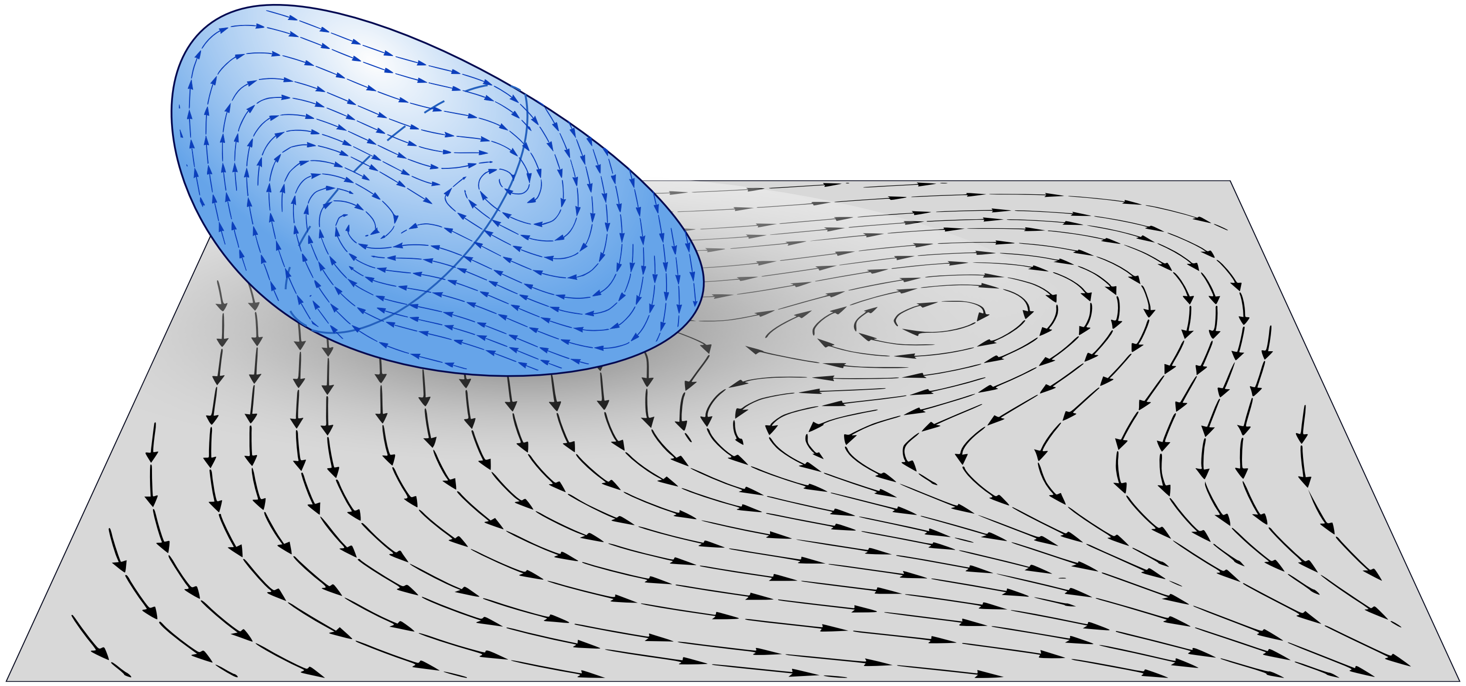
Joint work with Luis García Naranjo

UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO MATEMATICA
DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"



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Classic convex body rolling without slipping on a plane

Nonholonomic system

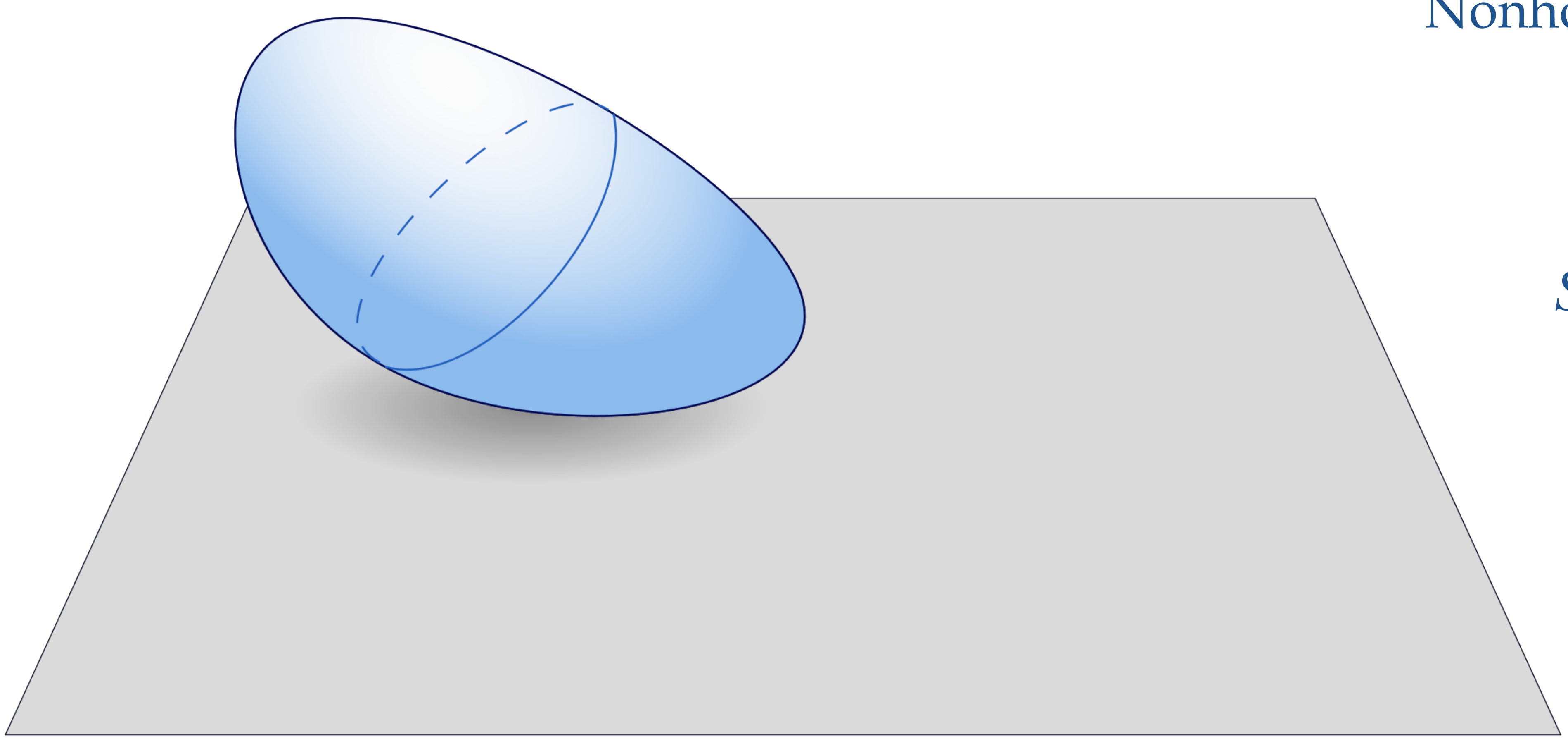
Configuration manifold: $Q = SO(3) \times \mathbb{R}^2$

$(B, \underline{x}) \in Q$

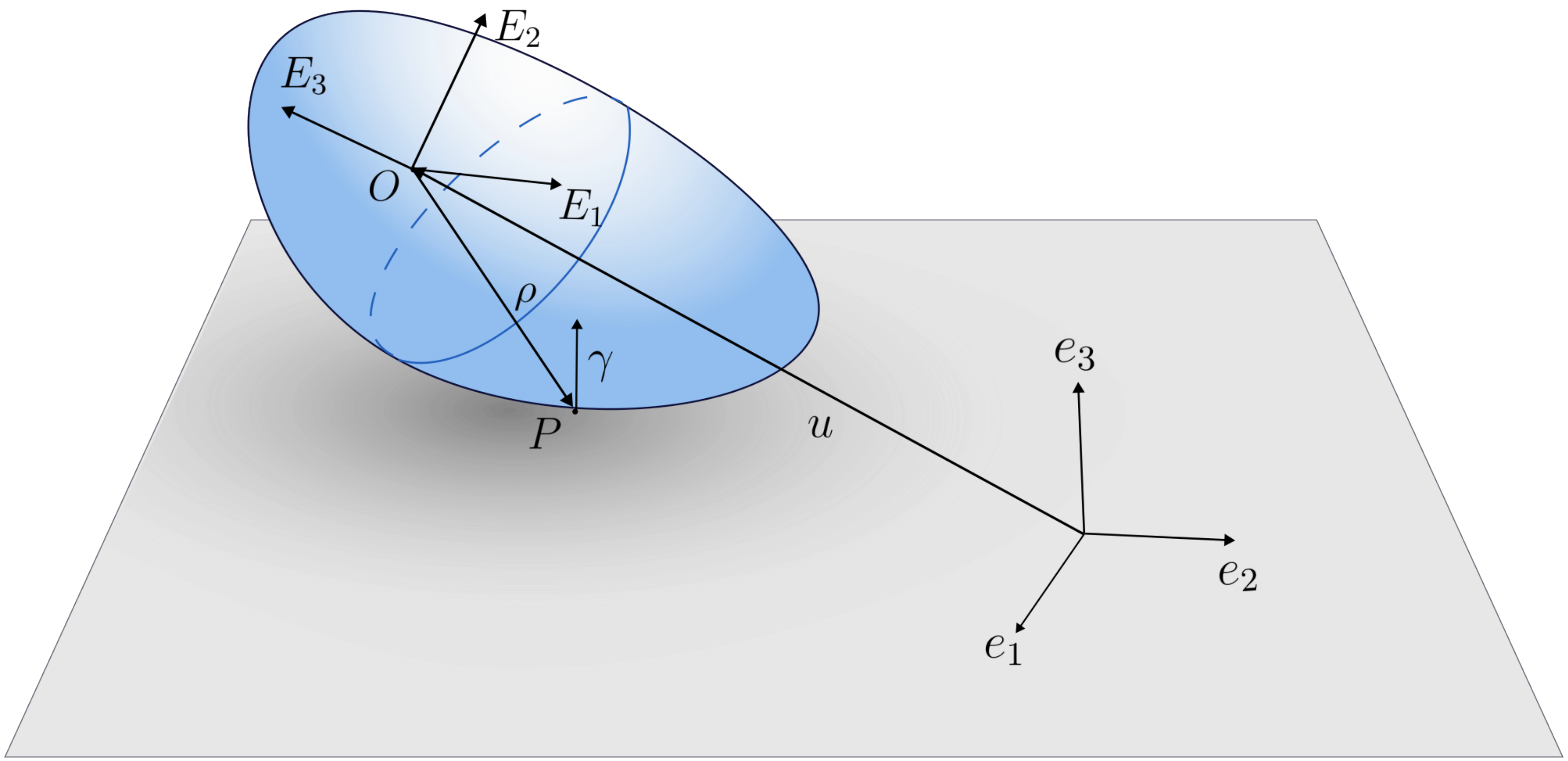
Nonholonomic linear constraint

Phase space: dim 8

$SE(2)$- symmetry on plane



Classic convex body rolling without slipping on a plane



$$\gamma = B^{-1} e_3$$

$$\rho = \rho(\gamma)$$

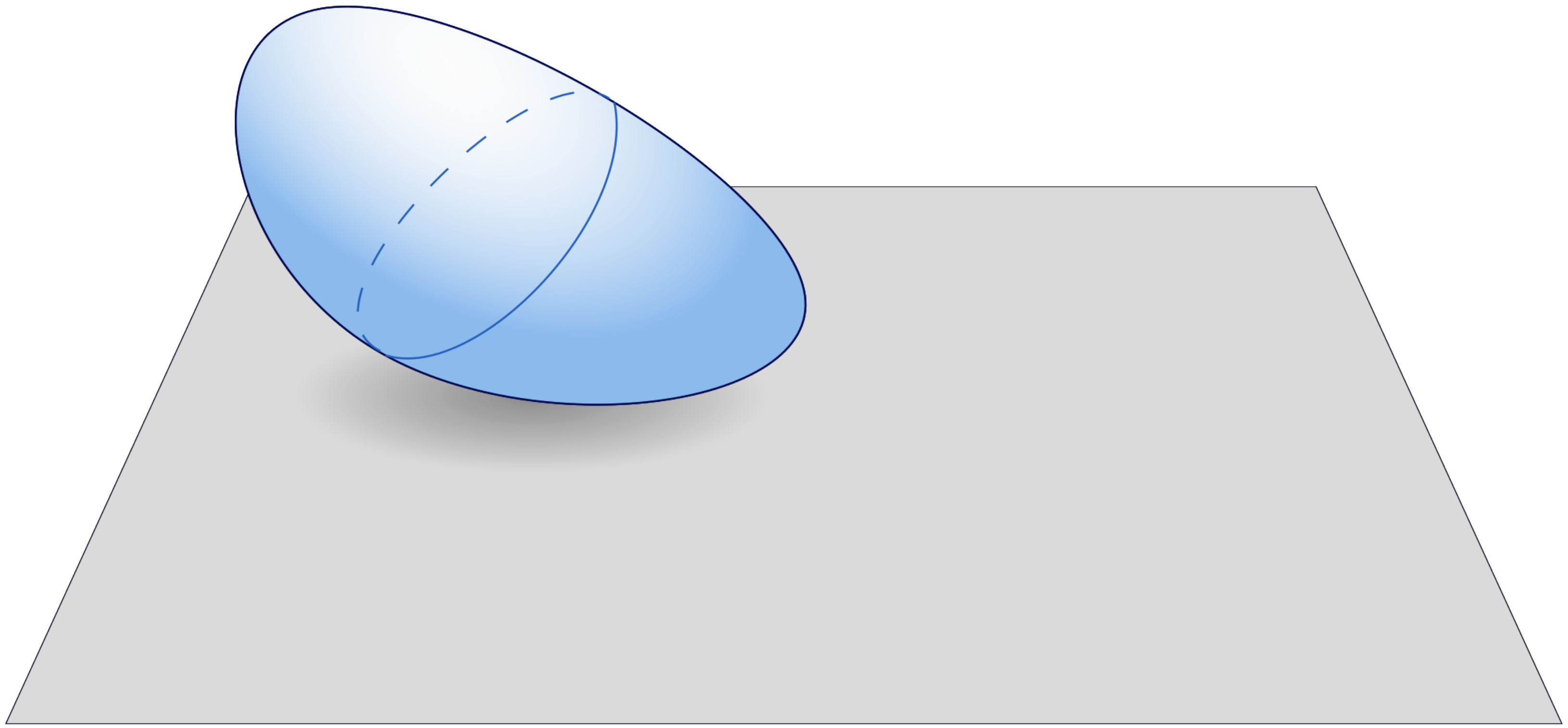
$$\underline{x} = u + B\rho$$

Equations of motion:

$$\dot{M} = M \times \Omega + m\dot{\rho} \times (\Omega \times \rho) + mg\rho \times \gamma$$

$$\dot{\gamma} = \Omega \times \gamma$$

$$M = \mathbb{I}\Omega + m\rho \times (\Omega \times \rho), \quad \rho = \rho(\gamma)$$



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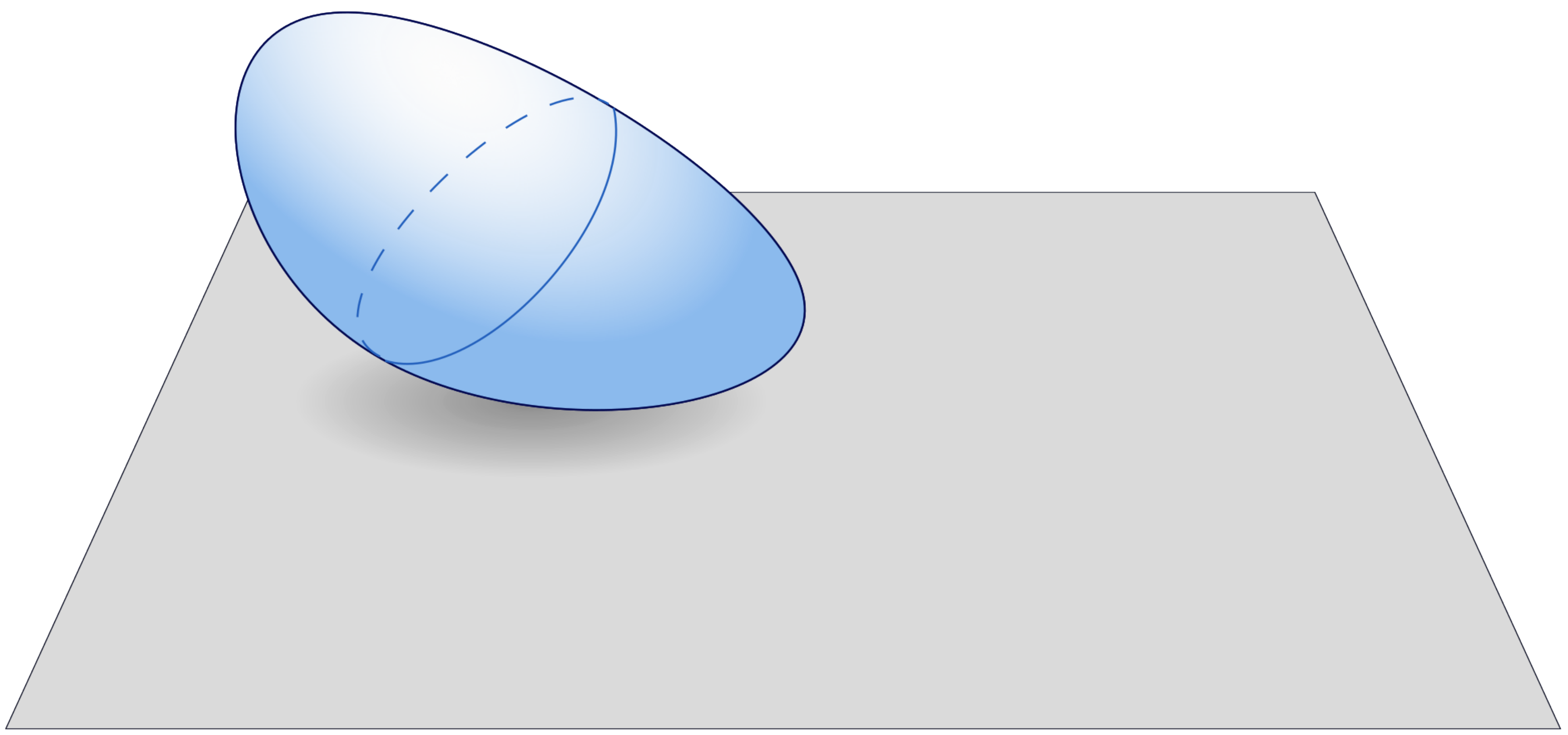
First integrals:

$$H = \frac{1}{2}\langle M, \Omega\rangle - mg\langle \rho, \gamma\rangle, \qquad \|\gamma\|^2 = 1$$



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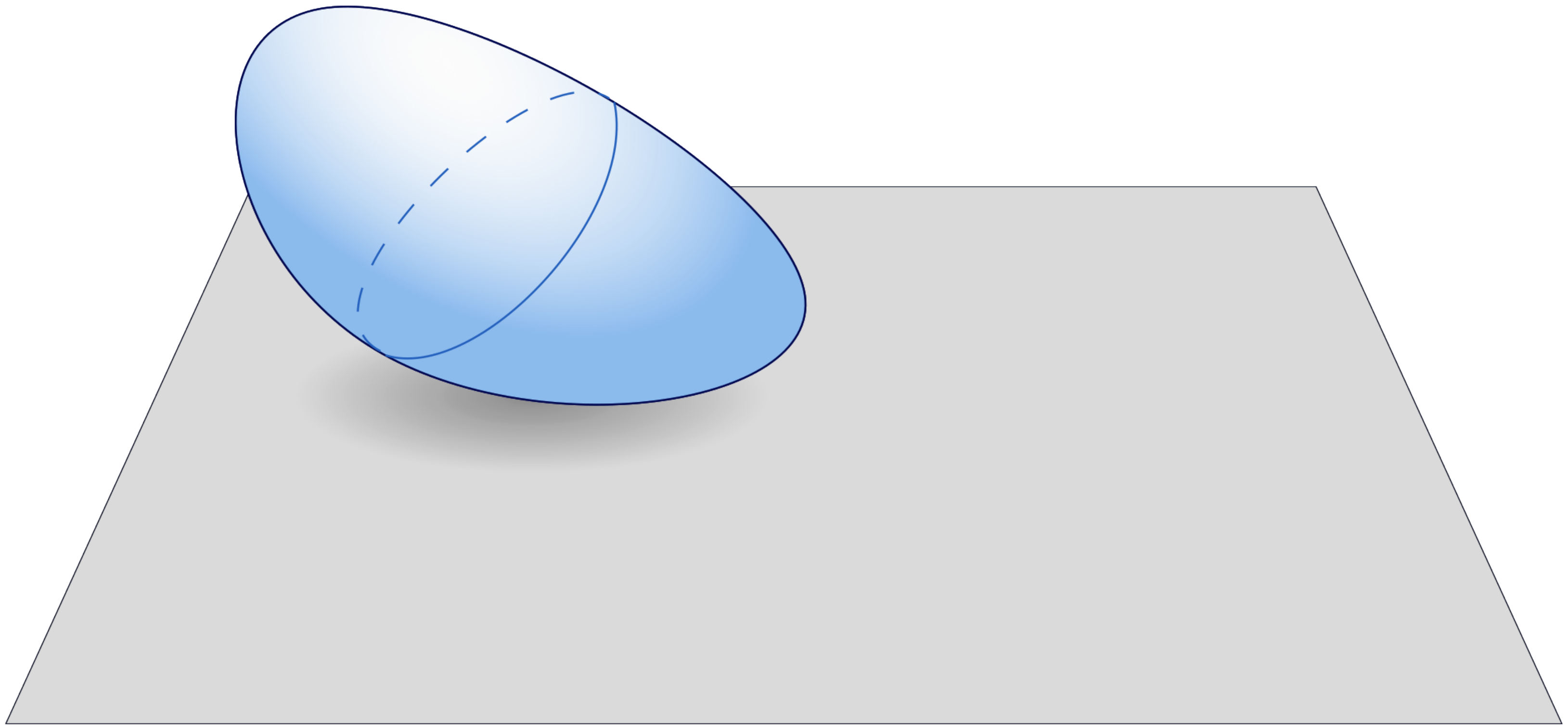
- No invariant measure
- Generally chaotic



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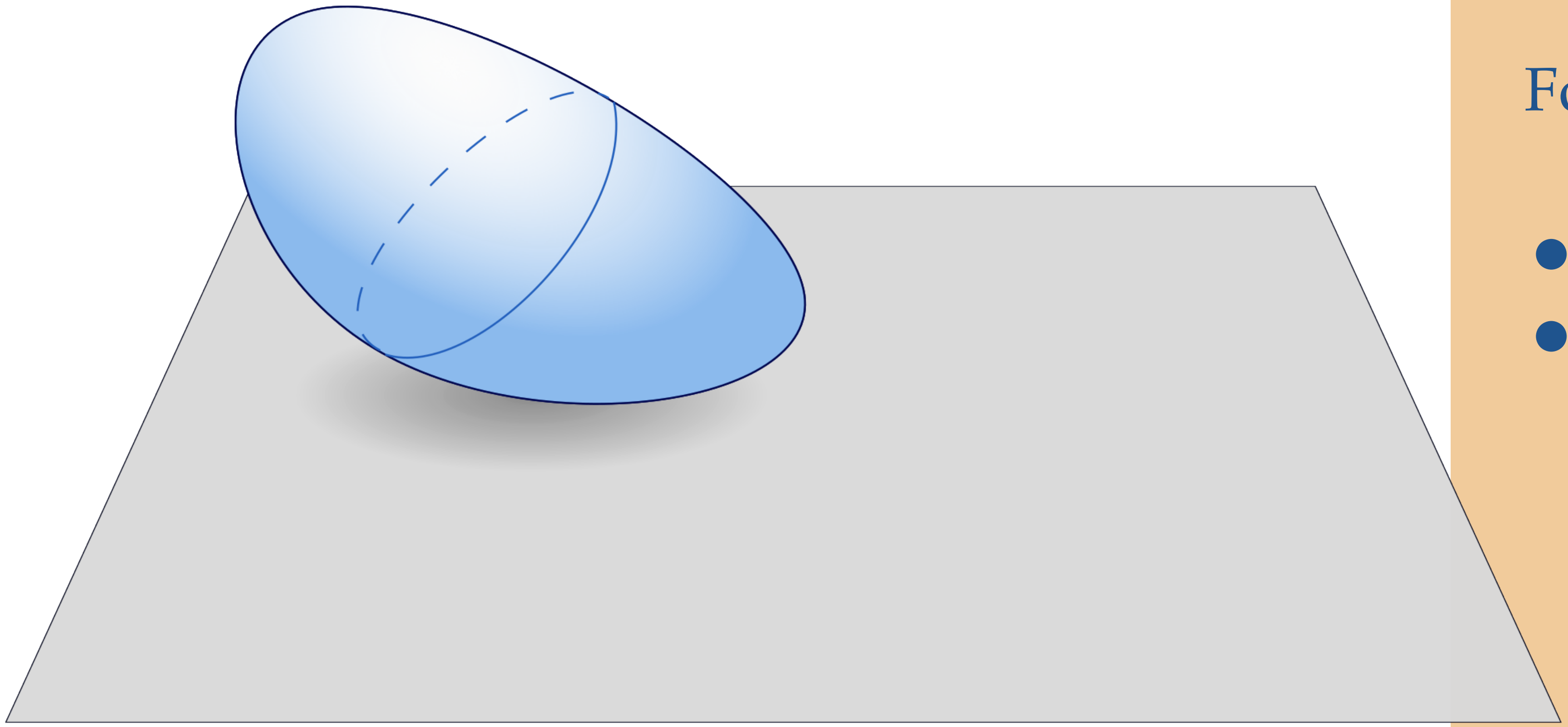
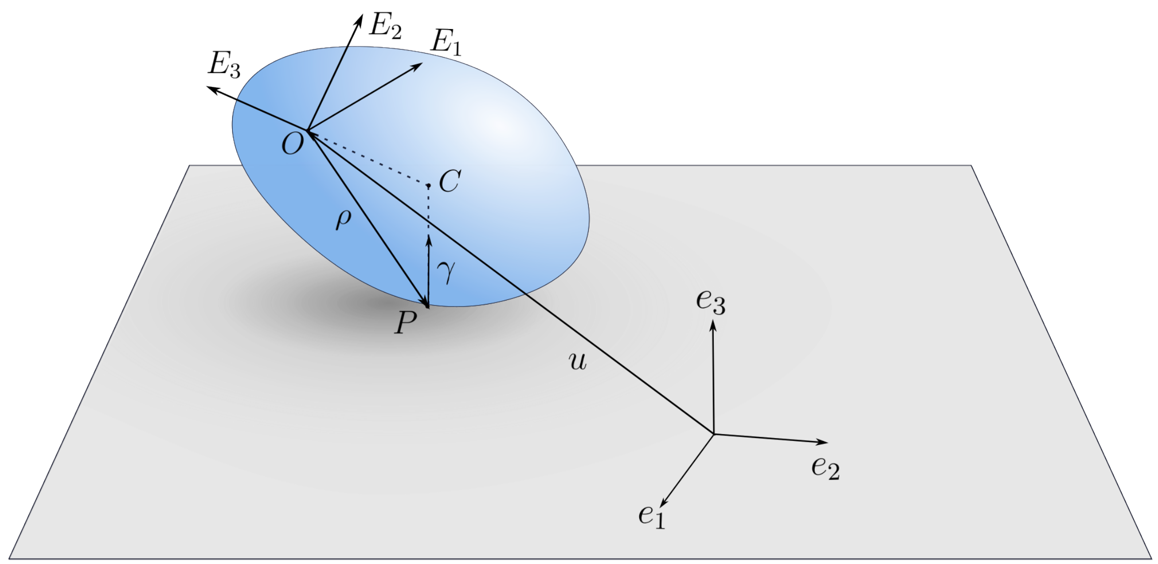


For it to be integrable we need:

- 2 additional first integrals
- invariant measure.

Integrable cases

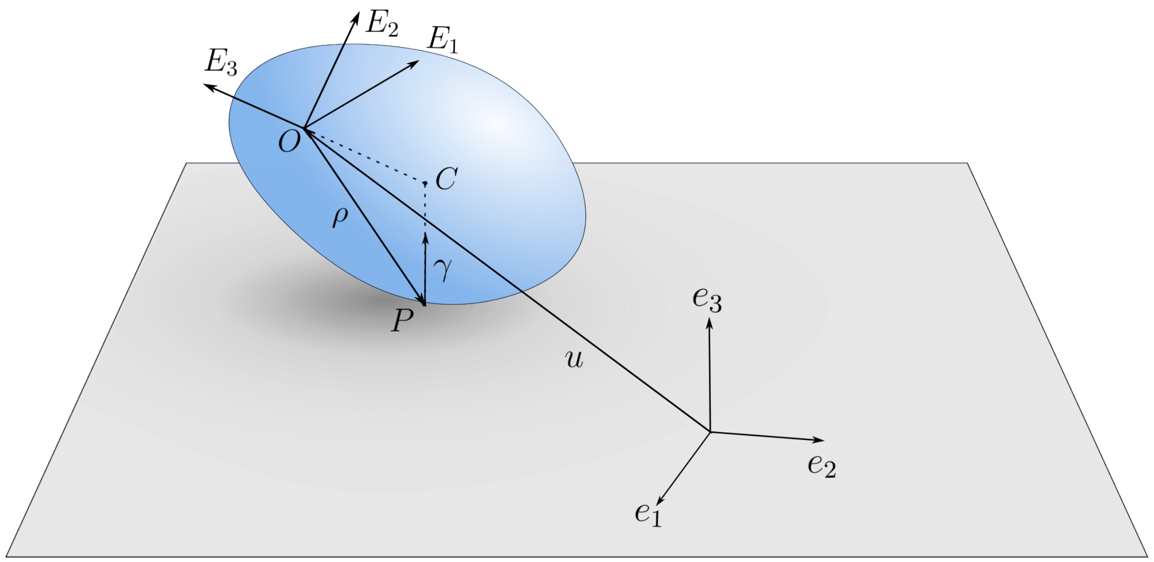
Integrable cases



Body of revolution (Chaplygin, 1897)

$I_1 = I_2 \neq I_3,\ O \neq C$

Integrable cases



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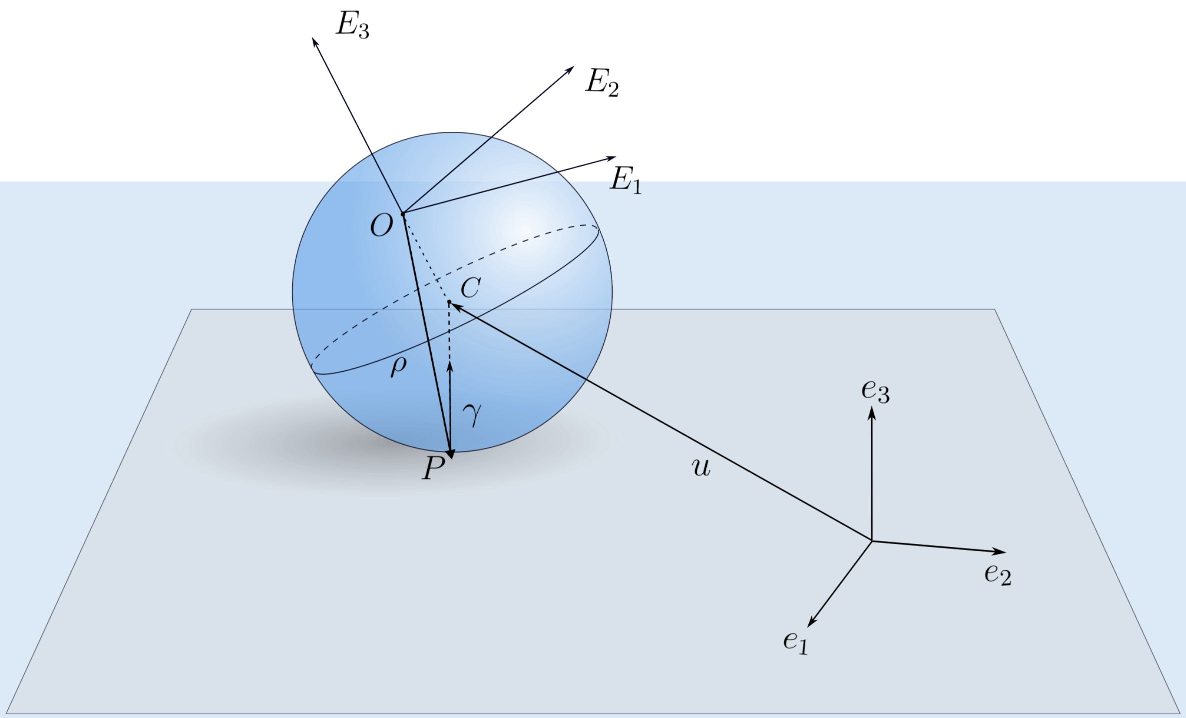
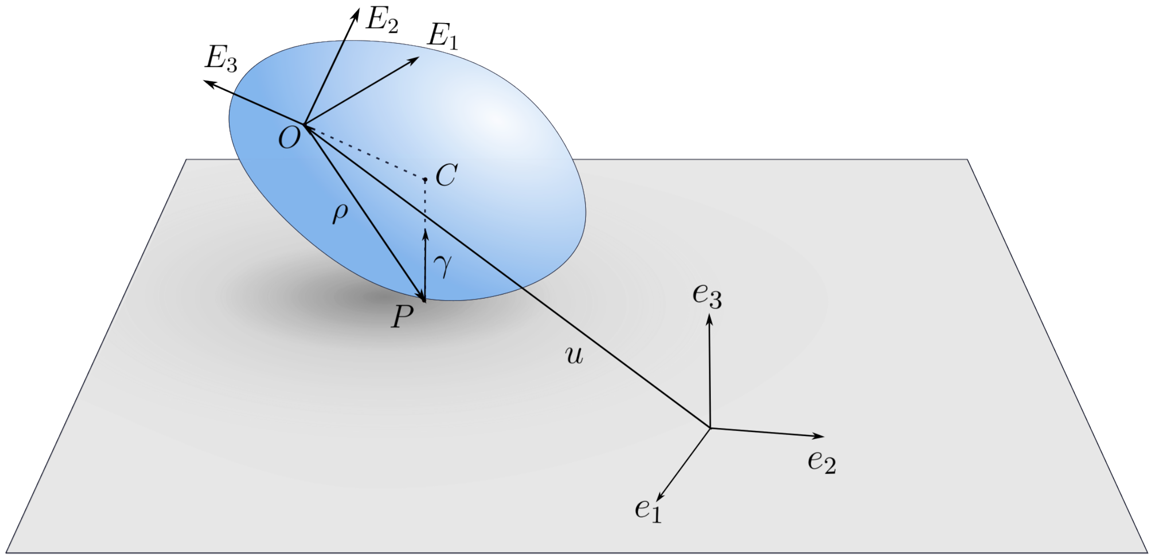


Routh's sphere (Routh, 1884)

$I_1 = I_2 \neq I_3,\ O \neq C$

$\rho(\gamma) = -r\gamma - lE_3$

Integrable cases



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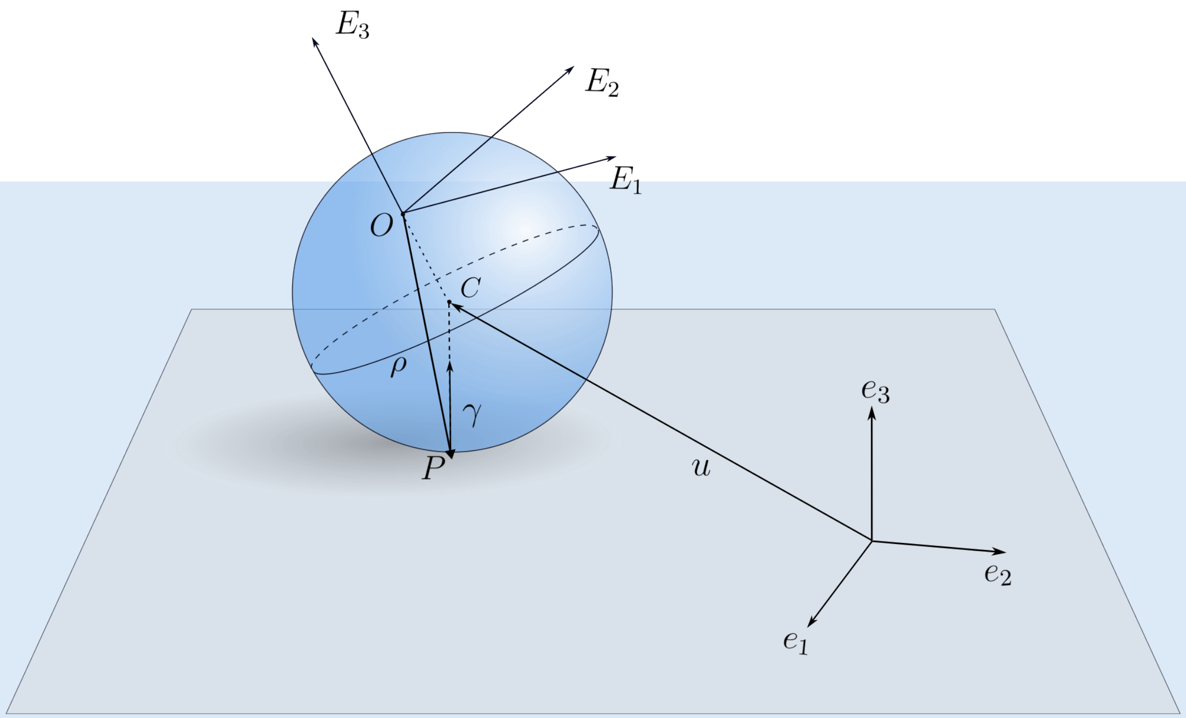

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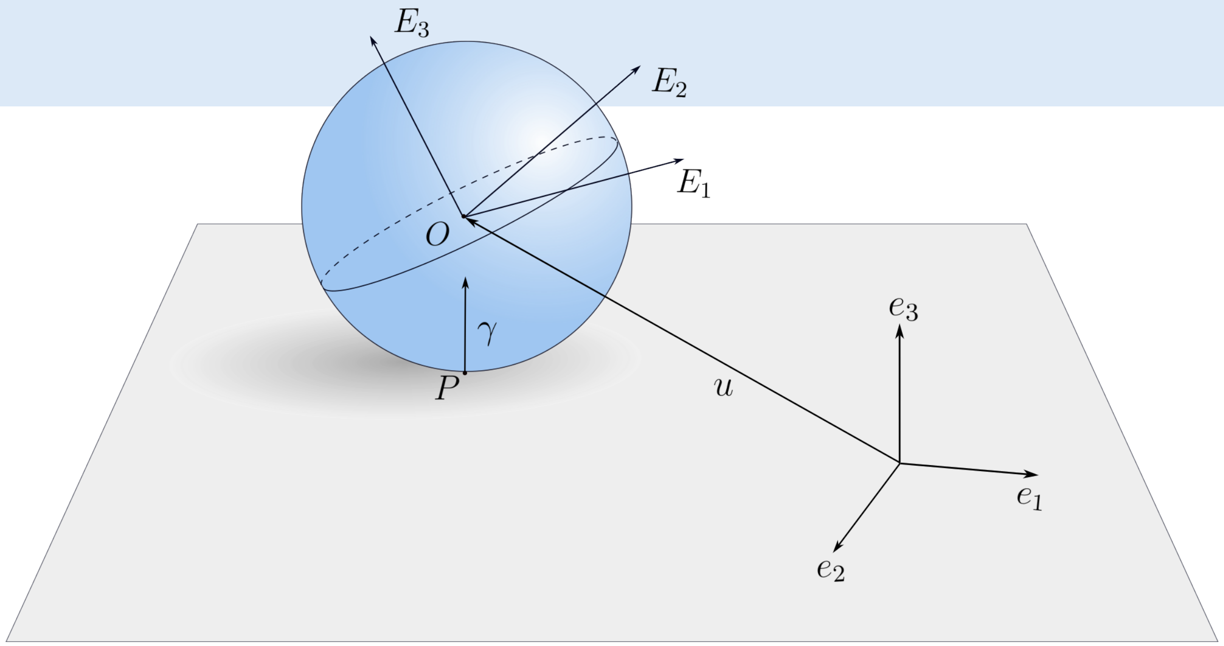


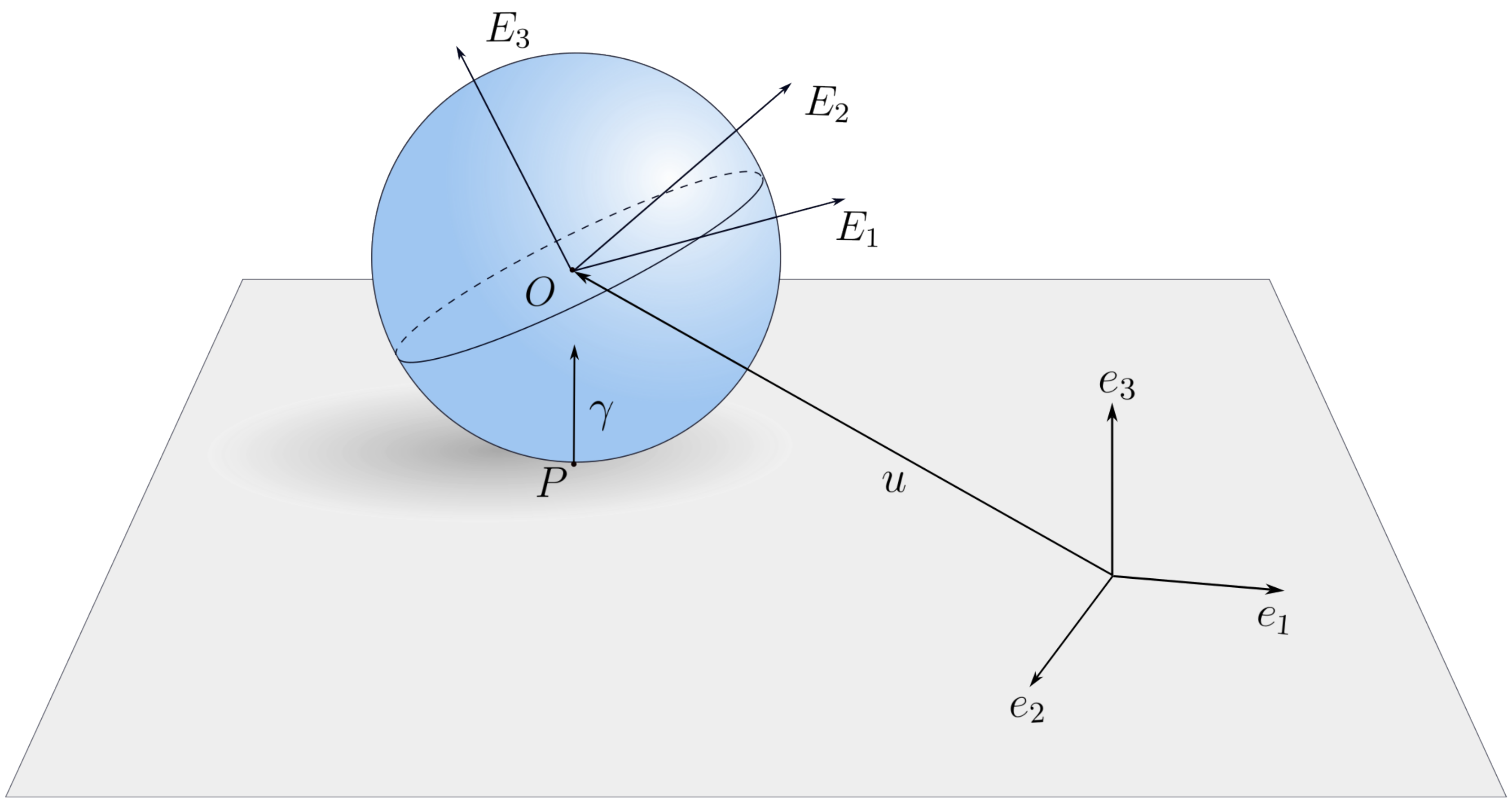
Chaplygin sphere (Chaplygin, 1903)

$I_1 \neq I_2 \neq I_3, \ O = C$

Integrable cases

Homogeneous sphere

$I_1 = I_2 = I_3, \; O = C$

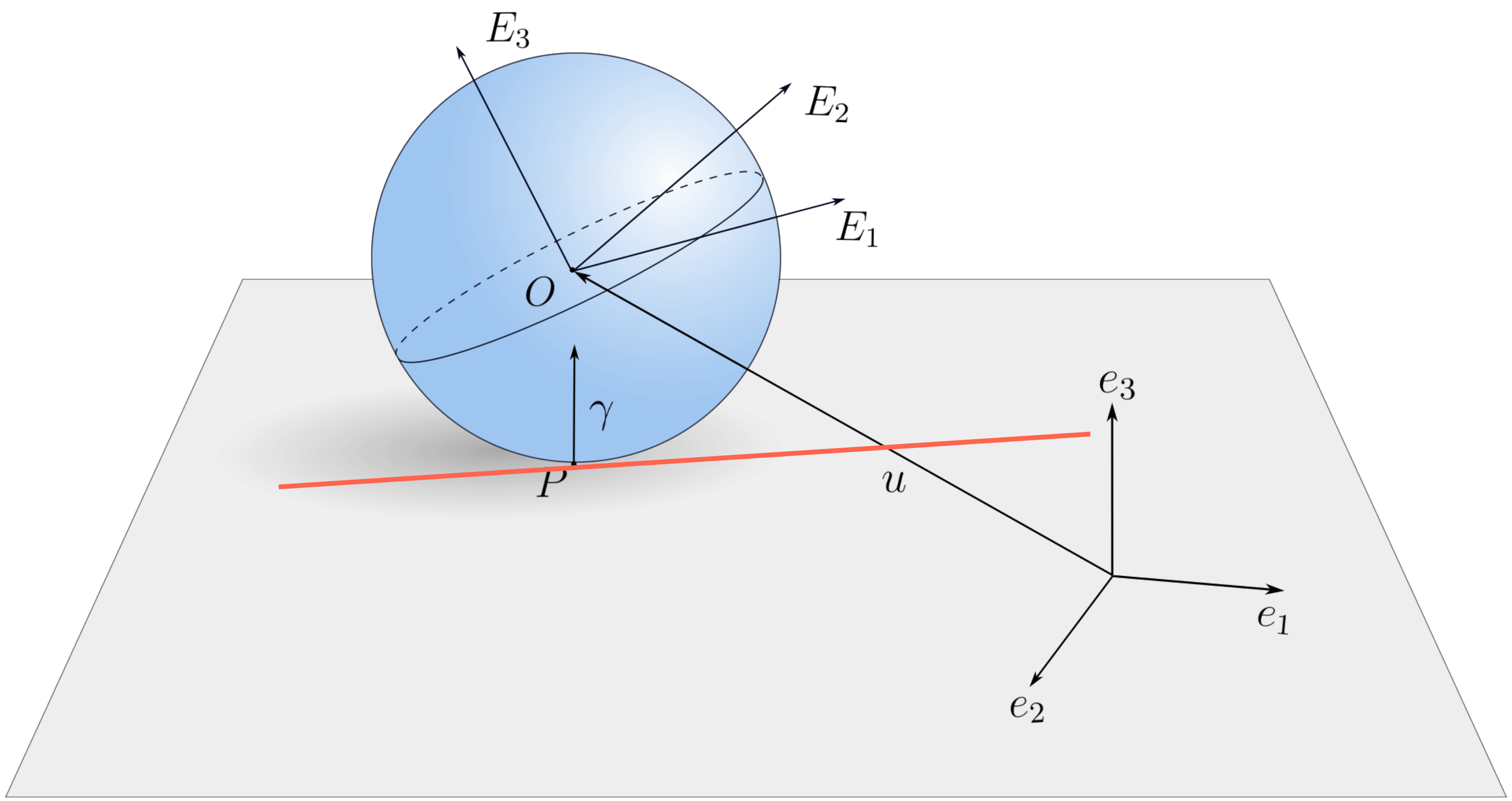


Integrable cases

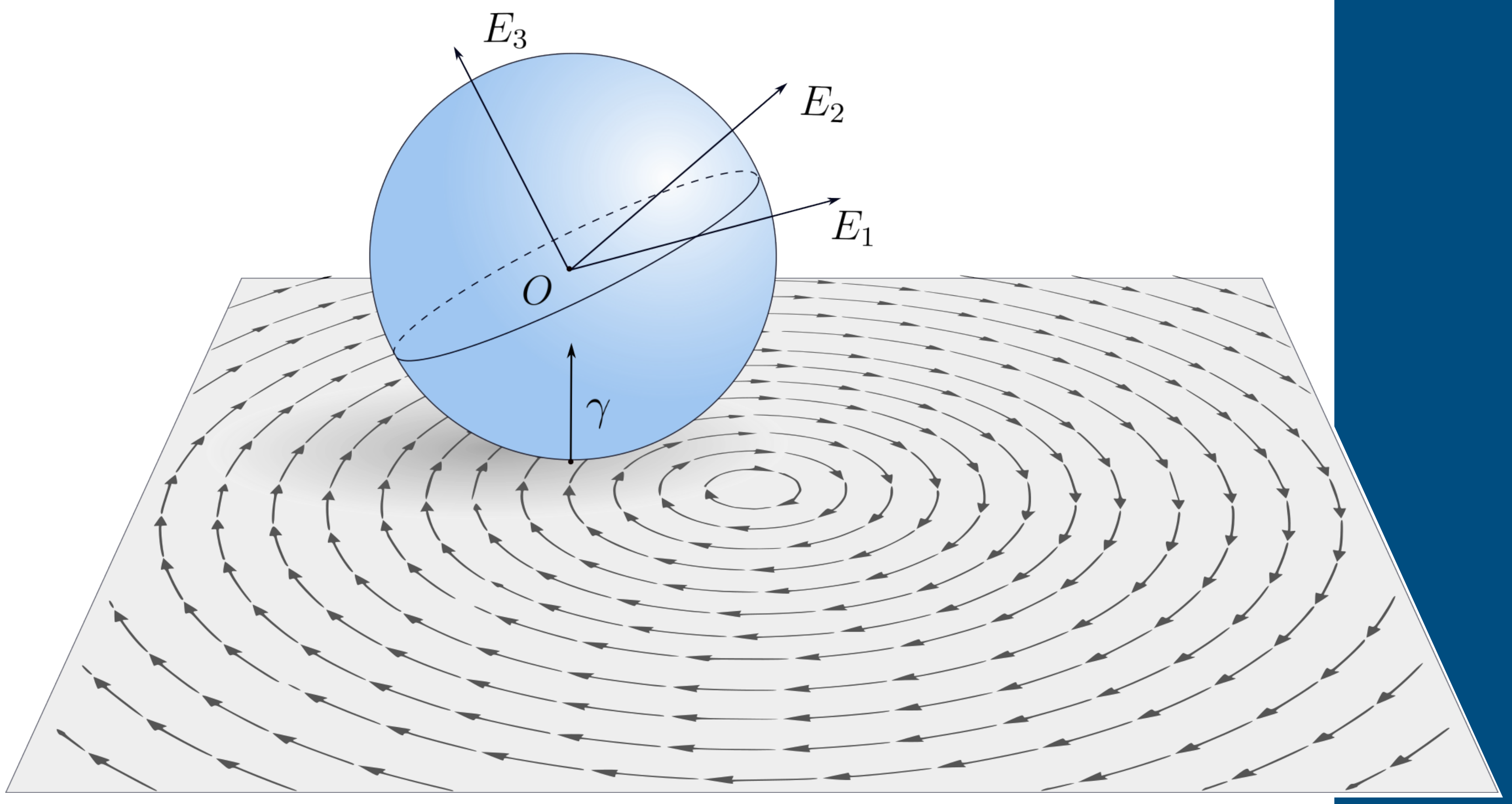
Homogeneous sphere

$I_1 = I_2 = I_3,\ O = C$

Moves in straight line



What happens if the plane is moving?



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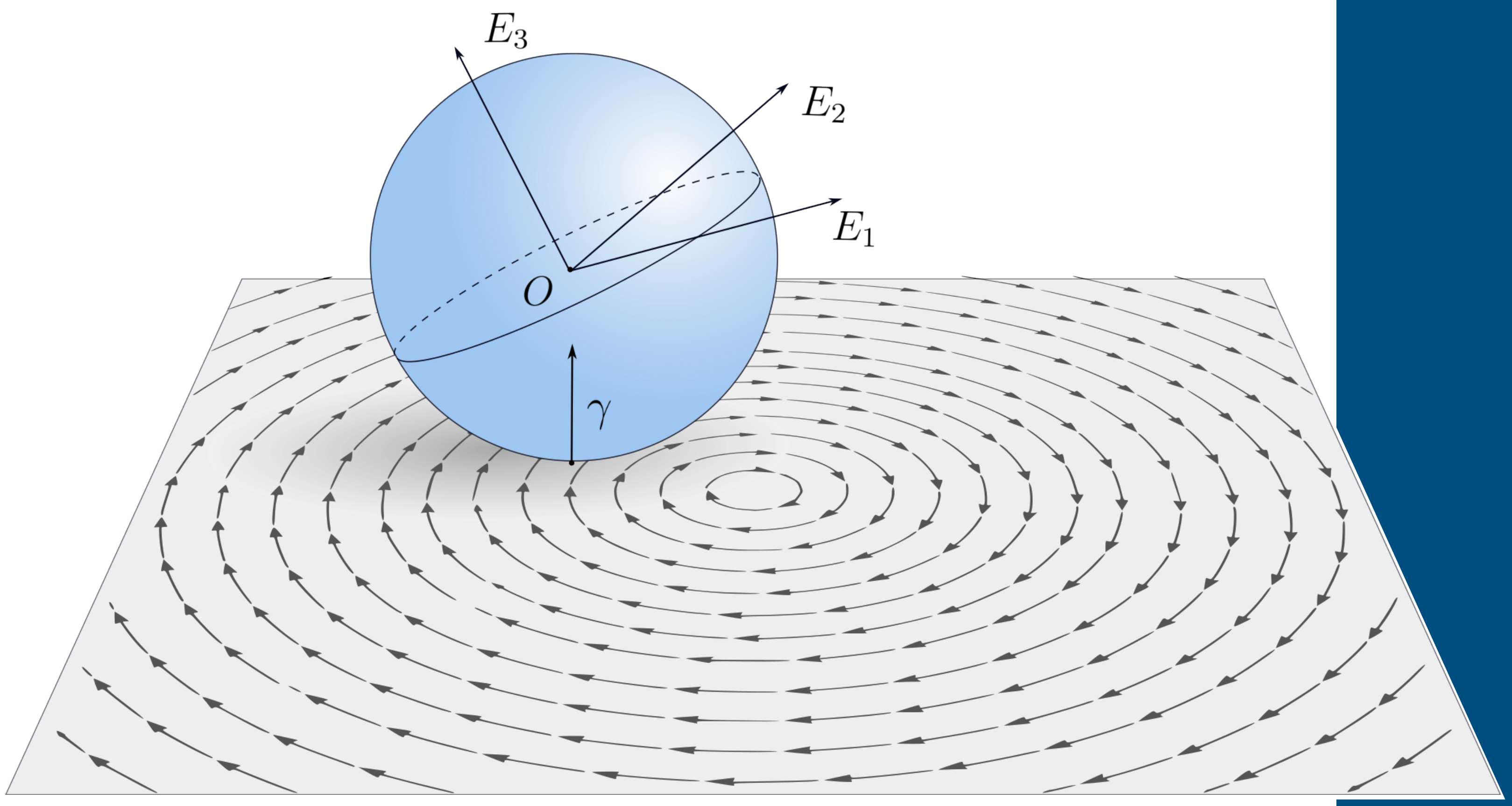
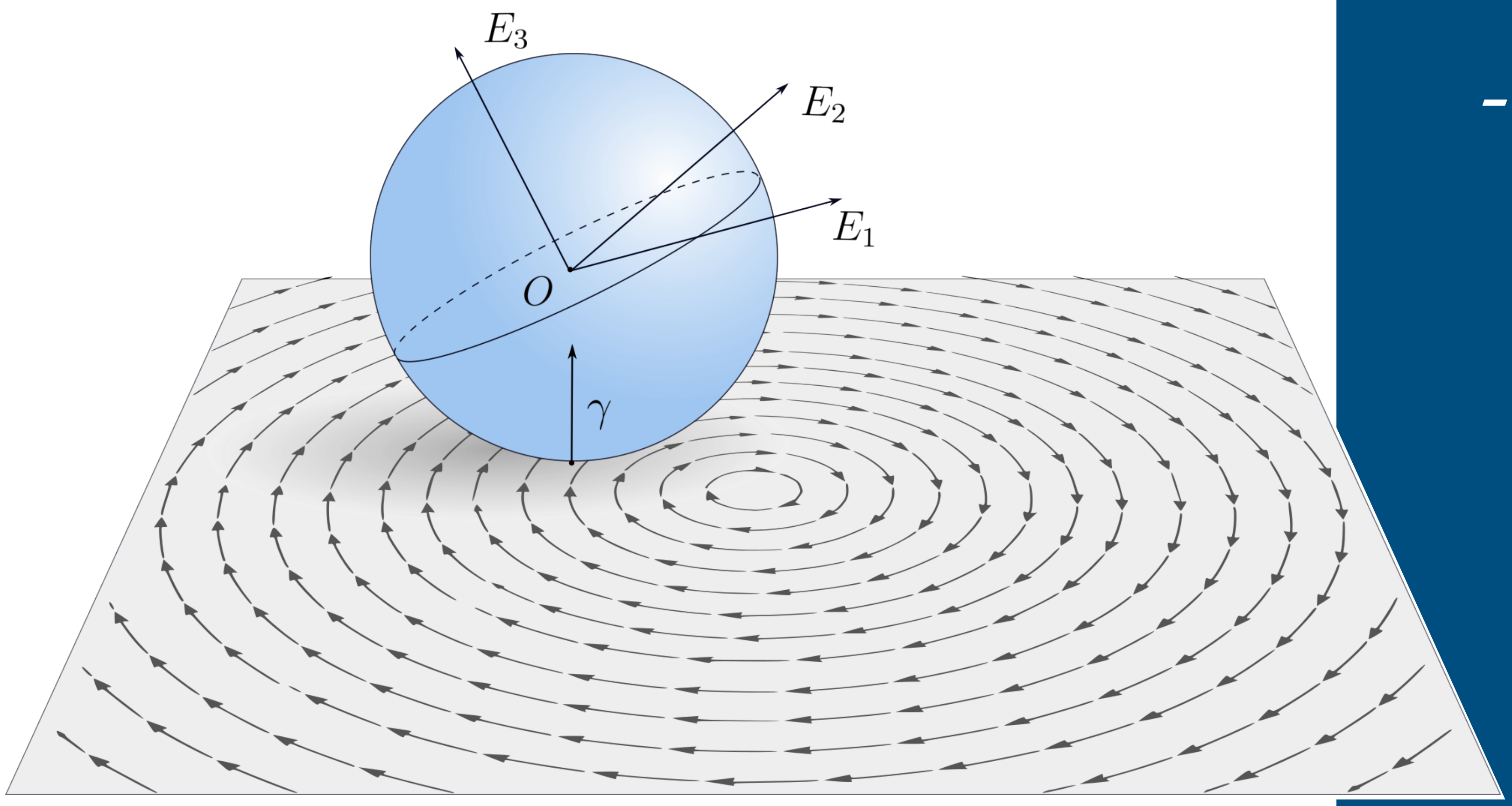


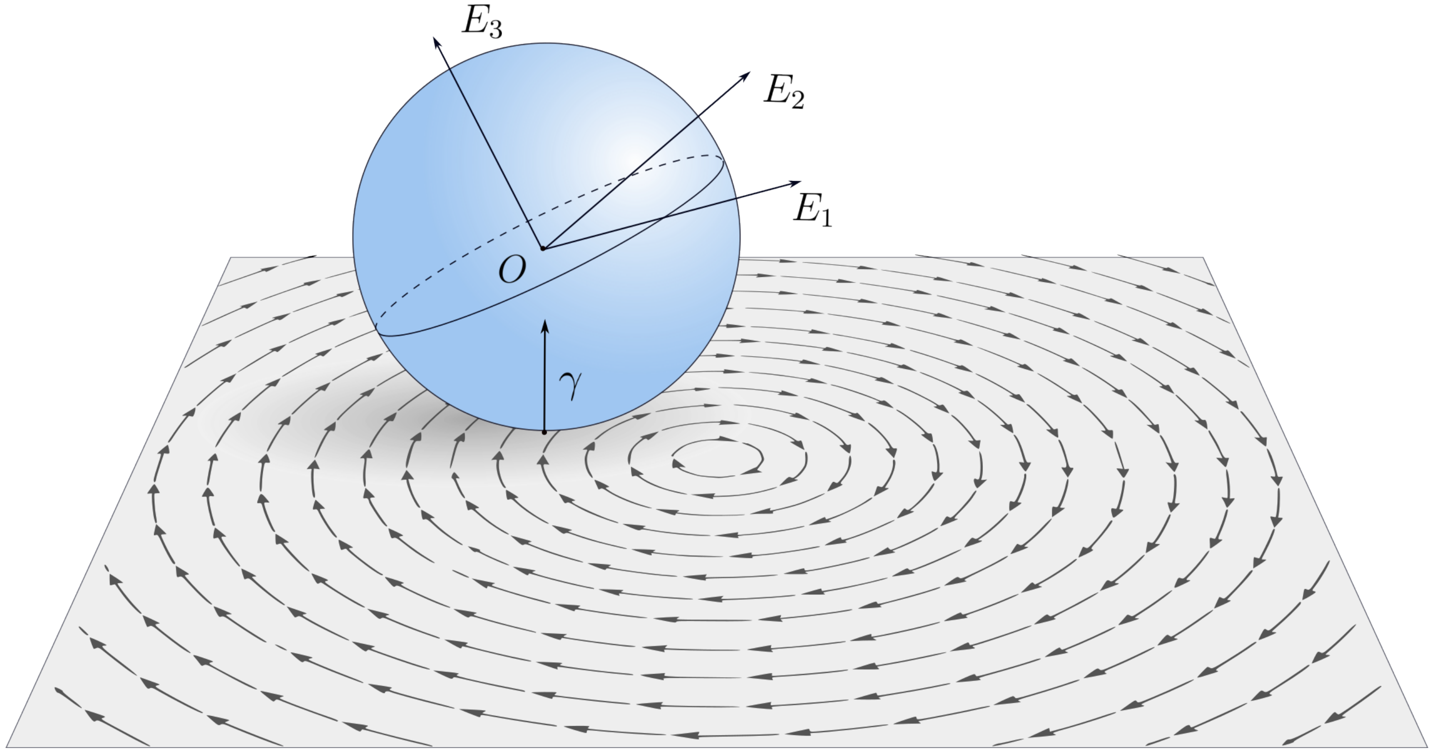
- Lose symmetry on the plane
- Lose energy first integral

What happens if the plane is moving?



- Lose symmetry on the plane
- Lose energy first integral
- Moving energy (Fassò, Sansonetto, 2015)

Homogeneous sphere on rotating plane



Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{U} = -r(\gamma \times \Omega) + U \times \Omega - \eta\gamma \times U$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$M = I\Omega + mr^2\gamma \times (\Omega \times \gamma) + \eta\gamma \times (\gamma \times U)$$

First integrals

$$\|M\|^2, \quad \langle M, \gamma \rangle, \qquad \|\gamma\|^2, \quad \langle U, \gamma \rangle = 0,$$

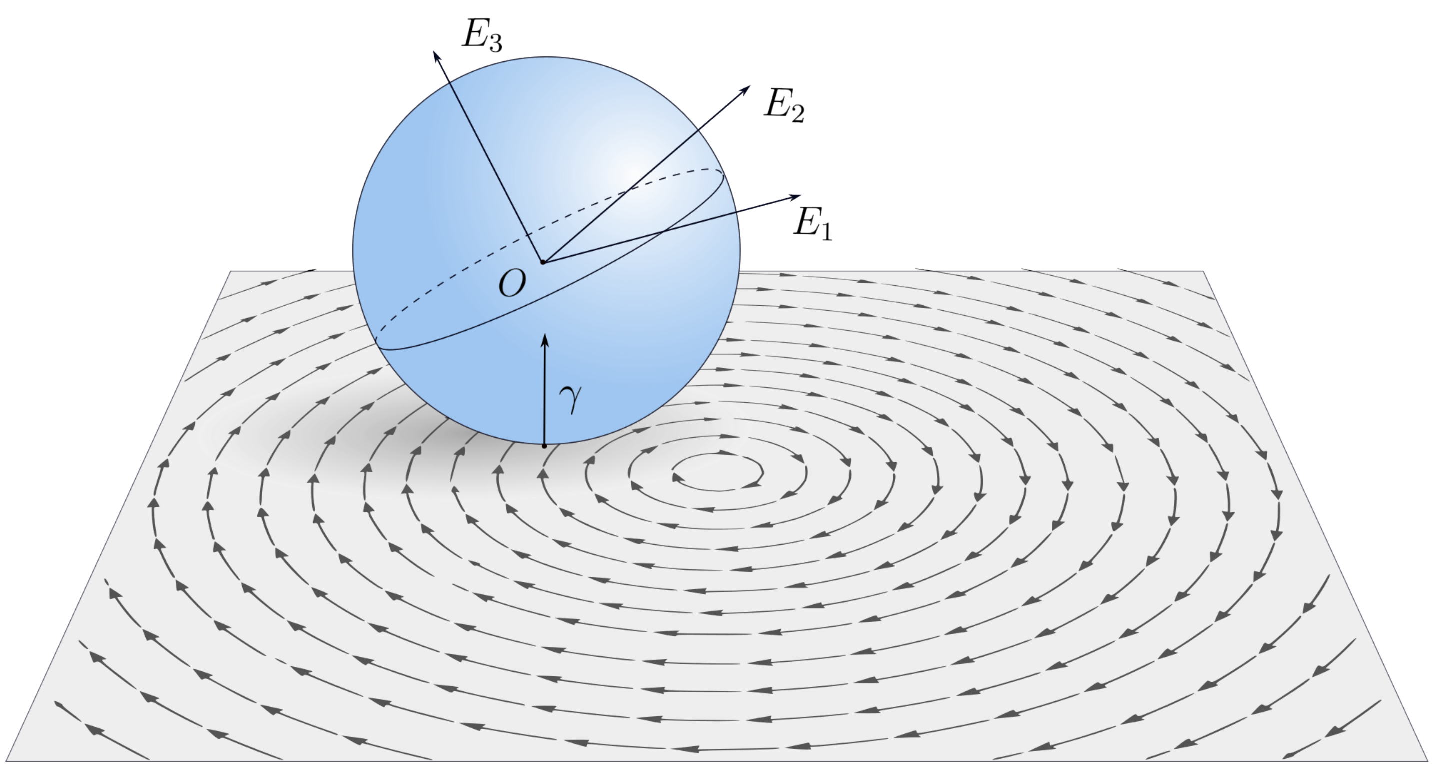
$$E_{mov}, \quad \|\Omega + \eta U\|^2,$$

Invariant measure

$$dMdxdyd\gamma$$

Homogeneous sphere on rotating plane

Integrable



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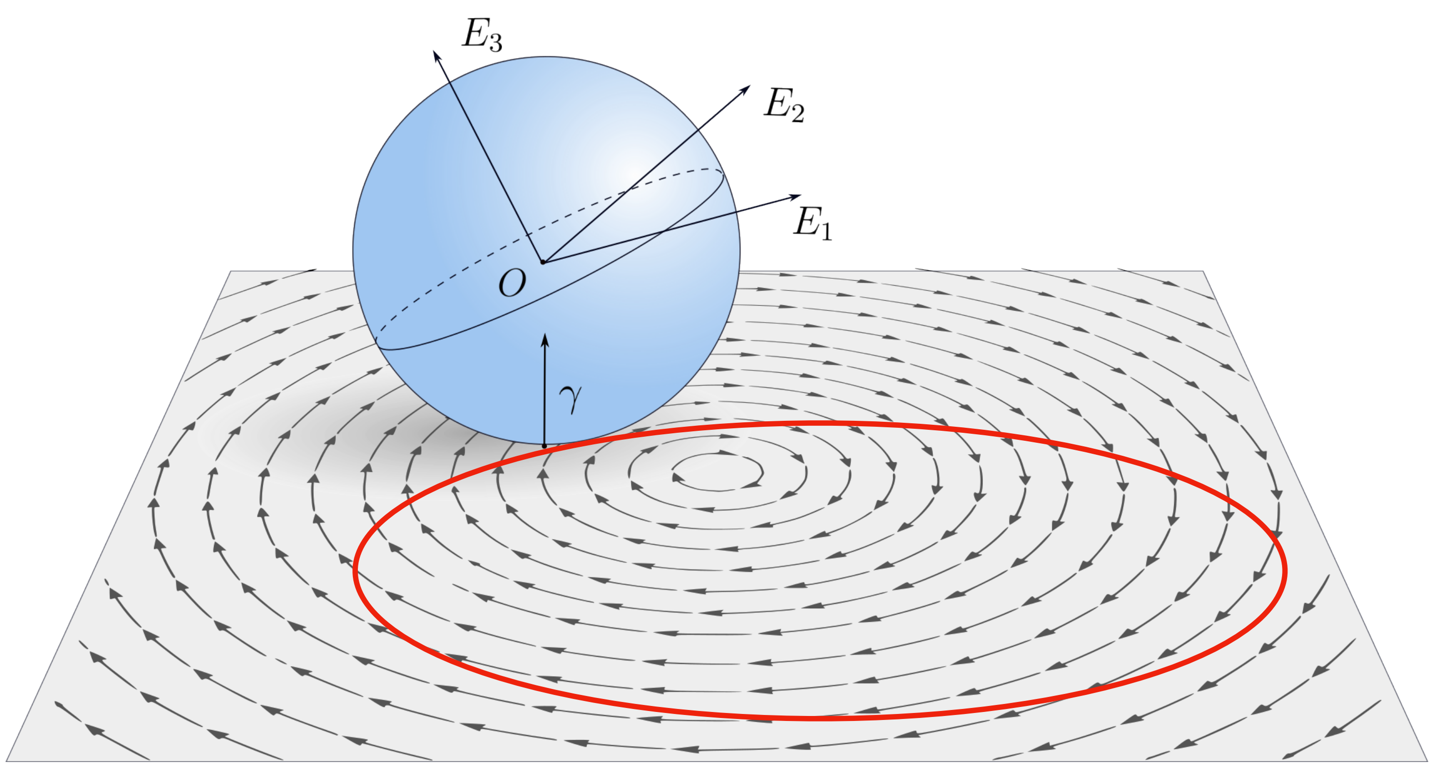
Invariant measure

$$dMdxdyd\gamma$$

Homogeneous sphere on rotating plane

Integrable

Moves in circle



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First integrals

$$\|M\|^2, \quad \langle M, \gamma \rangle, \qquad \|\gamma\|^2, \quad \langle U, \gamma \rangle = 0,$$

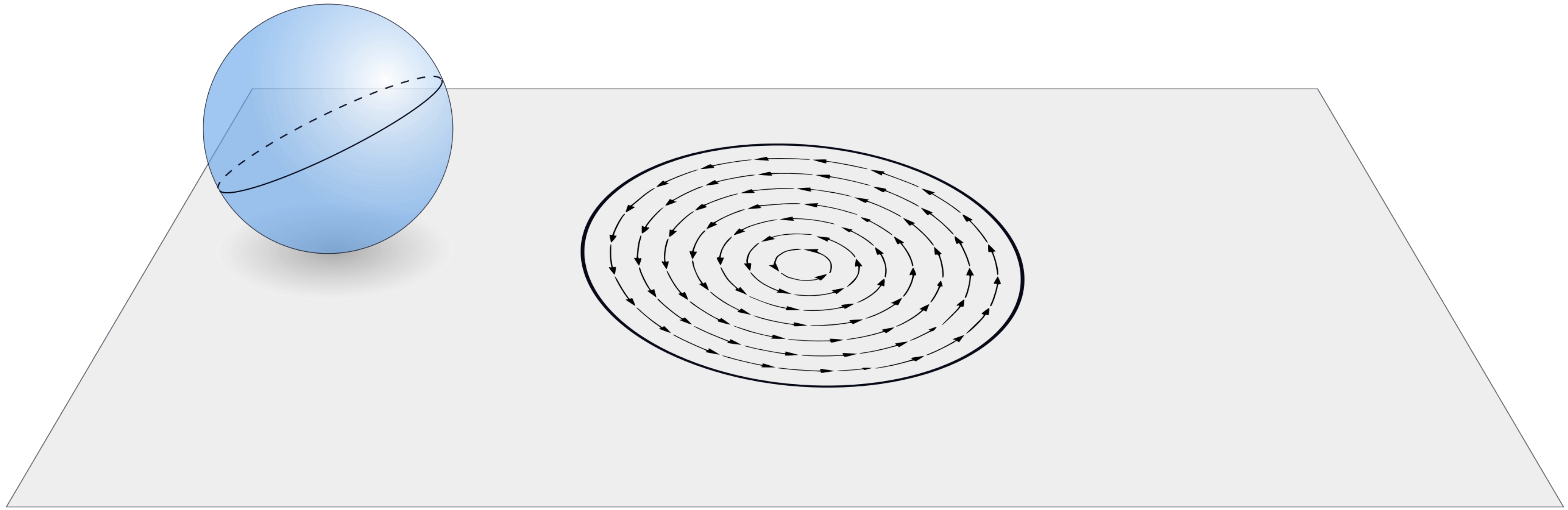
$$E_{mov}, \quad \|\Omega + \eta U\|^2,$$

Invariant measure

$$dMdxdyd\gamma$$

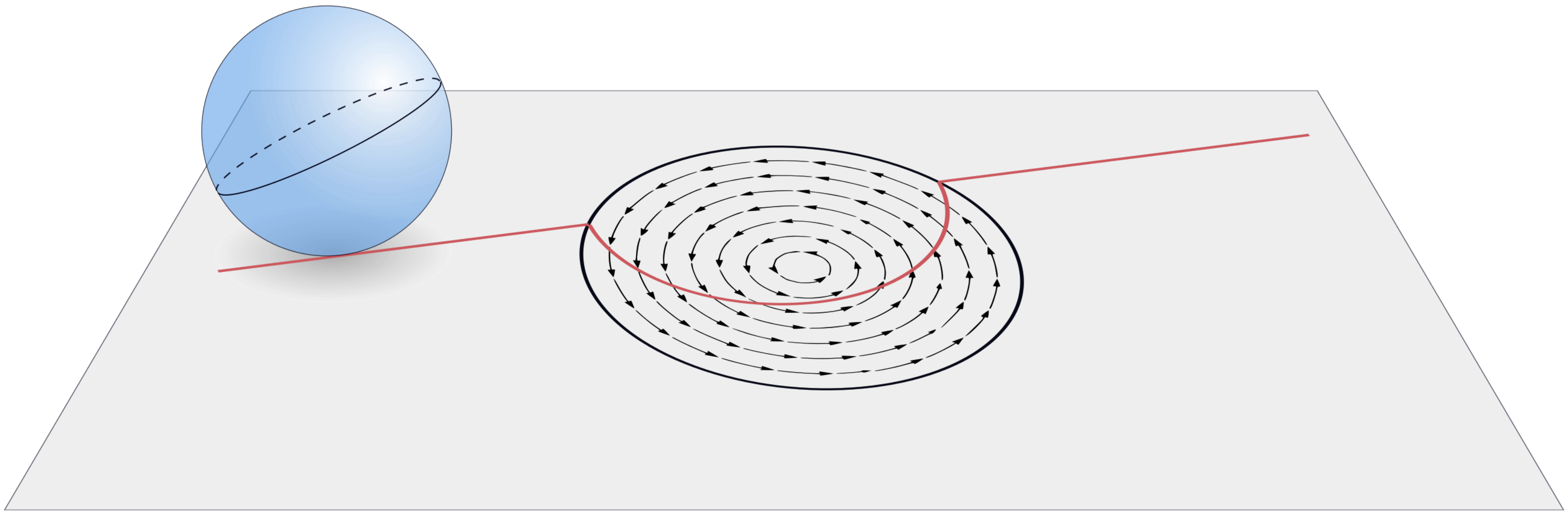
Homogeneous sphere on rotating plane

Anais-billiard phenomenon
(Lévy- Leblond, 1986)



Homogeneous sphere on rotating plane

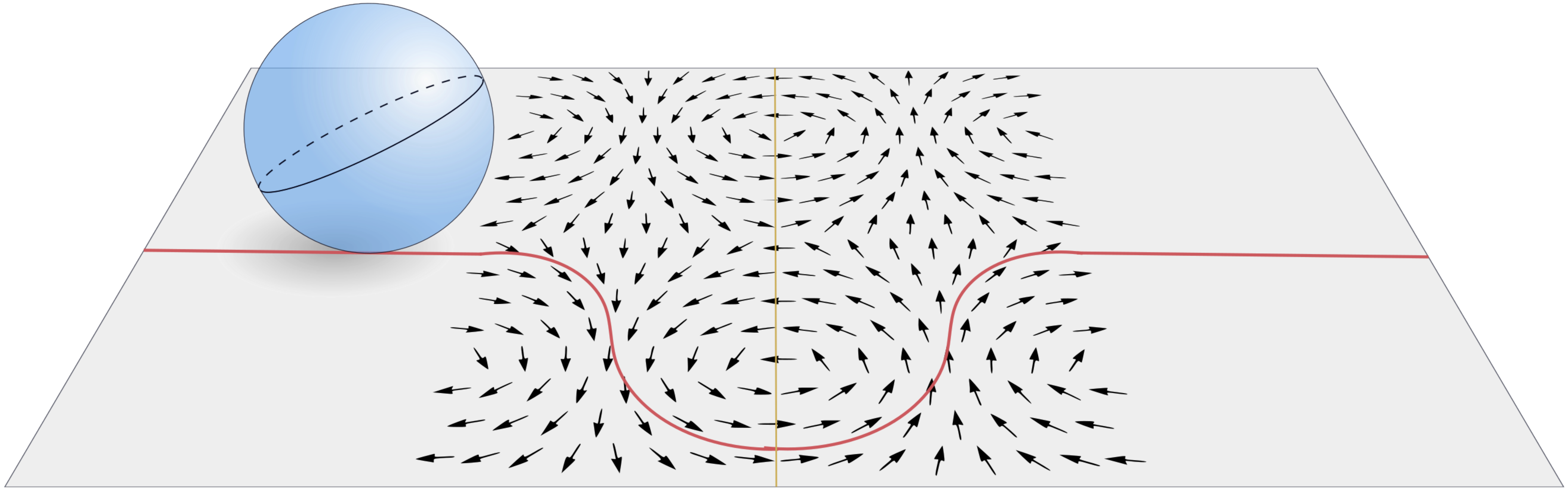
Anais-billiard phenomenon
(Lévy- Leblond, 1986)



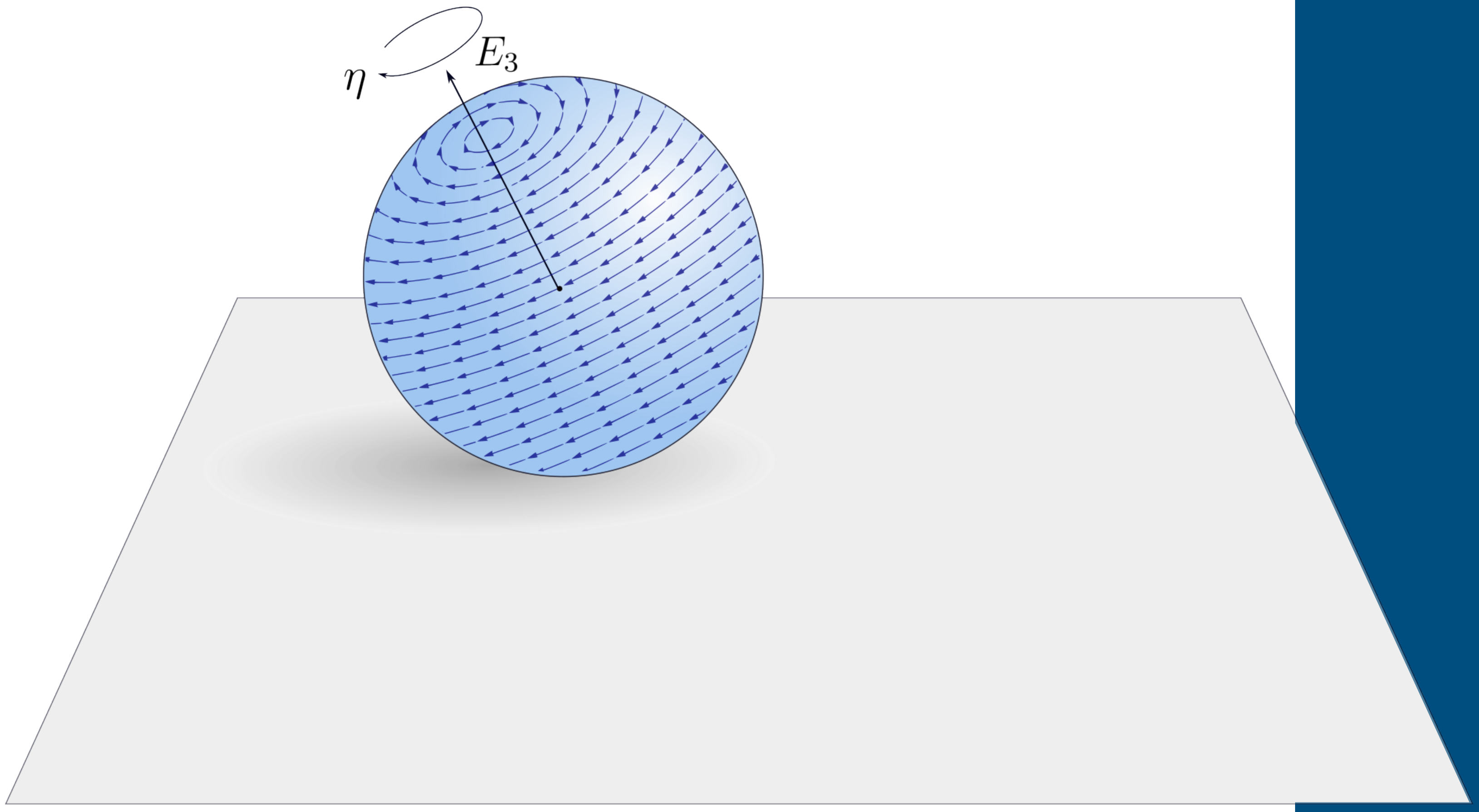
Homogeneous sphere in plane with symmetric vector field

Anais-billiard phenomenon
(Lévy- Leblond, 1986)

$$\begin{pmatrix} V_1(x,y) \\ V_2(x,y) \end{pmatrix} = \begin{pmatrix} V_1(-x,y) \\ -V_2(-x,y) \end{pmatrix}$$



What happens if the shell of the sphere is moving?



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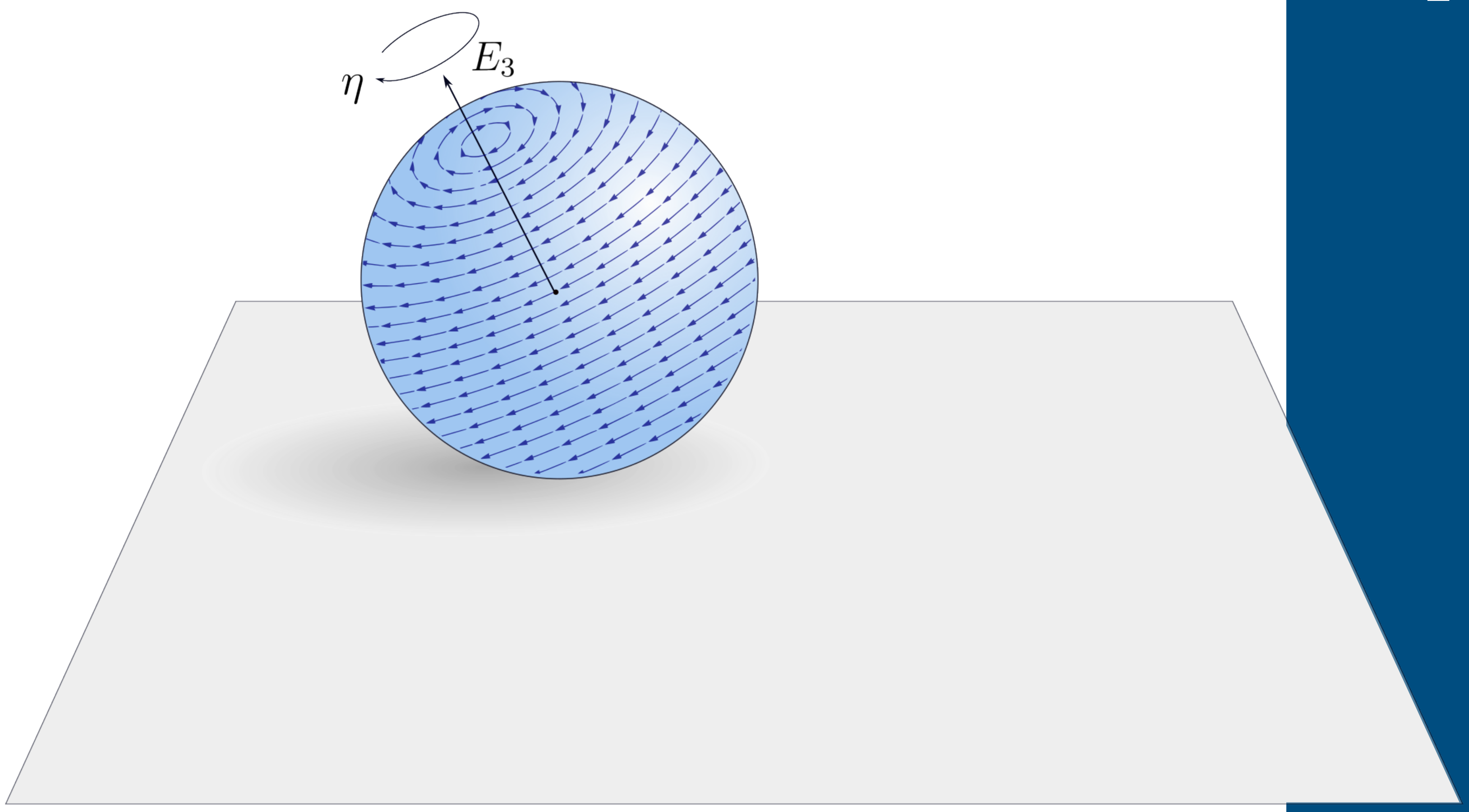
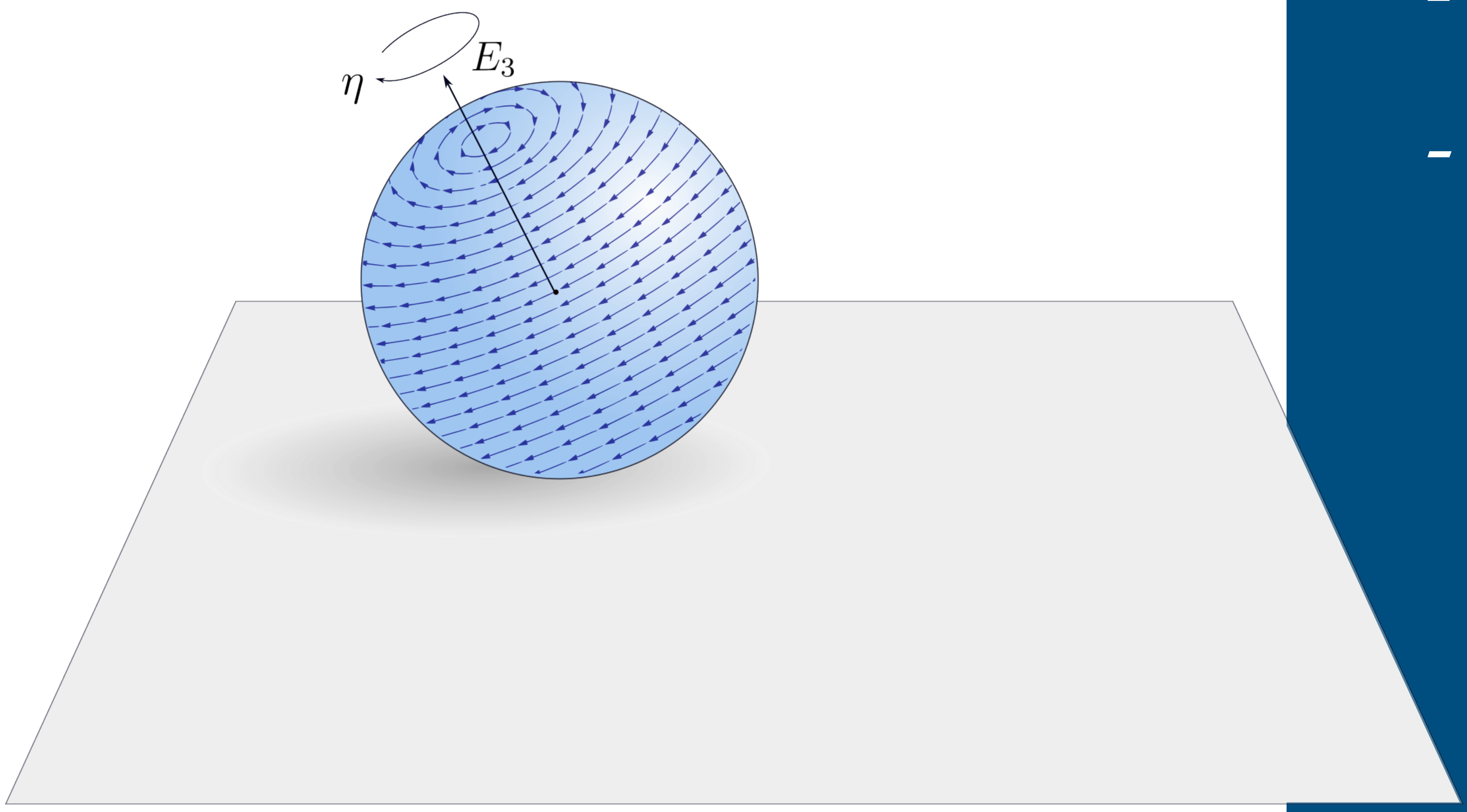


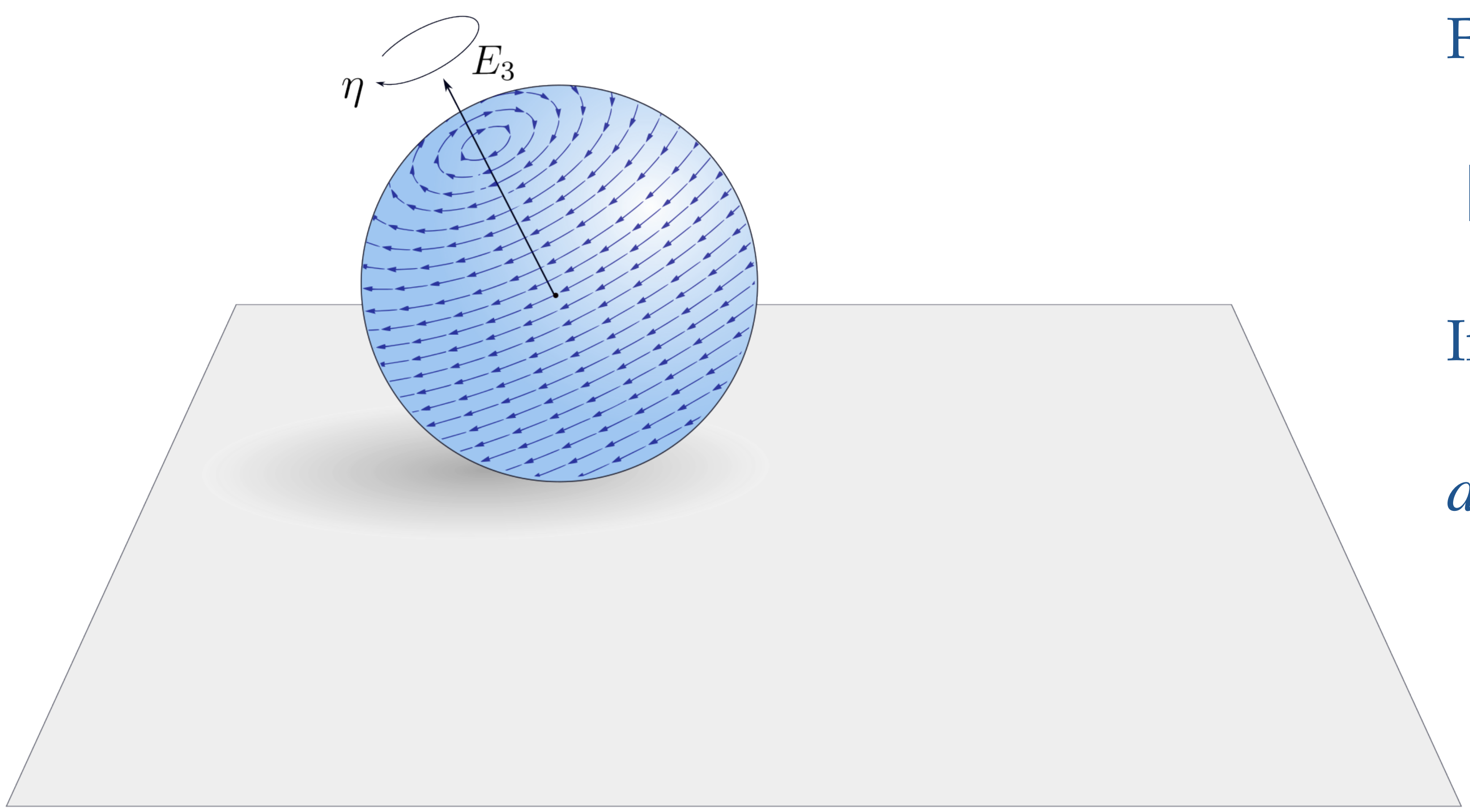
- Lose symmetry on the sphere
- Lose energy first integral

What happens if the shell of the sphere is moving?



- Lose symmetry on the sphere
- Lose energy first integral
- Moving energy

Homogeneous sphere with rotating shell



Equations of motion:

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$$\dot{\gamma} = \gamma \times \Omega$$

$$M = I\Omega + mr^2\gamma \times (\Omega \times \gamma) - mr^2\eta_1\gamma \times (\gamma \times E_3)$$

First integrals:

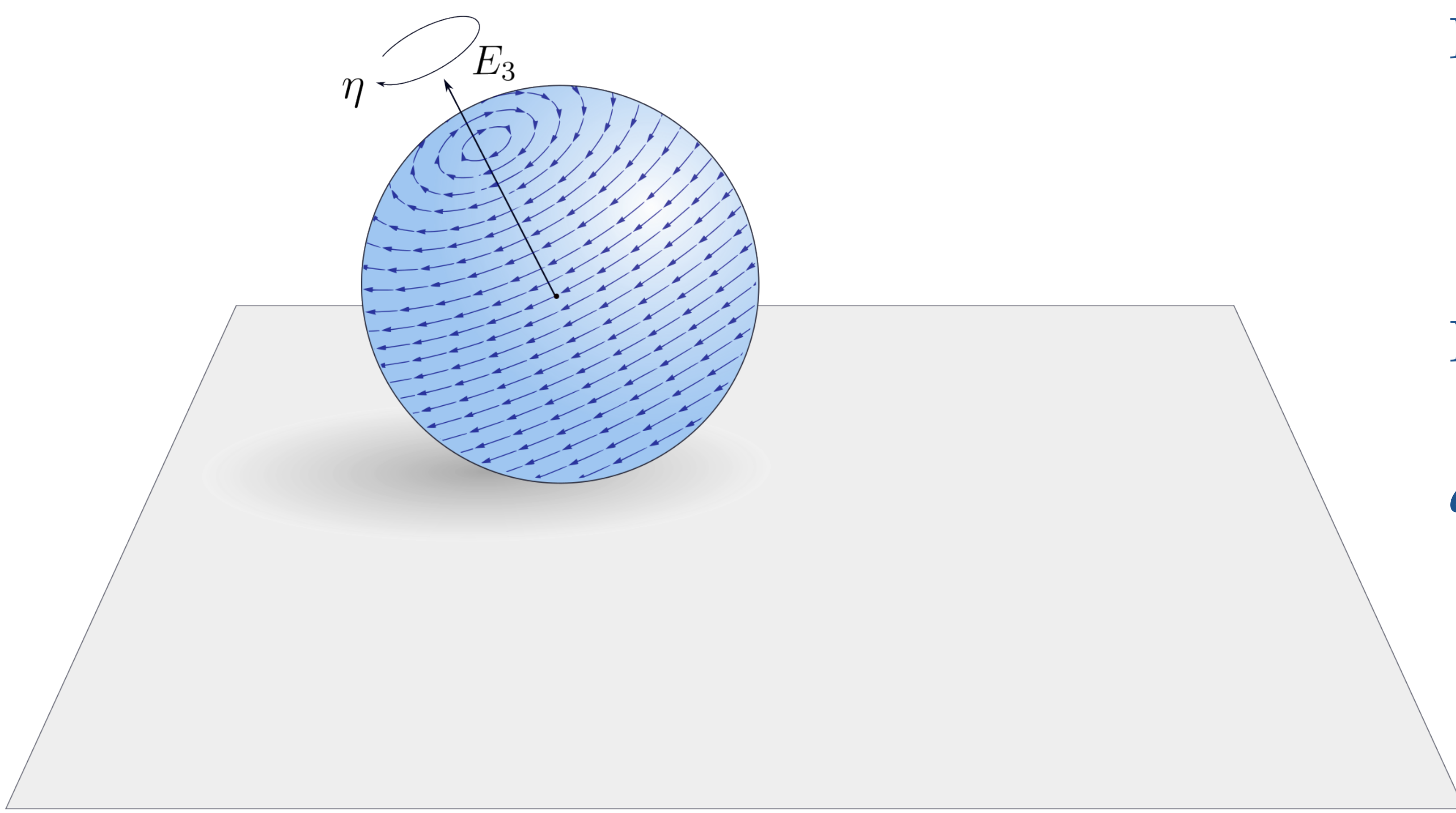
$$\|M\|^2, \quad \langle M, \gamma\rangle, \quad \|\gamma\|^2, \quad E_{mov}$$

Invariant measure:

$$dMdxdyd d\gamma$$

Homogeneous sphere with rotating shell

Integrable



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First integrals:

$$\|M\|^2, \quad \langle M, \gamma \rangle, \quad \|\gamma\|^2, \quad E_{mov}$$

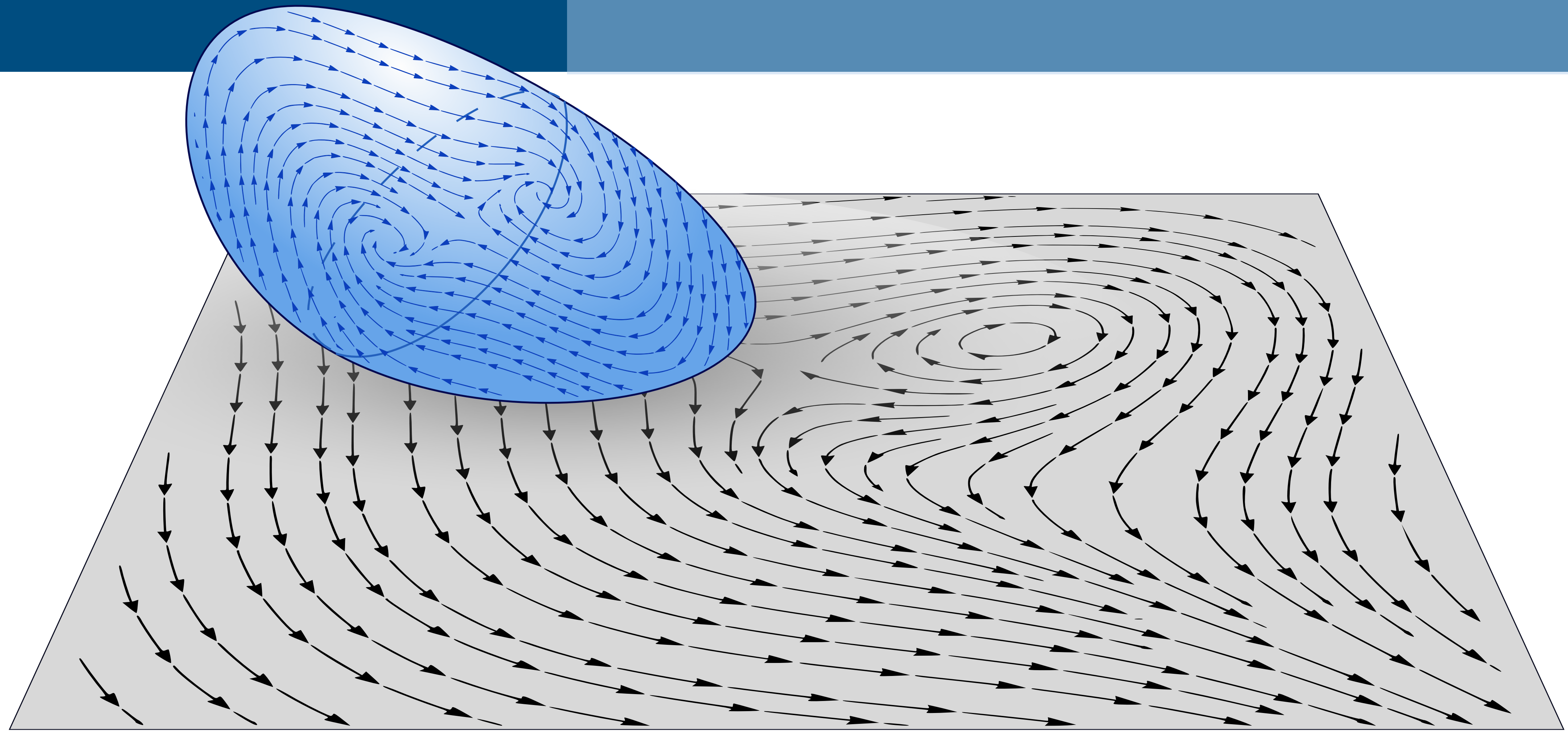
Invariant measure:

$$dMdxdyd d\gamma$$

Generalization

Convex body rolling on a plane

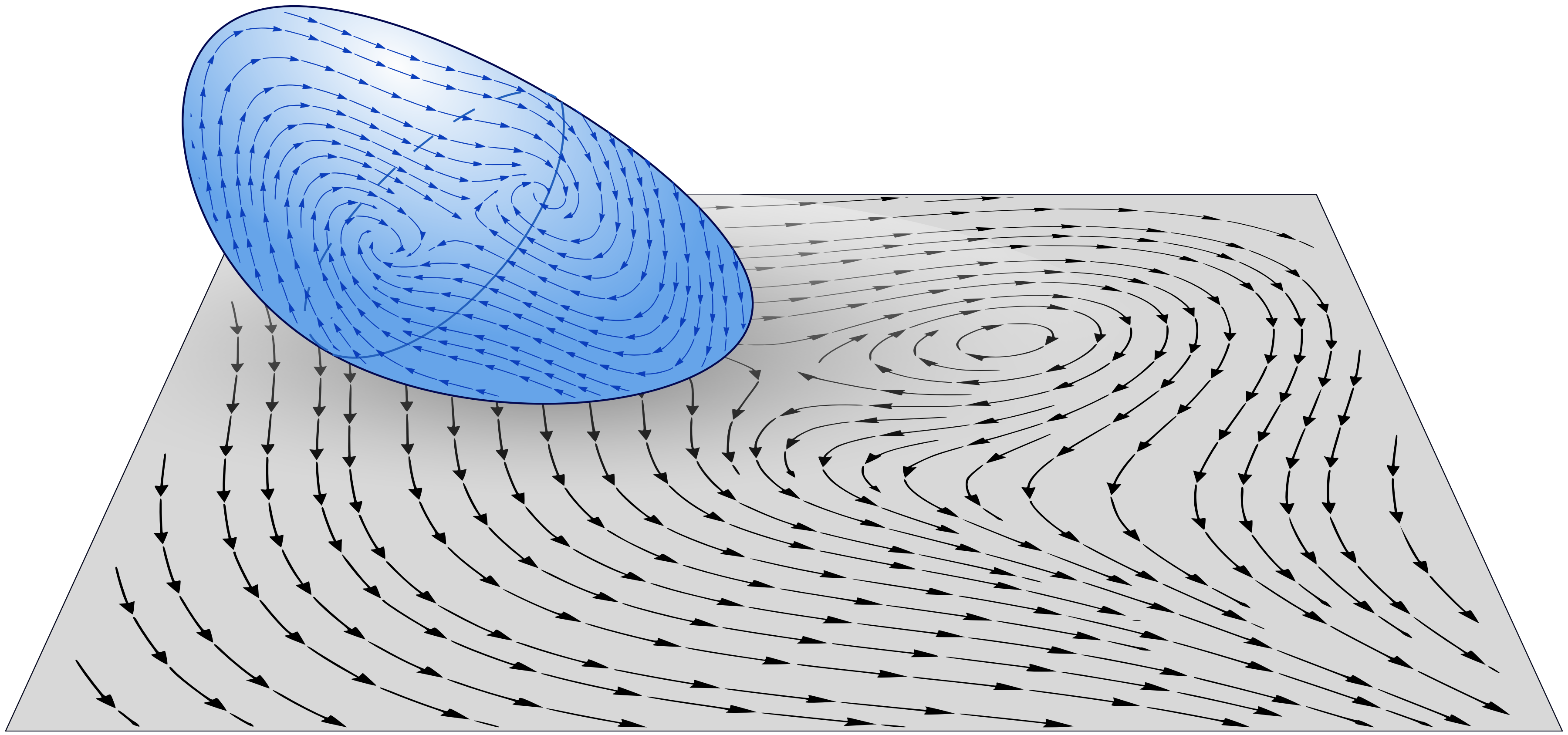
Add general vector fields to plane and body shell



Generalization

Goal:
Identify general principles leading to

- First integrals
- Moving energy
- Invariant measure



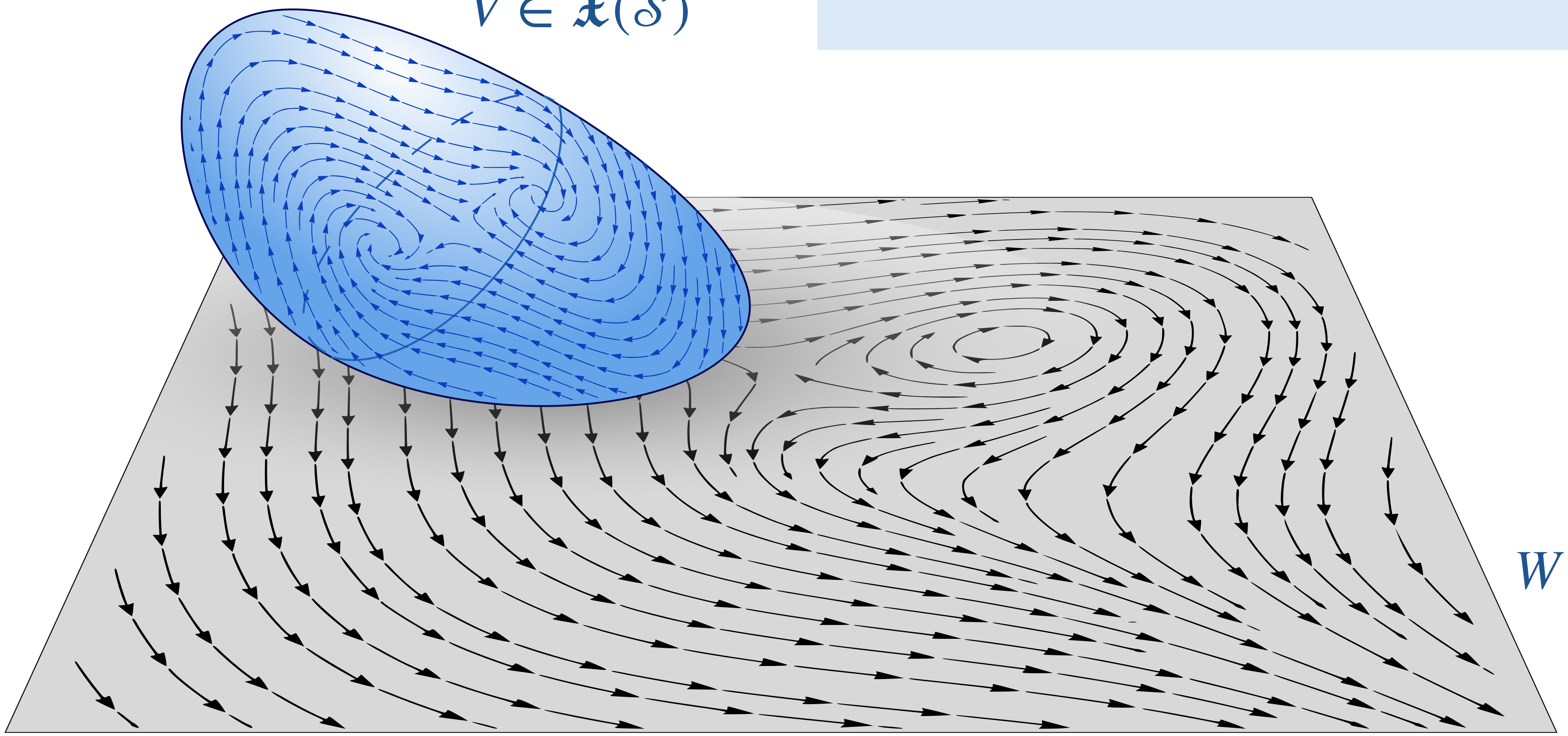
Generalization

Configuration manifold: $Q = SO(3) \times \mathbb{R}^2$

$(B, \underline{x}) \in Q$

Affine nonholonomic constraint:

$\dot{u} = B(\rho \times \Omega) + BV + W$



Generalization

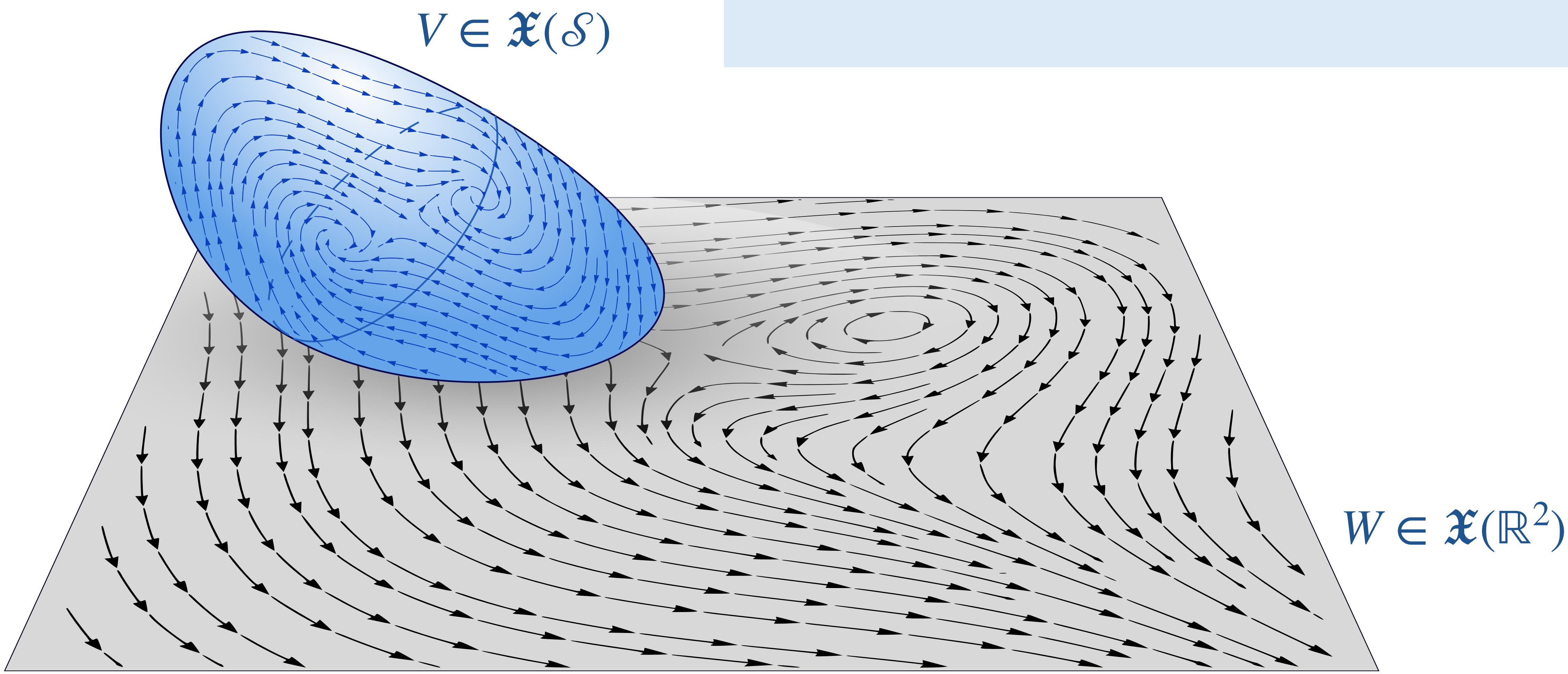
No symmetries on plane or body

Configuration manifold: $Q = SO(3) \times \mathbb{R}^2$

$(B, \underline{x}) \in Q$

Affine nonholonomic constraint:

$\dot{u} = B(\rho \times \Omega) + BV + W$



General system

Theorem (C., García Naranjo, 2023):

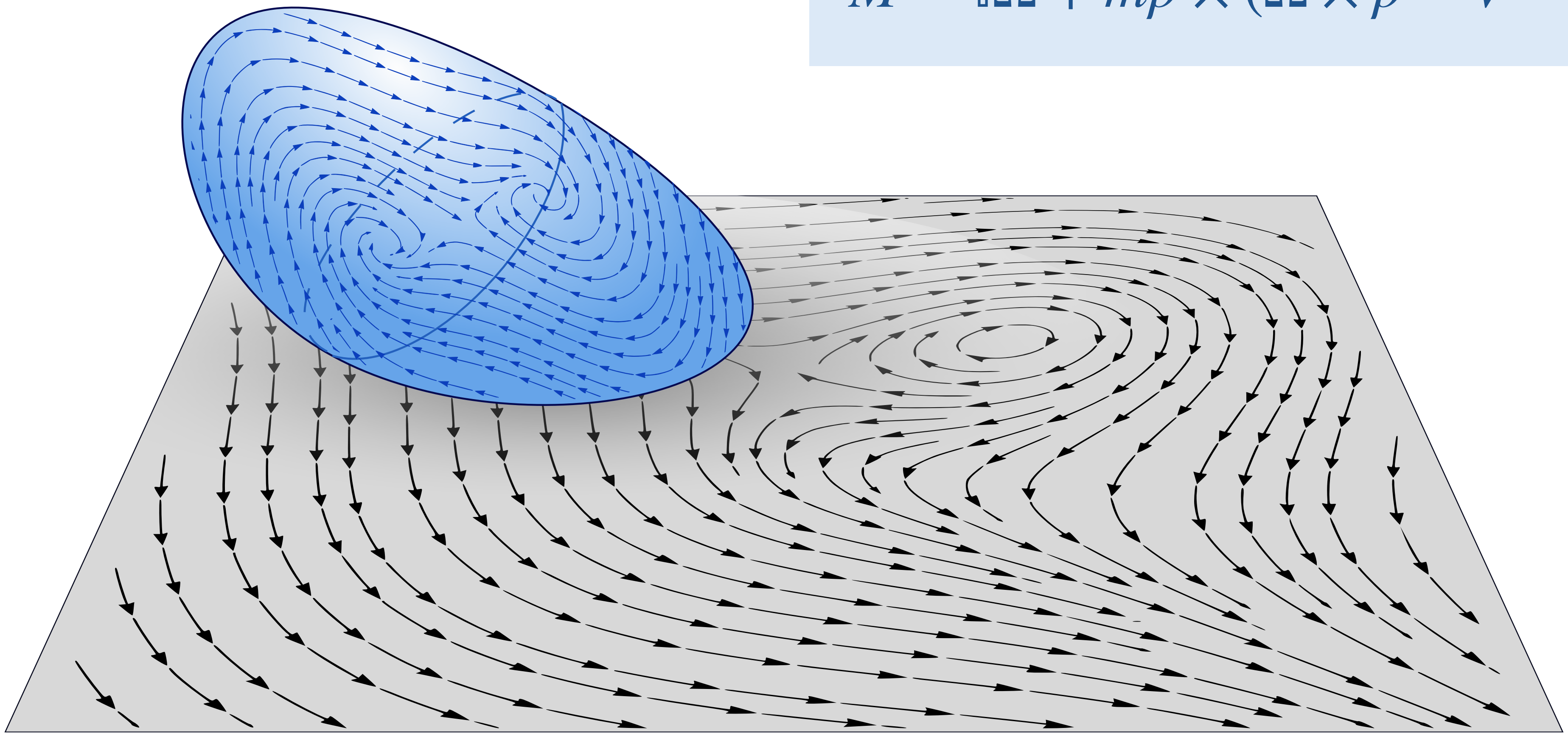
Equations of motion :

$$\dot{M} = M \times \Omega + m\dot{\rho} \times (\Omega \times \rho) + mg\rho \times \gamma \\ + m(V + B^{-1}W) \times (\dot{\rho} + \Omega \times \rho)$$

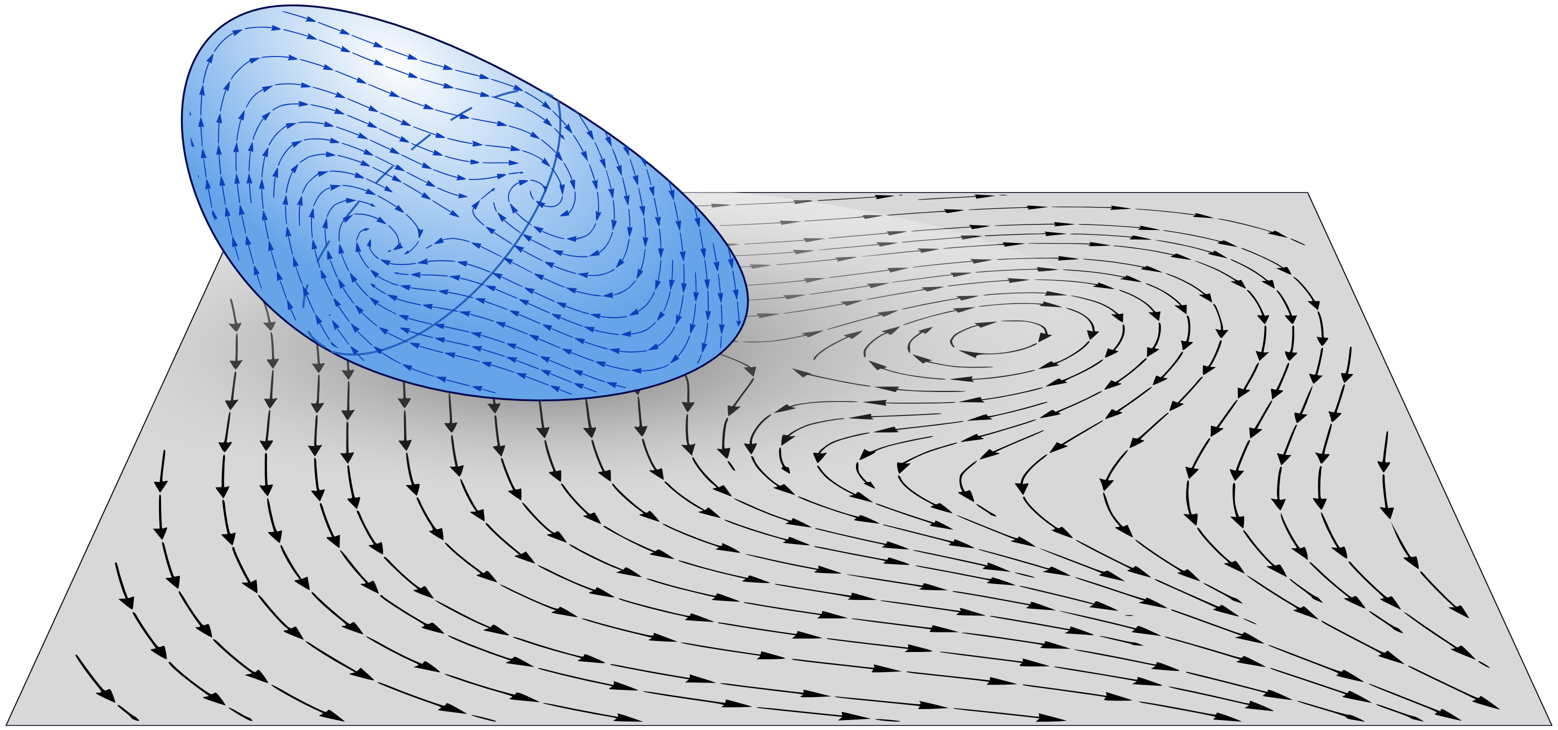
$$\dot{u} = B(\rho \times \Omega) + BV + W$$

$$\dot{B} = B\hat{\Omega}$$

$$M = \mathbb{I}\Omega + m\rho \times (\Omega \times \rho - V - B^{-1}W)$$

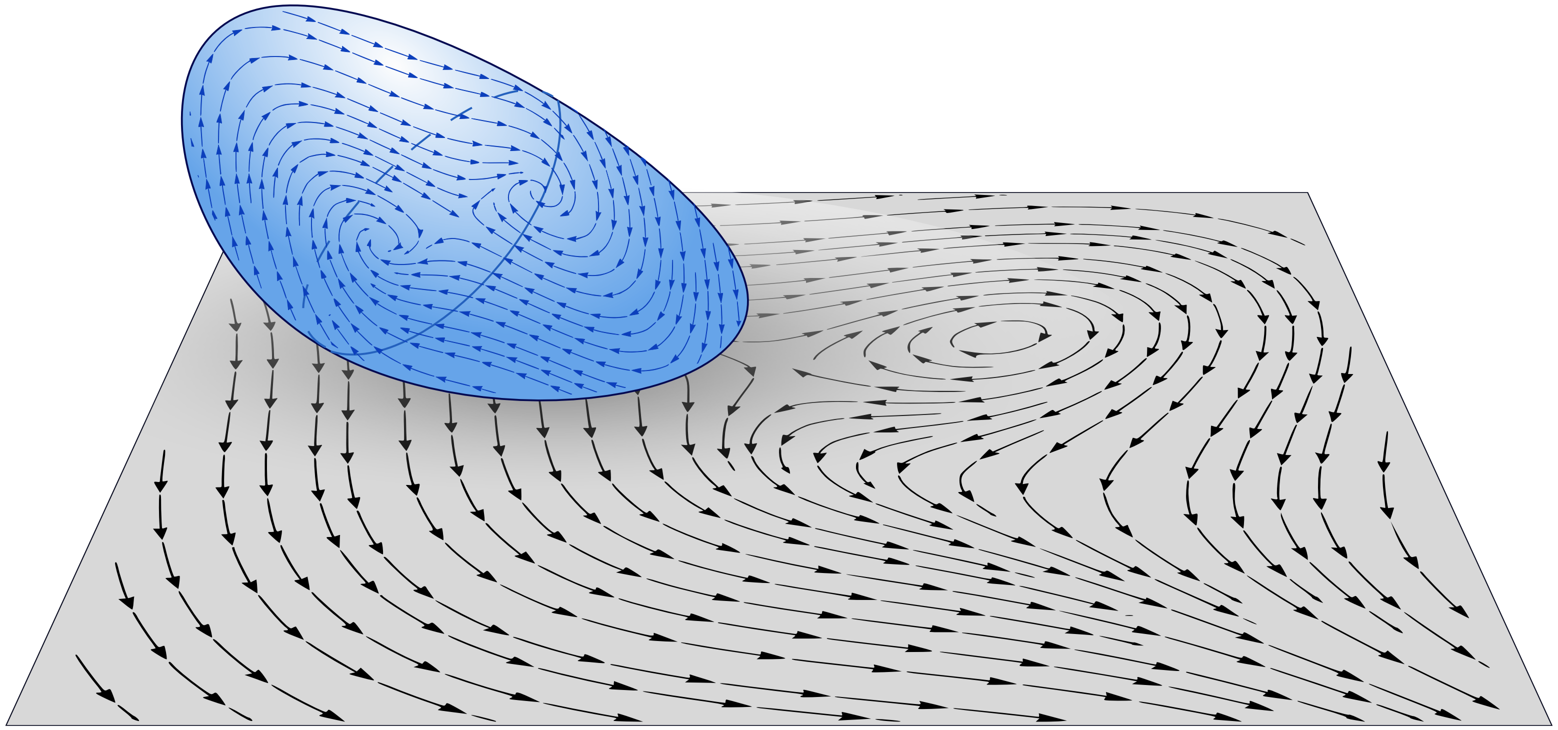


General system



General system

- **No invariant measure**
- **Generally chaotic**



General system

- No invariant measure
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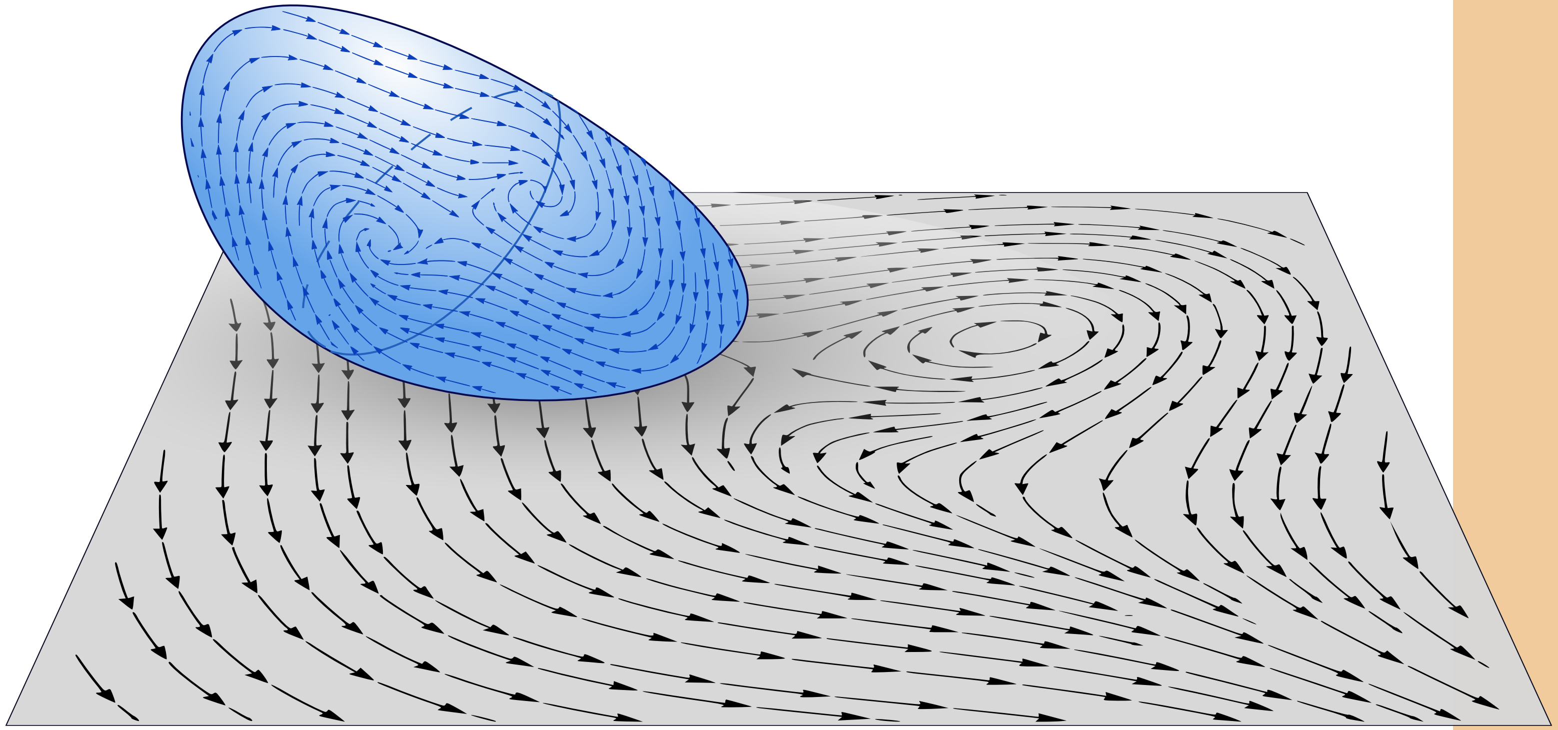
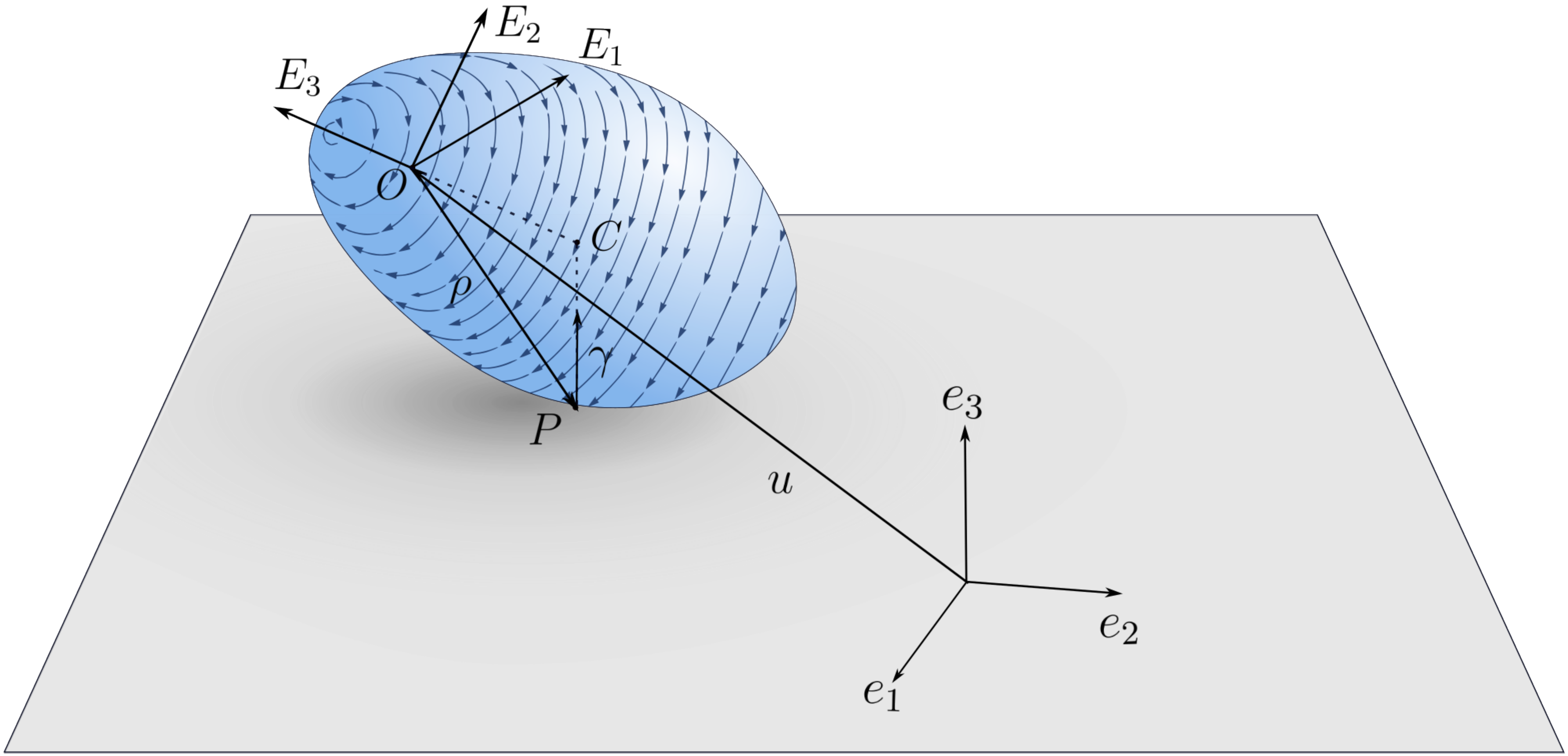


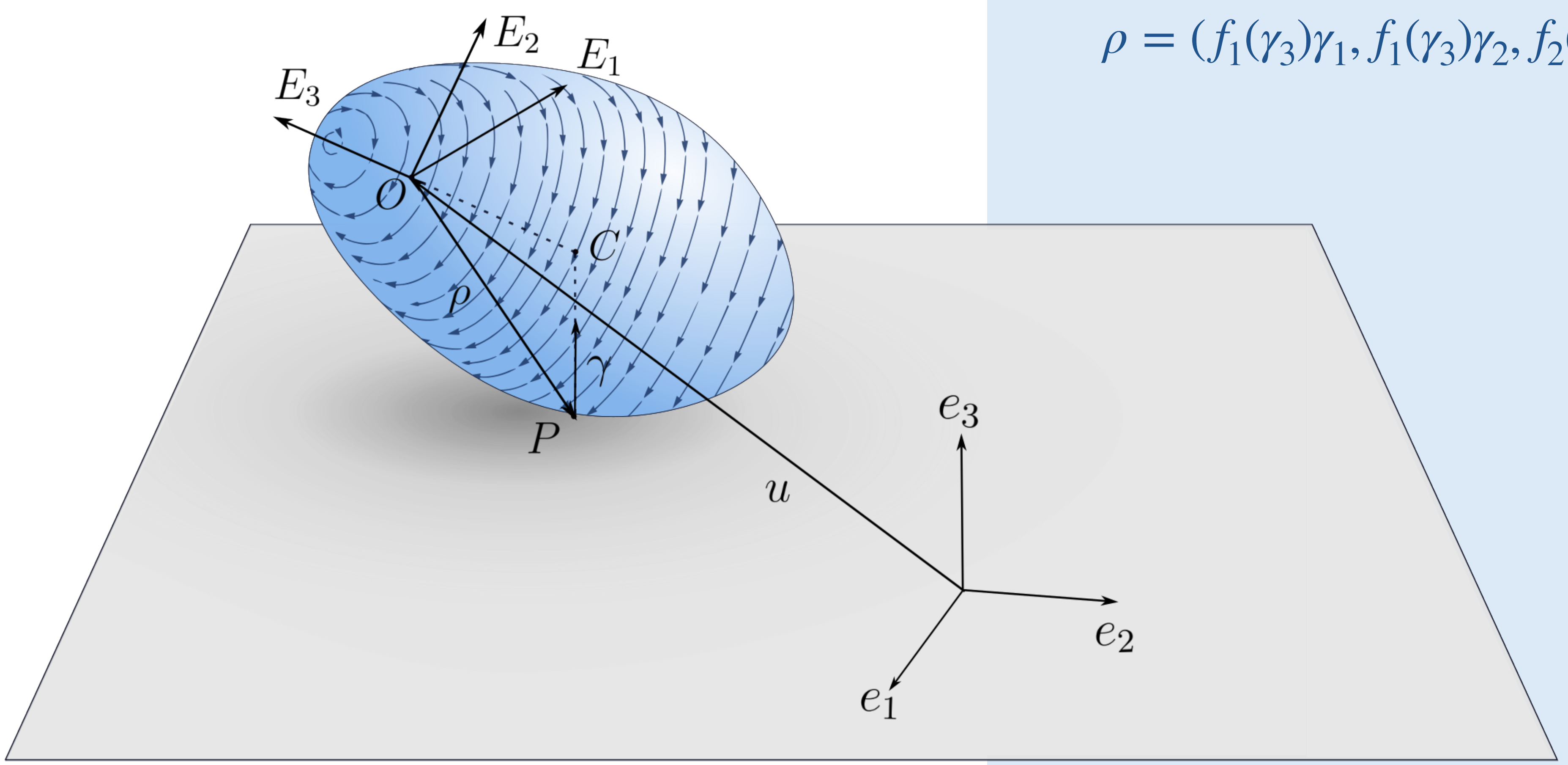
For it to be integrable we need extra symmetries that lead to a sufficient number of:

- First integrals
- Invariant measure

Body of revolution with rotating shell



Body of revolution with rotating shell



Equations of motion:

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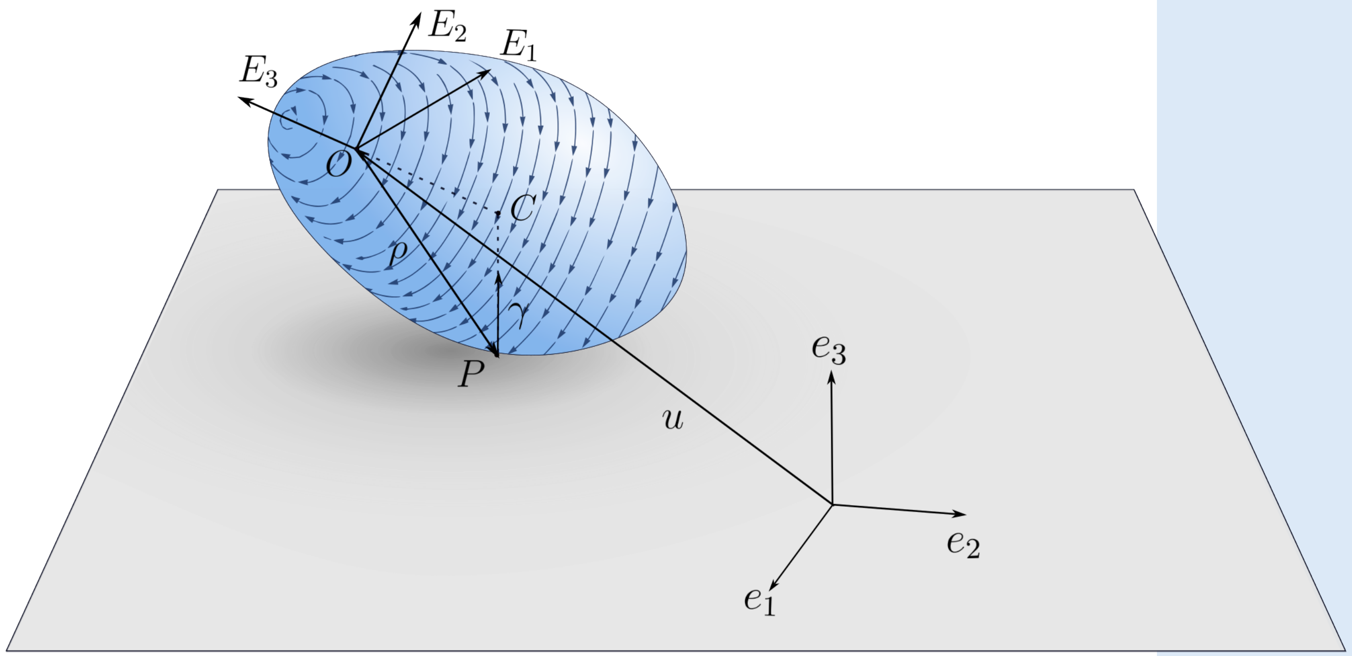
$$+ m\eta(\rho \times E_3) \times (\dot{\rho} + \Omega \times \rho)$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\eta\gamma \times (\rho \times E_3)$$

$$\rho = (f_1(\gamma_3)\gamma_1, f_1(\gamma_3)\gamma_2, f_2(\gamma_3))$$

Body of revolution with rotating shell



First integrals:

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = U^{-1}\left(\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \eta u\right),$$

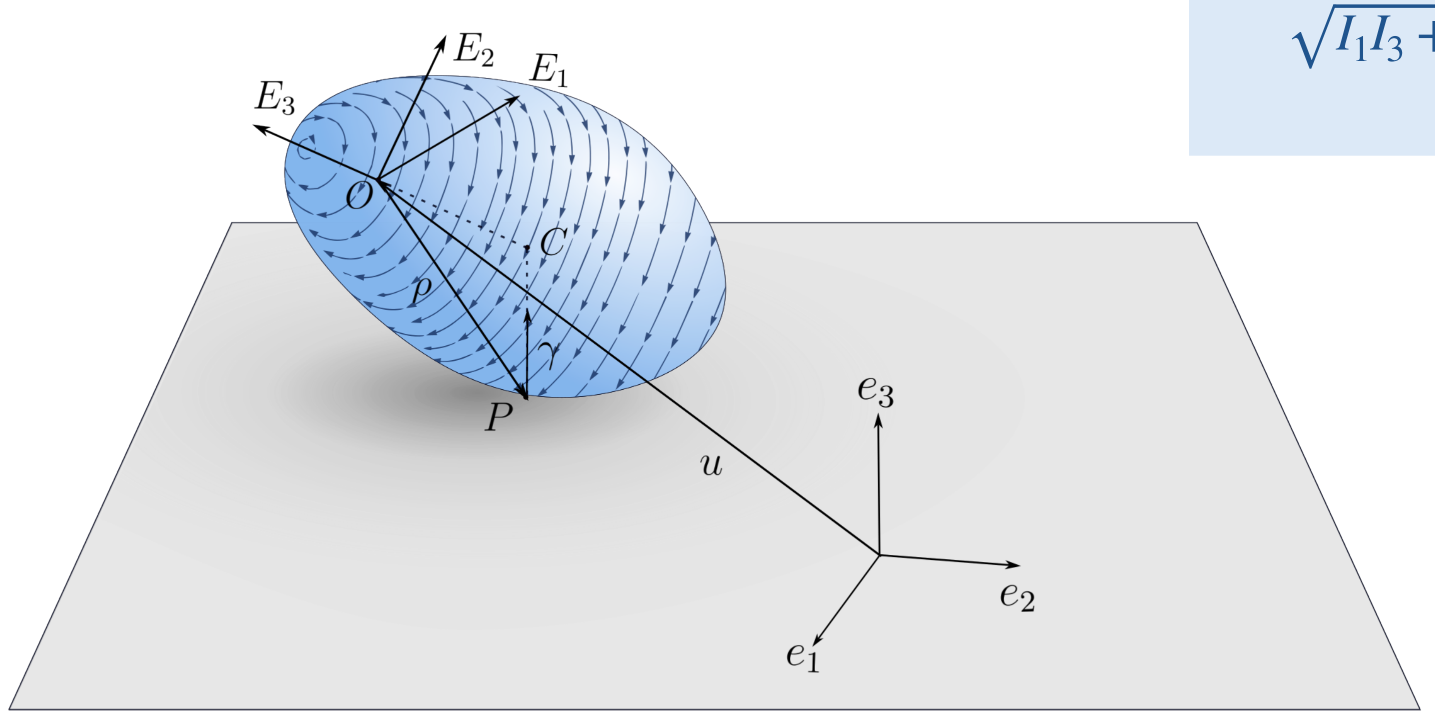
$$K_1 = \frac{\langle M, \rho\rangle}{f_1}, \quad K_2 = \frac{\Omega_3}{\sqrt{I_1 I_3 + m\langle \rho, \mathbb{I}\rho\rangle}},$$

U solution matrix of $U' = G(\gamma_3)U, \quad U(0) = Id$

u solution of $u' = Gu + b.$

$G = G(\gamma_3), b = b(\gamma_3)$

Body of revolution with rotating shell



Theorem (C., García Naranjo, 2023):

First integrals:

$||\gamma||^2, J_1, J_2$

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}(\Omega + \eta E_3), \Omega + \eta E_3 \rangle + \frac{m}{2}\|\rho \times (\Omega + \eta E_3)\|^2 - mr\langle \rho, \gamma \rangle$$

Invariant measure:

$$\frac{1}{\sqrt{I_1 I_3 + m\langle \rho, \mathbb{I}\rho \rangle}} dM d\gamma$$

Body of revolution with rotating shell

Integrable

Theorem (C., García Naranjo, 2023):

First integrals:

$||\gamma||^2, J_1, J_2$

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}(\Omega + \eta E_3), \Omega + \eta E_3 \rangle + \frac{m}{2}\|\rho \times (\Omega + \eta E_3)\|^2 - mr\langle \rho, \gamma \rangle$$

Invariant measure:

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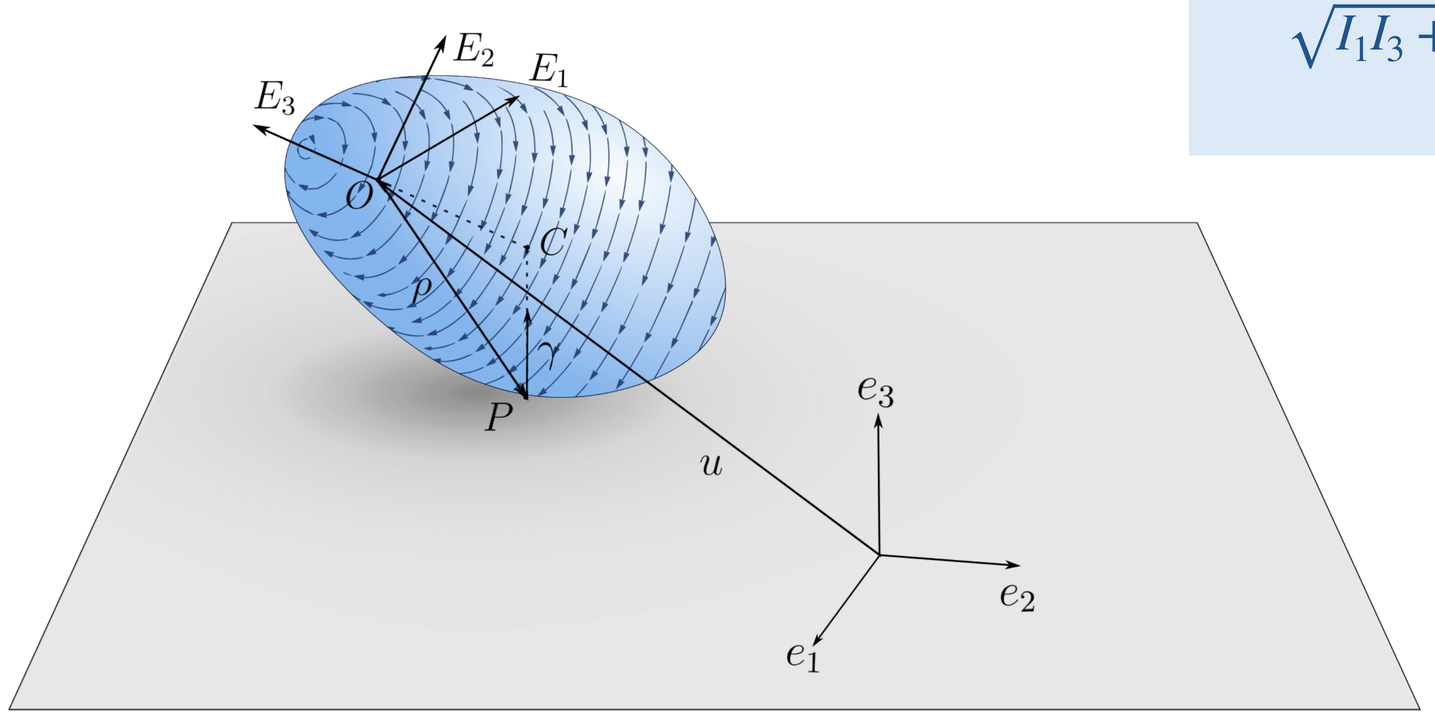
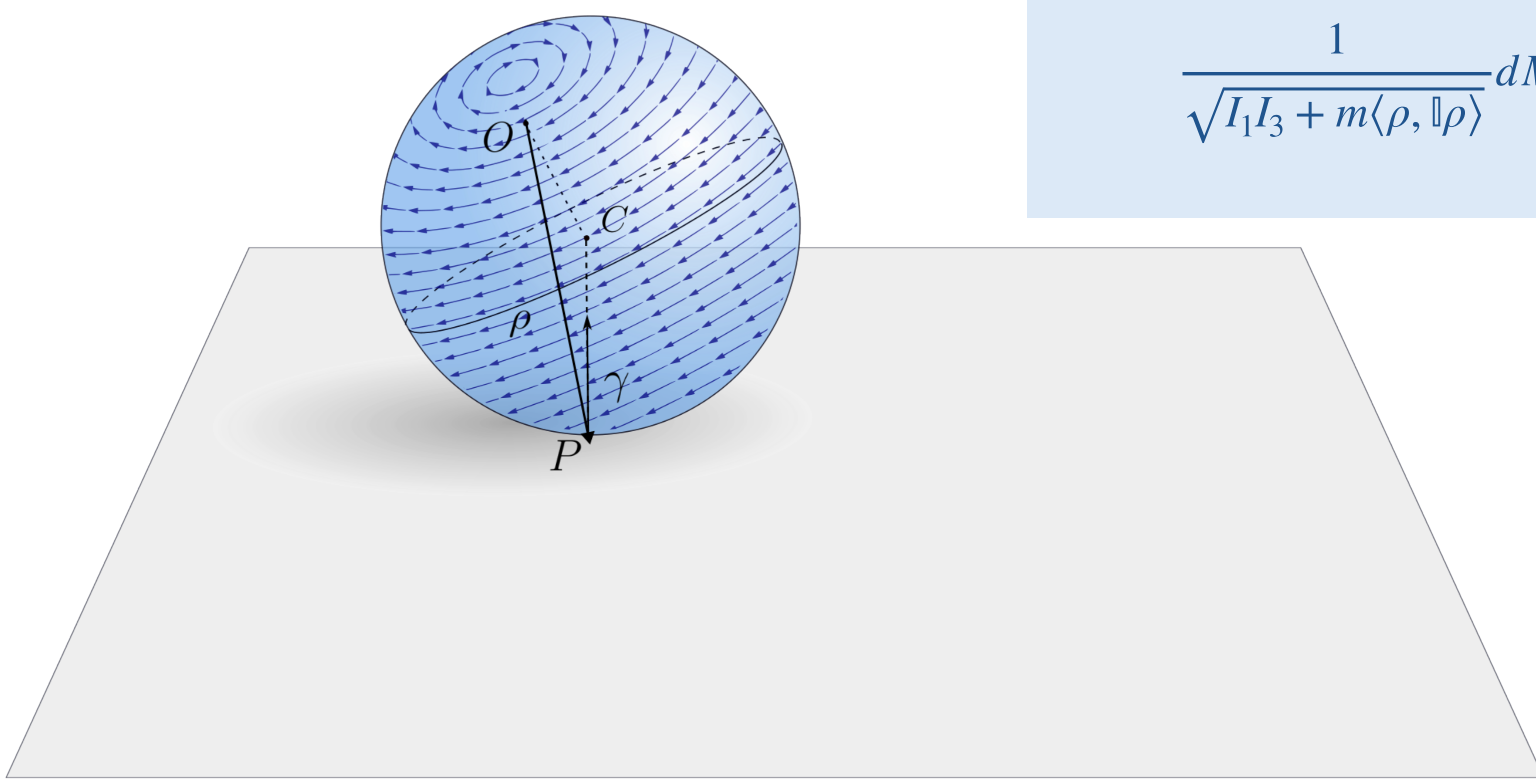


Routh's sphere with rotating shell

$$\rho(\gamma) = -r\gamma - lE_3$$



First integrals

$$||\gamma||^2,$$

$$J_1 = \frac{1}{r}\langle M, \rho\rangle,$$

$$J_2 = \Omega_3 + \eta\left(\frac{I_1 I_3\sqrt{m}}{(I_1 - I_3)^{3/2}}\arctan\left(\frac{\mu'}{r\sqrt{m(I_1 - I_3)}}\right) - \frac{I_1}{I_1 - I_3}\mu\right),$$

$$\mu = \sqrt{I_1 I_3 + m\langle \rho, \mathbb{I}\rho\rangle}$$

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}(\Omega + \eta E_3), \Omega + \eta E_3\rangle + \frac{m}{2}||\rho \times (\Omega + \eta E_3)||^2 - mr\langle \rho, \gamma\rangle$$

Invariant measure

$$\frac{1}{\sqrt{I_1 I_3 + m\langle \rho, \mathbb{I}\rho\rangle}} dM d\gamma$$

Routh's sphere with rotating shell

Integrable

$$\rho(\gamma) = -r\gamma - lE_3$$

First integrals

$$||\gamma||^2,$$

$$J_1 = \frac{1}{r}\langle M, \rho\rangle,$$

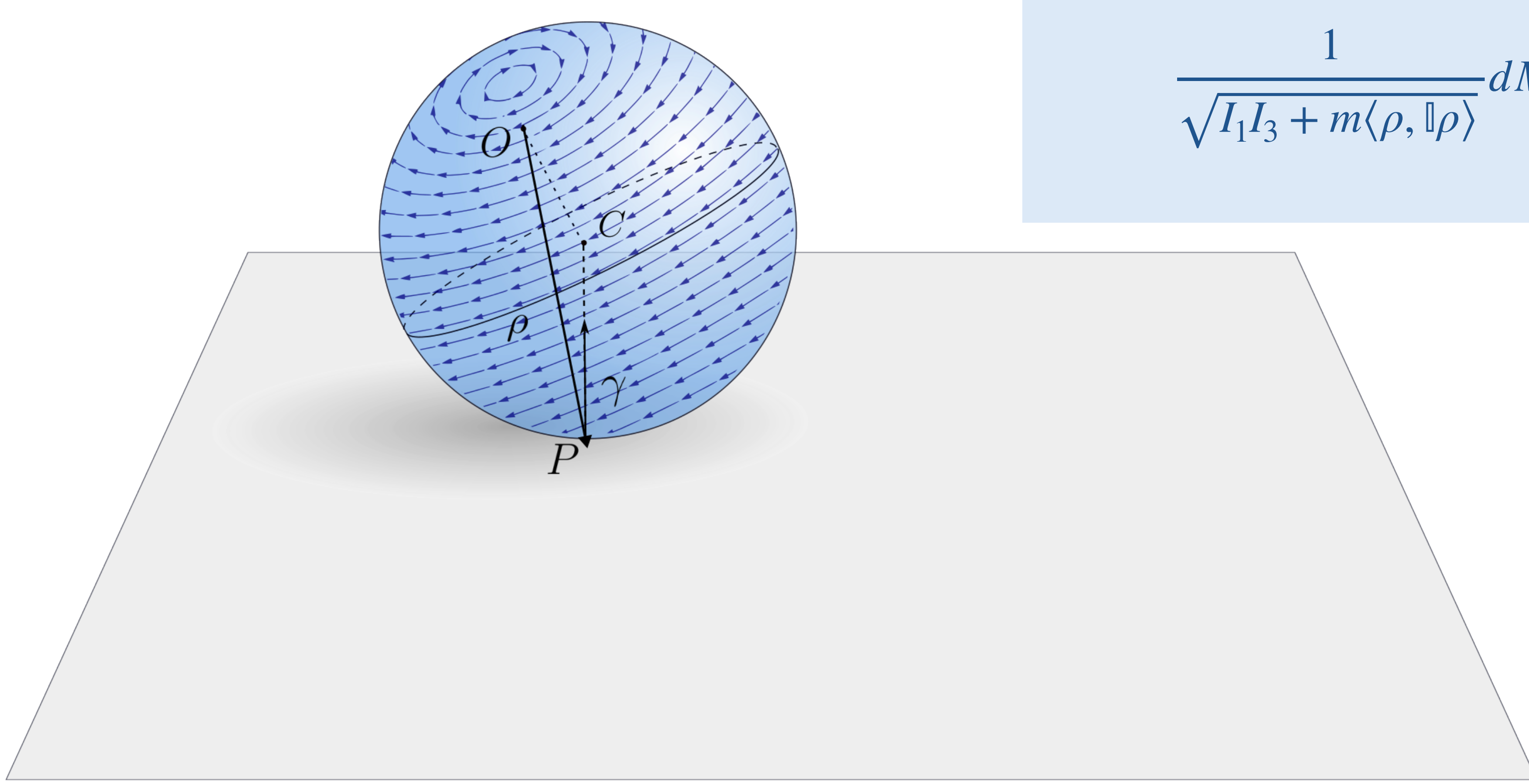
$$J_2 = \Omega_3 + \eta\left(\frac{I_1 I_3\sqrt{m}}{(I_1 - I_3)^{3/2}}\arctan\left(\frac{\mu'}{r\sqrt{m(I_1 - I_3)}}\right) - \frac{I_1}{I_1 - I_3}\mu\right),$$

$$\mu = \sqrt{I_1 I_3 + m\langle\rho, \mathbb{I}\rho\rangle}$$

$$E_{mov} = \frac{1}{2}\langle\mathbb{I}(\Omega + \eta E_3), \Omega + \eta E_3\rangle + \frac{m}{2}||\rho \times (\Omega + \eta E_3)||^2 - mr\langle\rho, \gamma\rangle$$

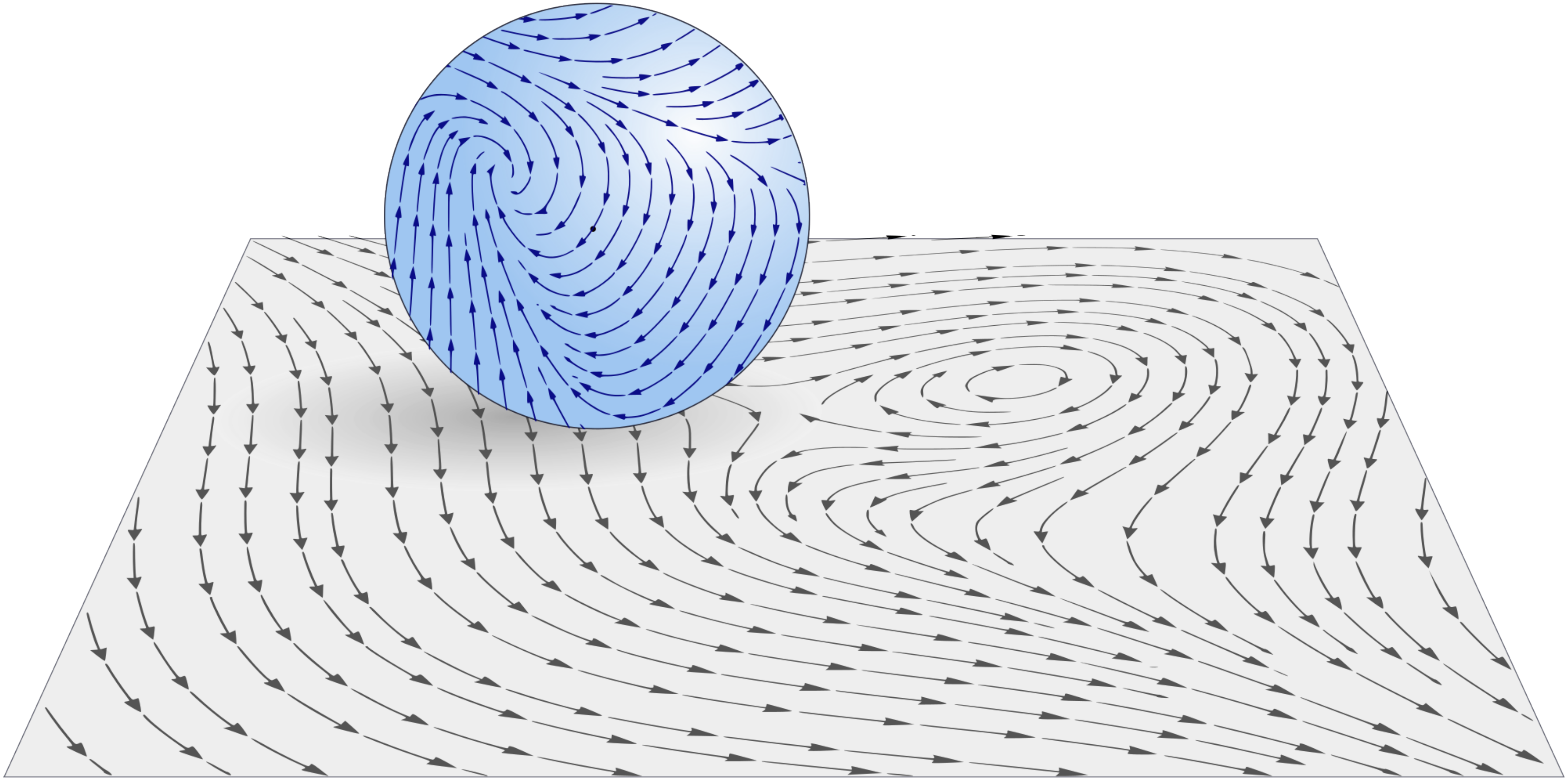
Invariant measure

$$\frac{1}{\sqrt{I_1 I_3 + m\langle\rho, \mathbb{I}\rho\rangle}}dMd\gamma$$



Chaplygin sphere

$I_1 \neq I_2 \neq I_3$



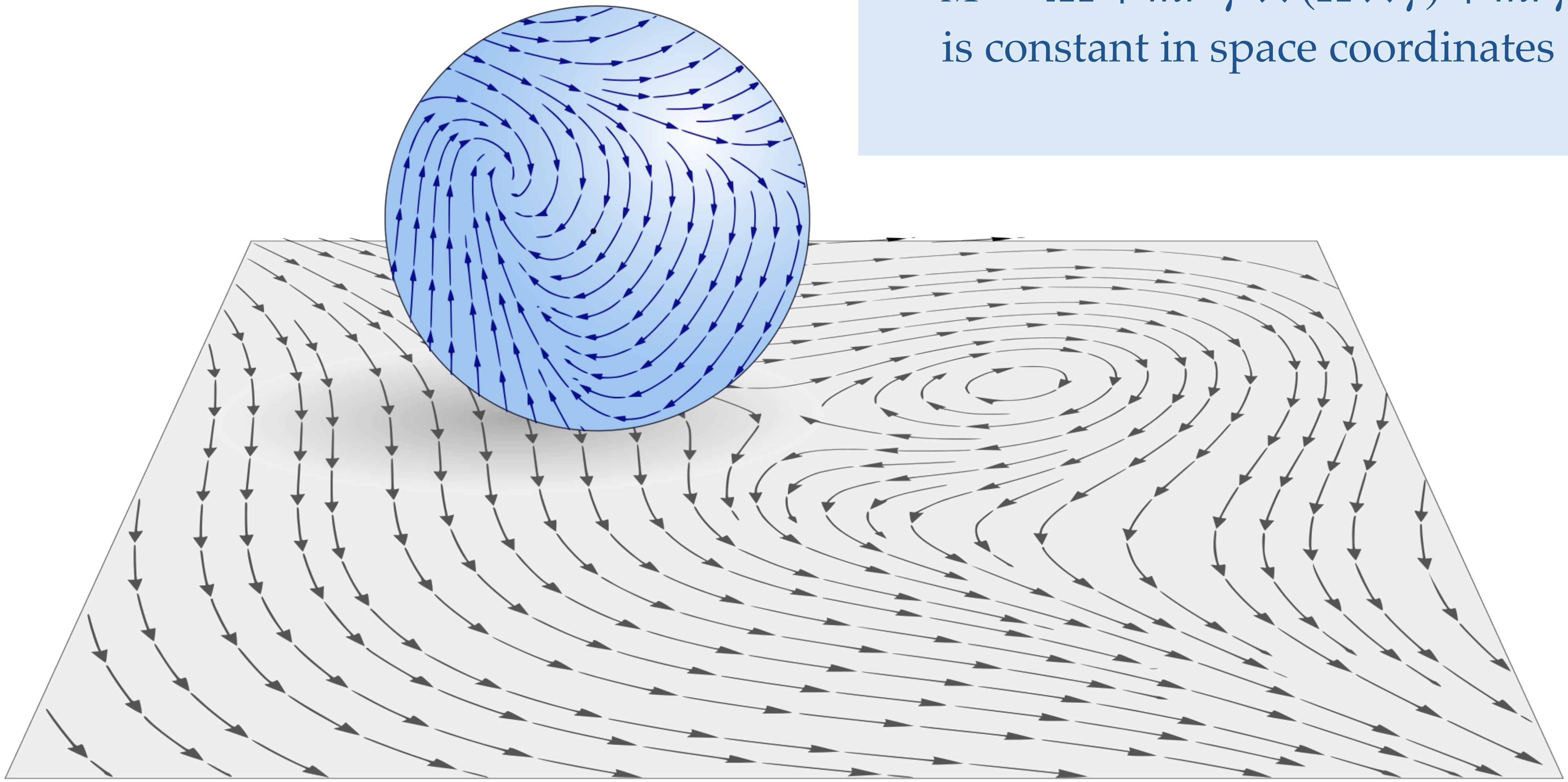
Chaplygin sphere

$I_1 \neq I_2 \neq I_3$

Equations of motion:

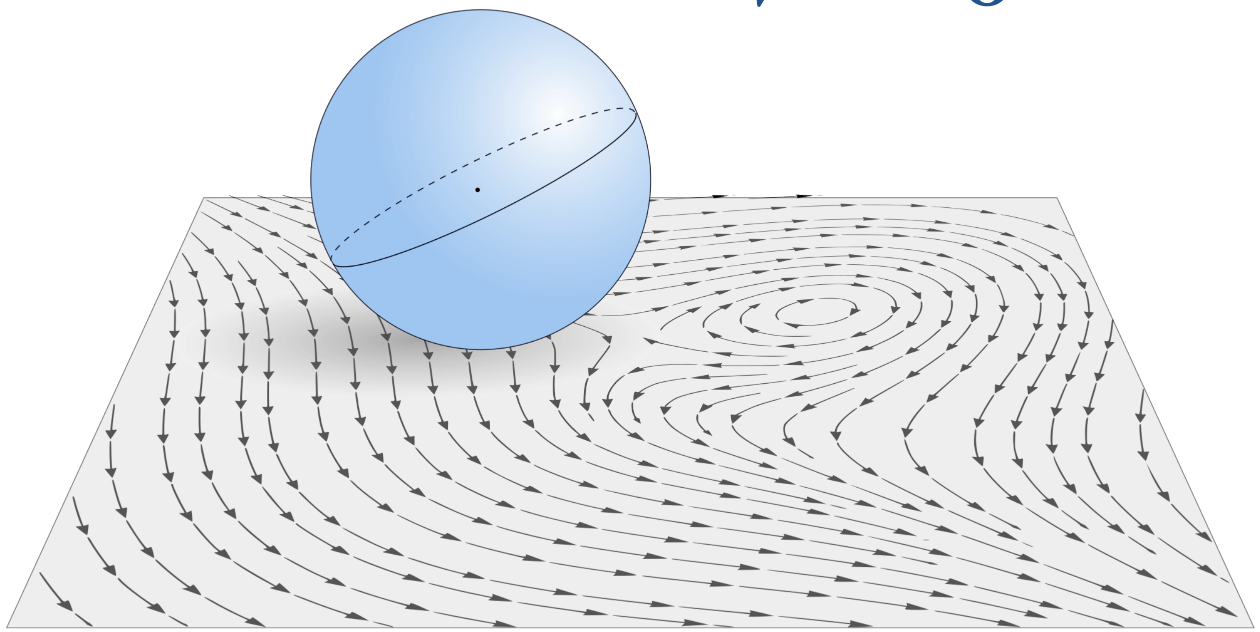
$$\dot{M} = M \times \Omega$$
$$\dot{u} = -rB(\gamma \times \Omega) + BV + W$$
$$\dot{B} = B\hat{\Omega}$$

Theorem (C. García Naranjo, 2023):
$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times (V + B^{-1}W)$
is constant in space coordinates

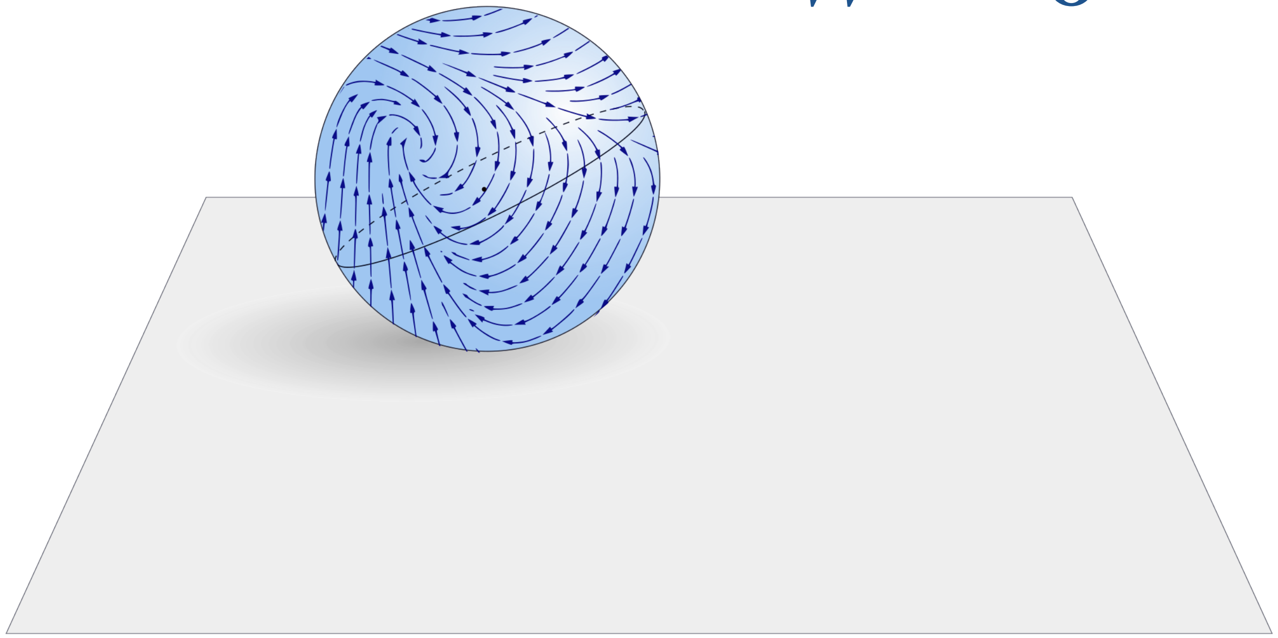


Chaplygin sphere

$V = 0$

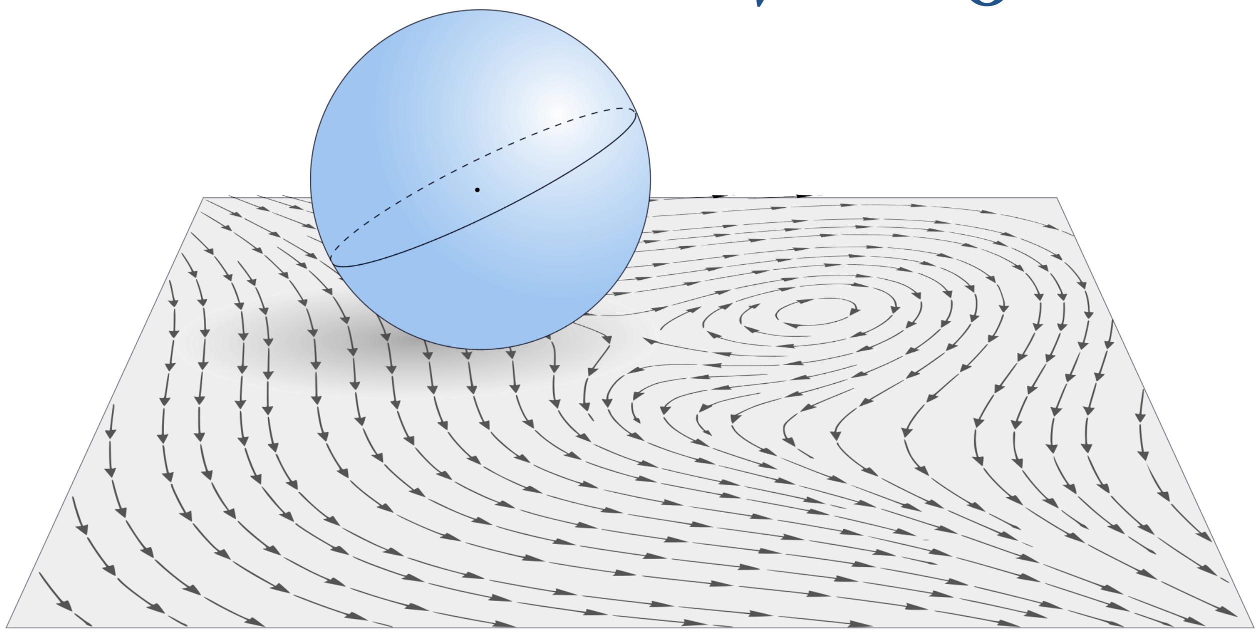


$W = 0$



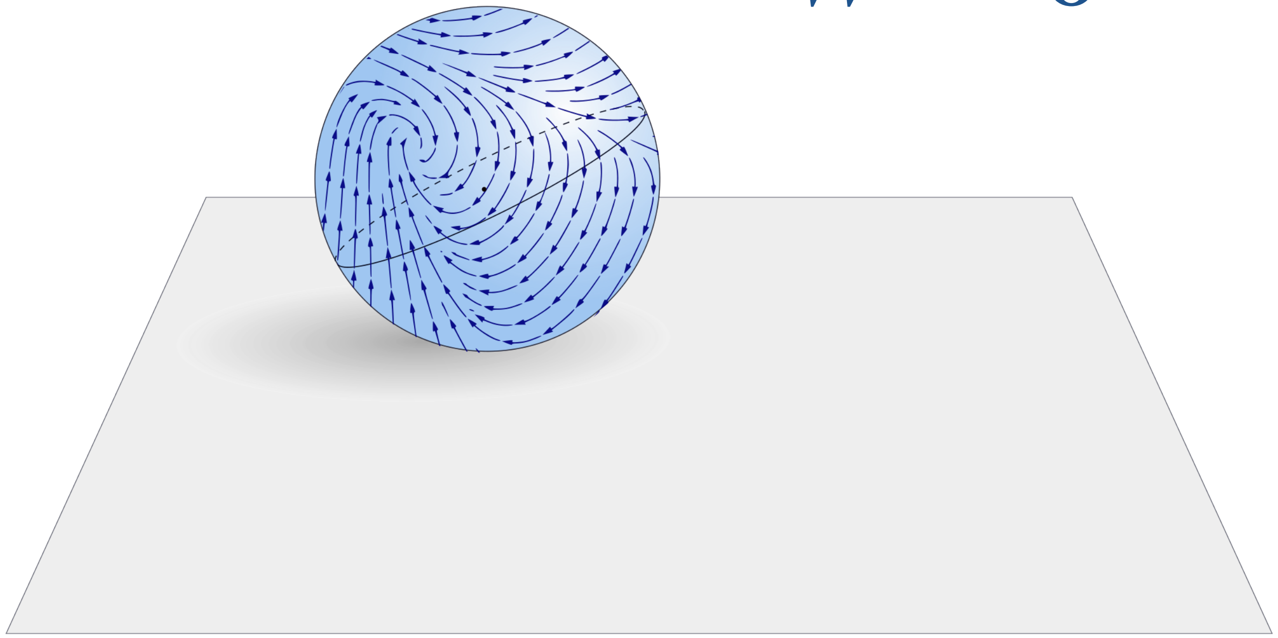
Chaplygin sphere

$V = 0$



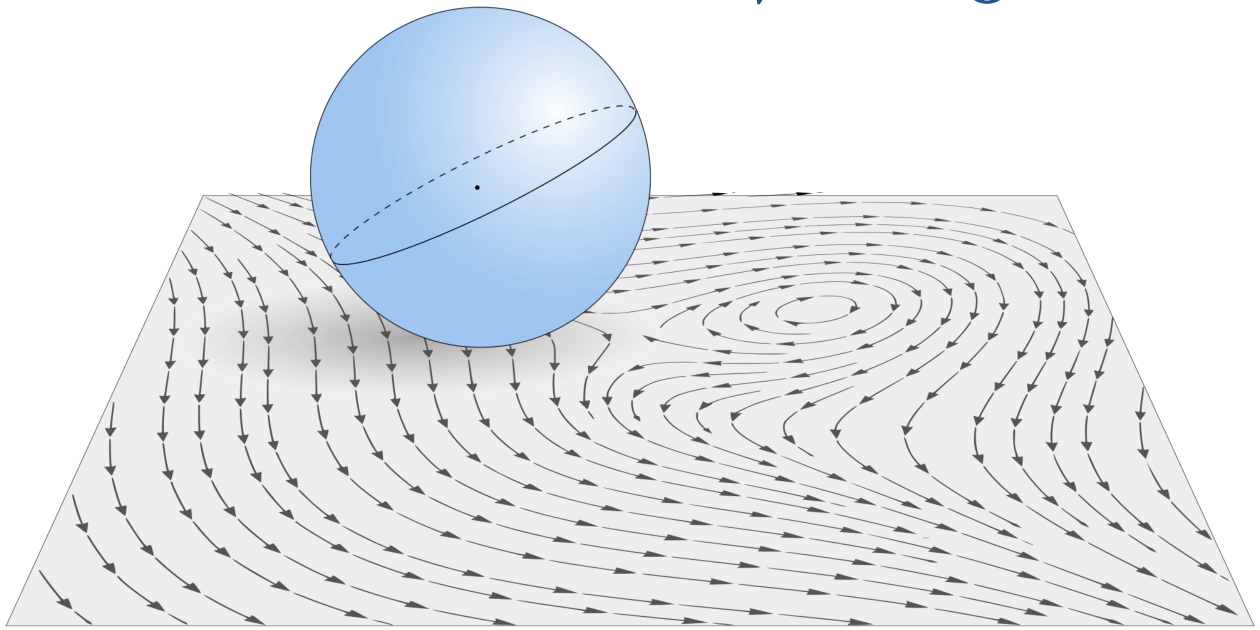
First integrals
Invariant measure

$W = 0$



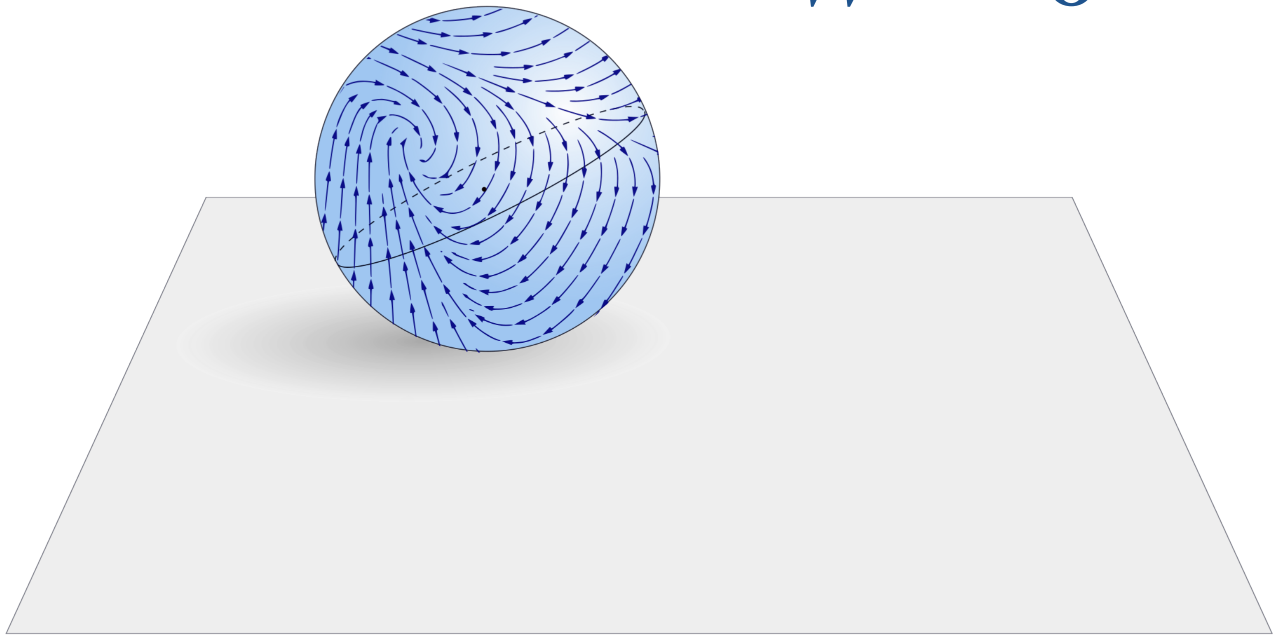
Chaplygin sphere

$V = 0$



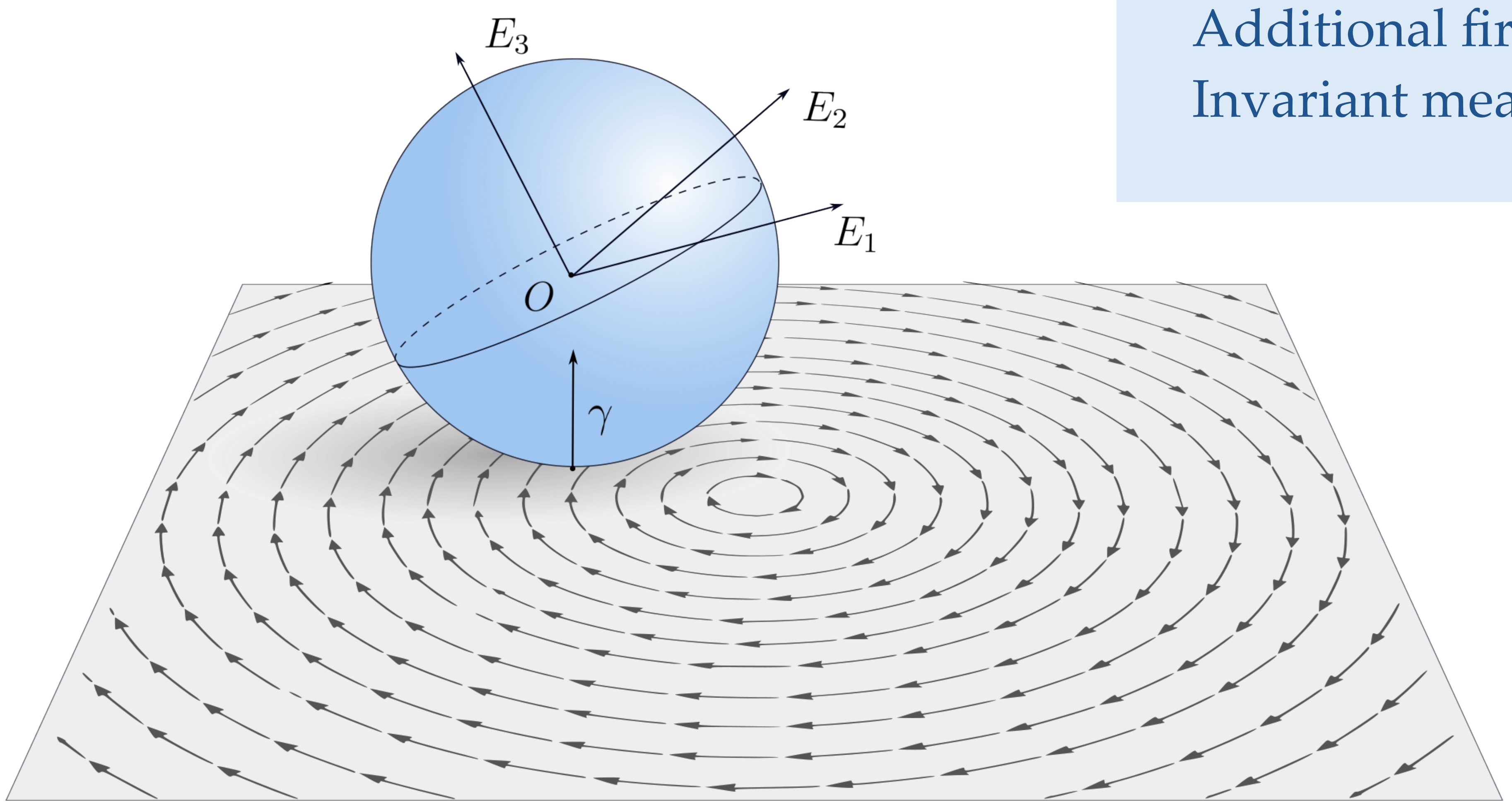
First integrals
Invariant measure

$W = 0$



SE(2)-symmetry
First integrals

Chaplygin sphere $V = 0$



Examples:

Rotating plane:

Bizyaev, Borisov, Mamaev, 2018

Vibrating plane:

Kilin, Pivovarova, 2021

Additional first integral- moving energy

Invariant measure

Chaplygin sphere $V = 0$

Generally chaotic

Examples:

Rotating plane:

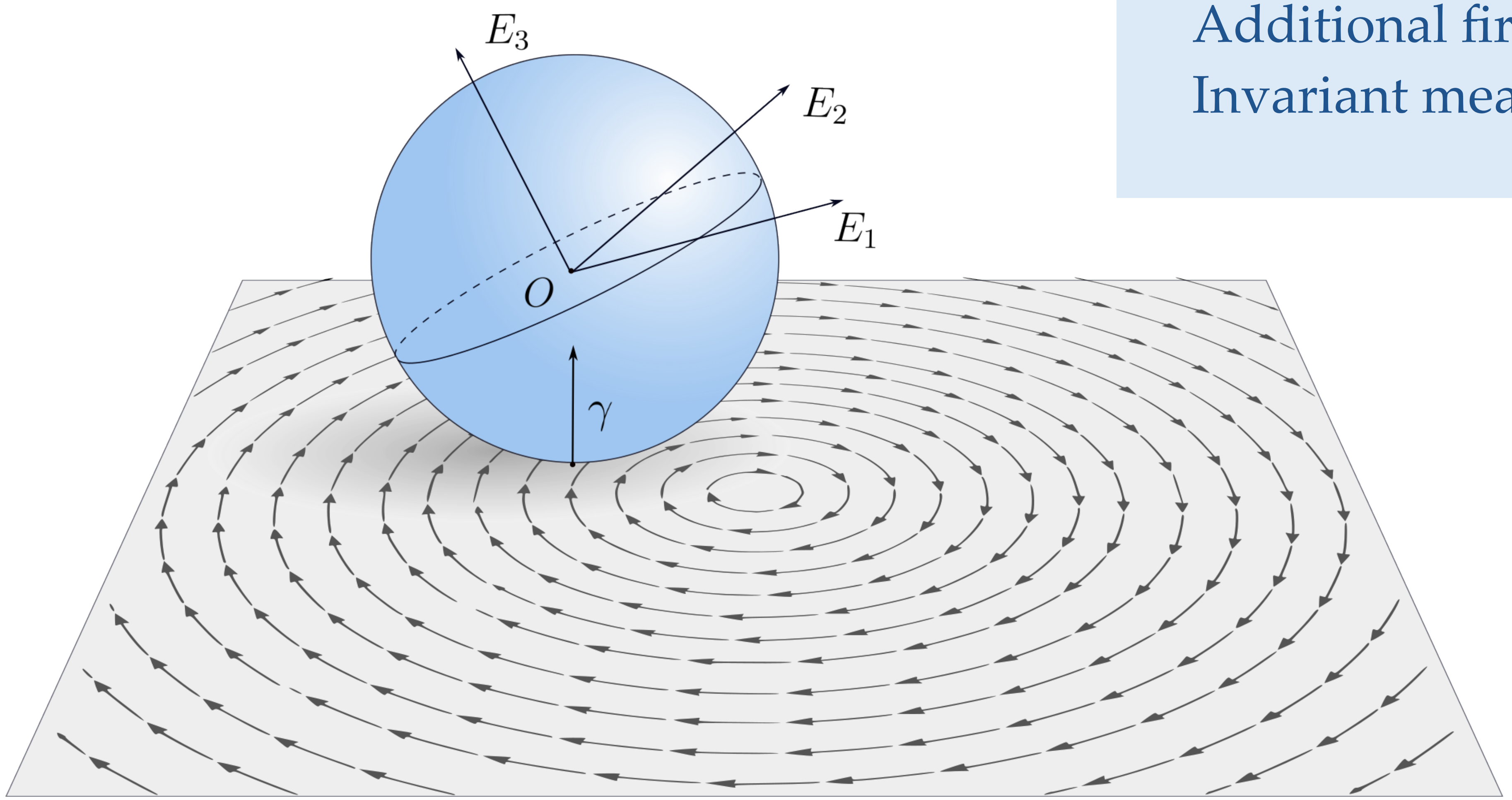
Bizyaev, Borisov, Mamaev, 2018

Vibrating plane:

Kilin, Pivovarova, 2021

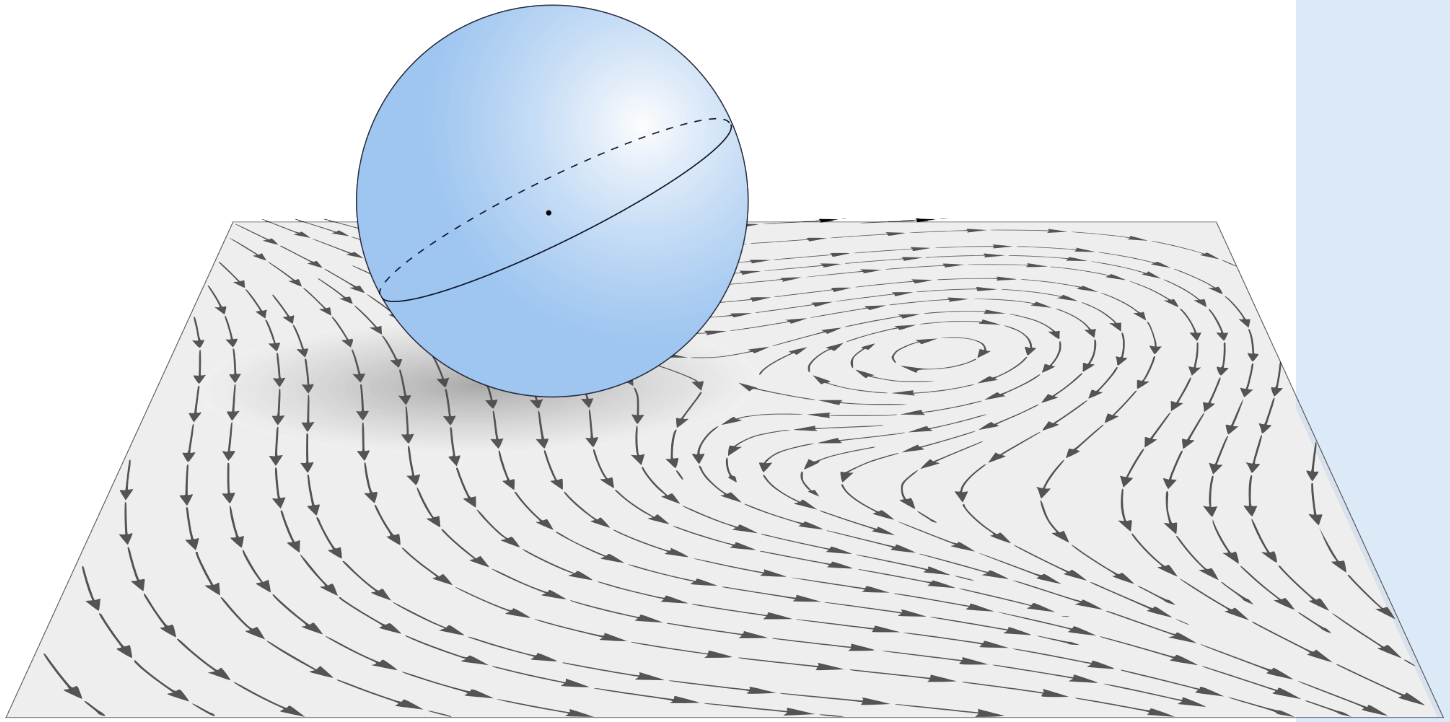
Additional first integral- moving energy

Invariant measure



Chaplygin sphere $V = 0$

$W \in \mathfrak{X}(\mathbb{R}^2)$



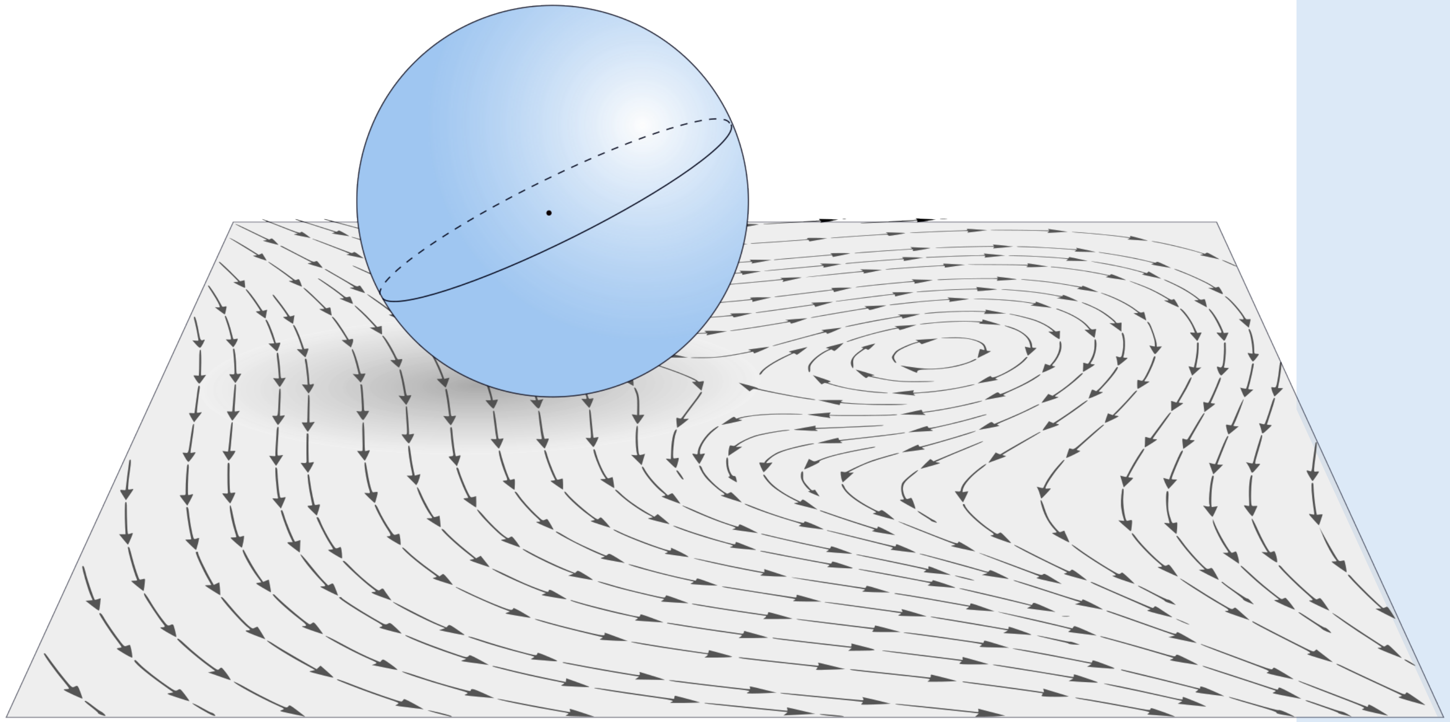
Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{u} = -rB(\gamma \times \Omega) + W$$
$$\dot{B} = B\hat{\Omega}$$

$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times (B^{-1}W)$$

Chaplygin sphere $V = 0$

$W \in \mathfrak{X}(\mathbb{R}^2)$



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$$\dot{B} = B\hat{\Omega}$$

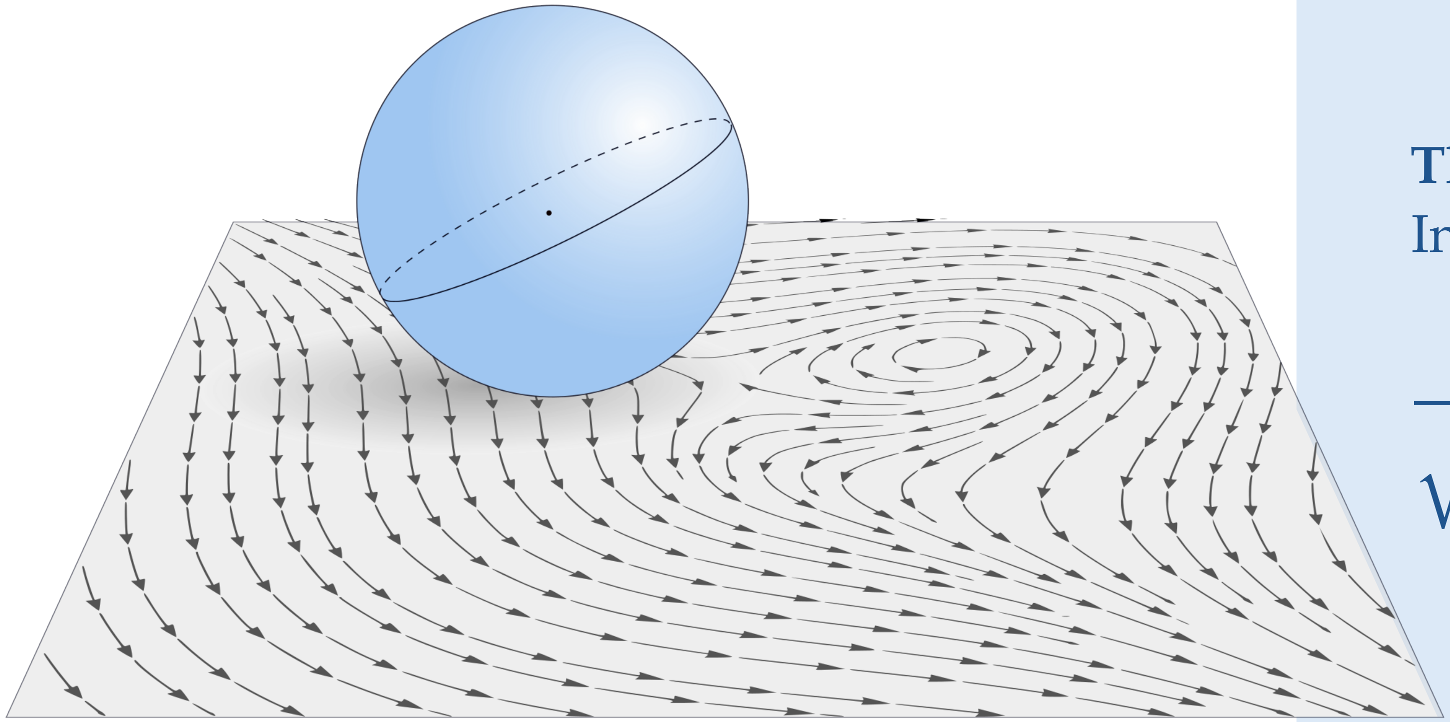
$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times (B^{-1}W)$$

First integrals

$$\langle M, \alpha \rangle, \ \langle M, \beta \rangle, \ \langle M, \gamma \rangle$$

Chaplygin sphere $V = 0$

$W \in \mathfrak{X}(\mathbb{R}^2)$



Equations of motion:

$$\begin{aligned} \dot{M} &= M \times \Omega \\ \dot{u} &= -\, rB(\gamma \times \Omega) + W \\ \dot{B} &= B\hat{\Omega} \end{aligned}$$

$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times (B^{-1}W)$$

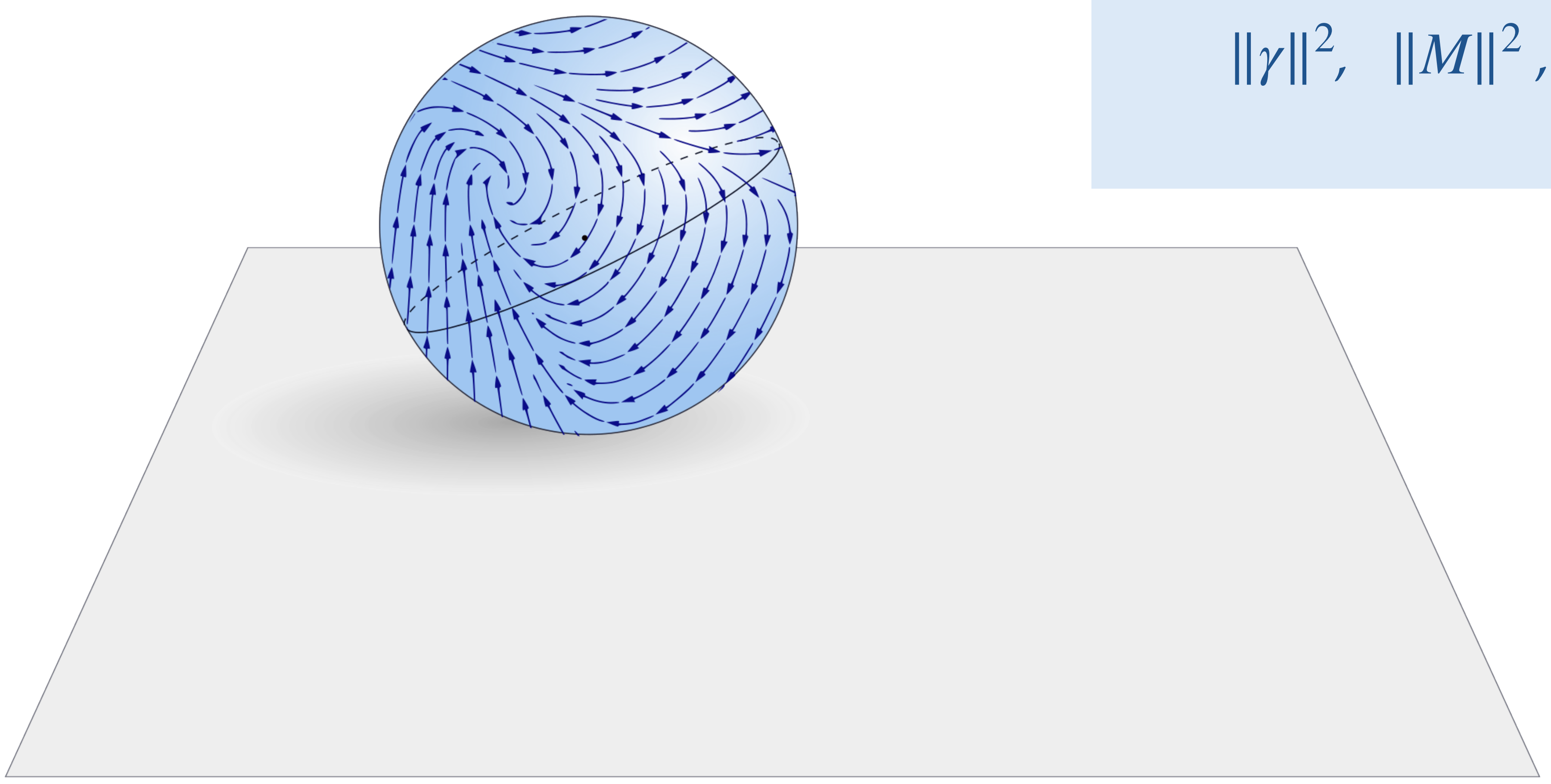
First integrals

$$\langle M, \alpha \rangle, \ \langle M, \beta \rangle, \ \langle M, \gamma \rangle$$

Theorem (C. García Naranjo, 2023):
Invariant measure if $\mathrm{div}_{\mathbb{R}^2} \mathrm{W} = 0$

$$\frac{1}{\sqrt{1 - mr^2\langle \gamma, (\mathbb{I} + mr^2)^{-1}\gamma \rangle}} dMdxdyd\alpha d\beta d\gamma$$

Chaplygin sphere $W = 0$



Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{\gamma} = \gamma \times \Omega$$

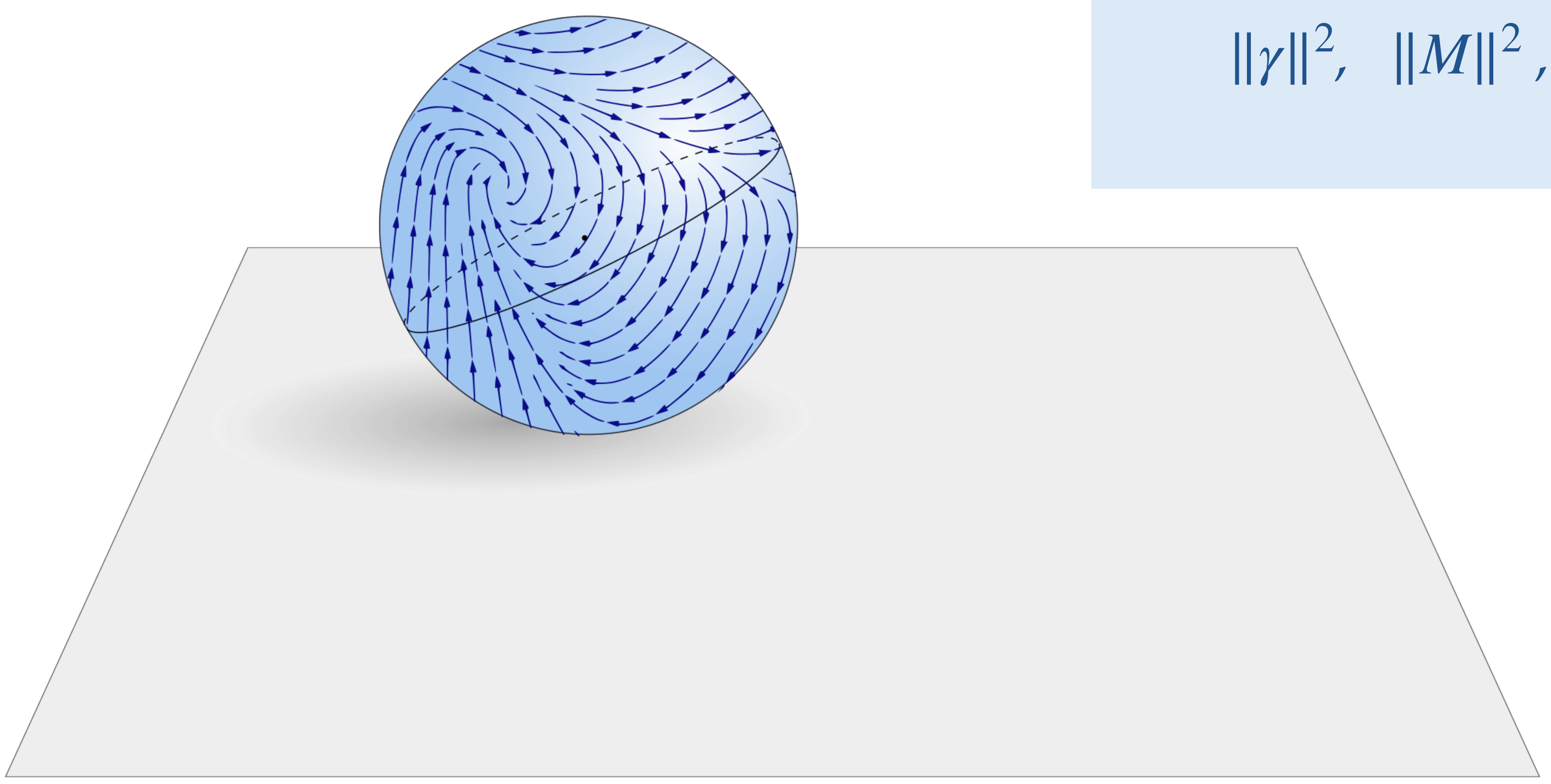
$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times V$$

First integrals

$$\|\gamma\|^2, \;\; \|M\|^2, \;\; \langle M, \gamma \rangle$$

Chaplygin sphere $W = 0$

No invariant measure
No moving energy



Equations of motion:

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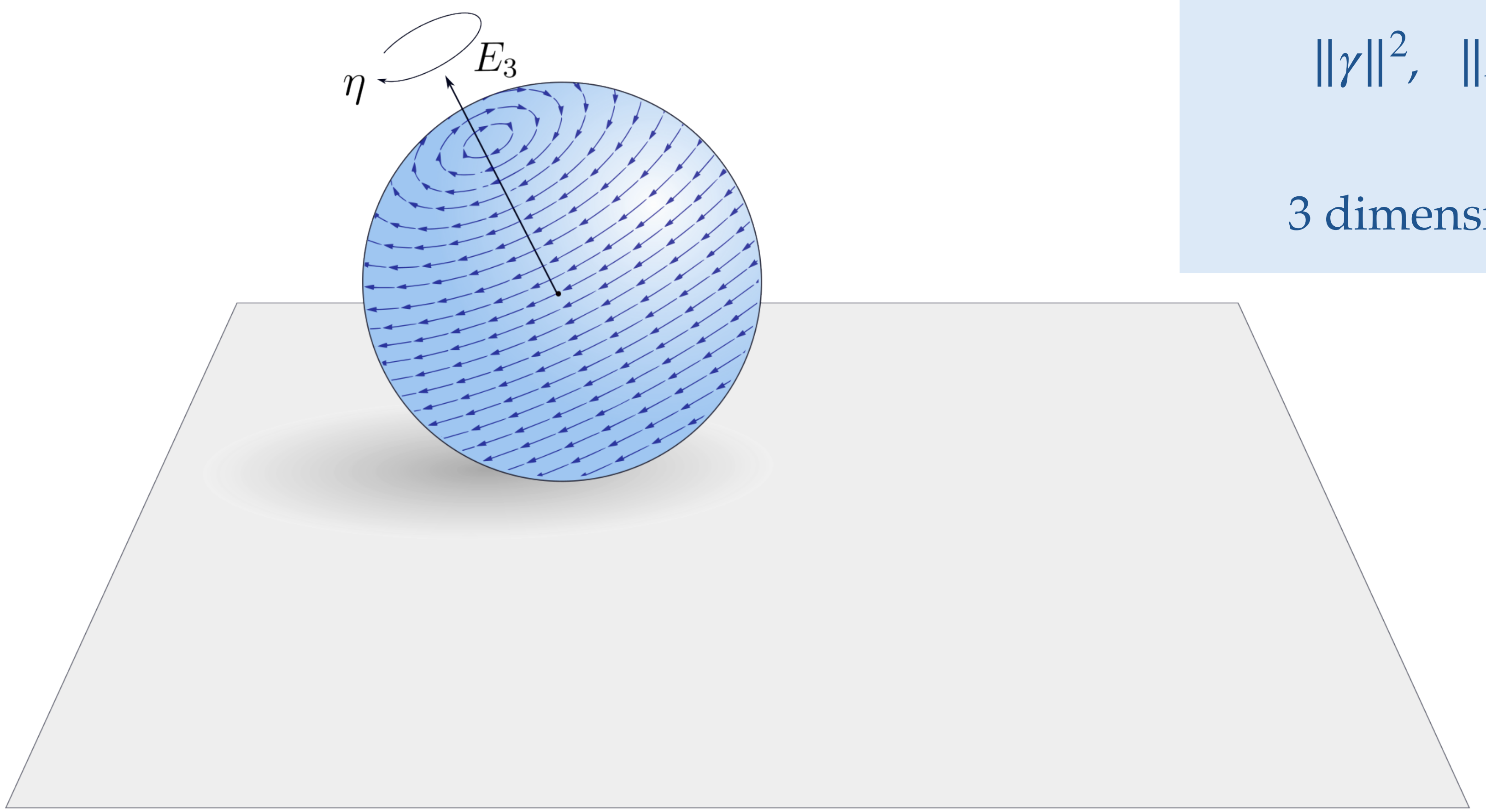
$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times V$$

First integrals

$$\|\gamma\|^2, \quad \|M\|^2, \quad \langle M, \gamma \rangle$$

Chaplygin sphere with rotating shell

$$V = -r\eta\gamma \times E_3$$



Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) - m^2\eta\gamma \times (\gamma \times E_3)$$

First integrals:

$$\|\gamma\|^2, \quad \|M\|^2, \quad \langle M, \gamma \rangle$$

3 dimensional level sets

Chaplygin sphere with rotating shell

$$V = -r\eta\gamma \times E_3$$

Generally chaotic

Equations of motion:

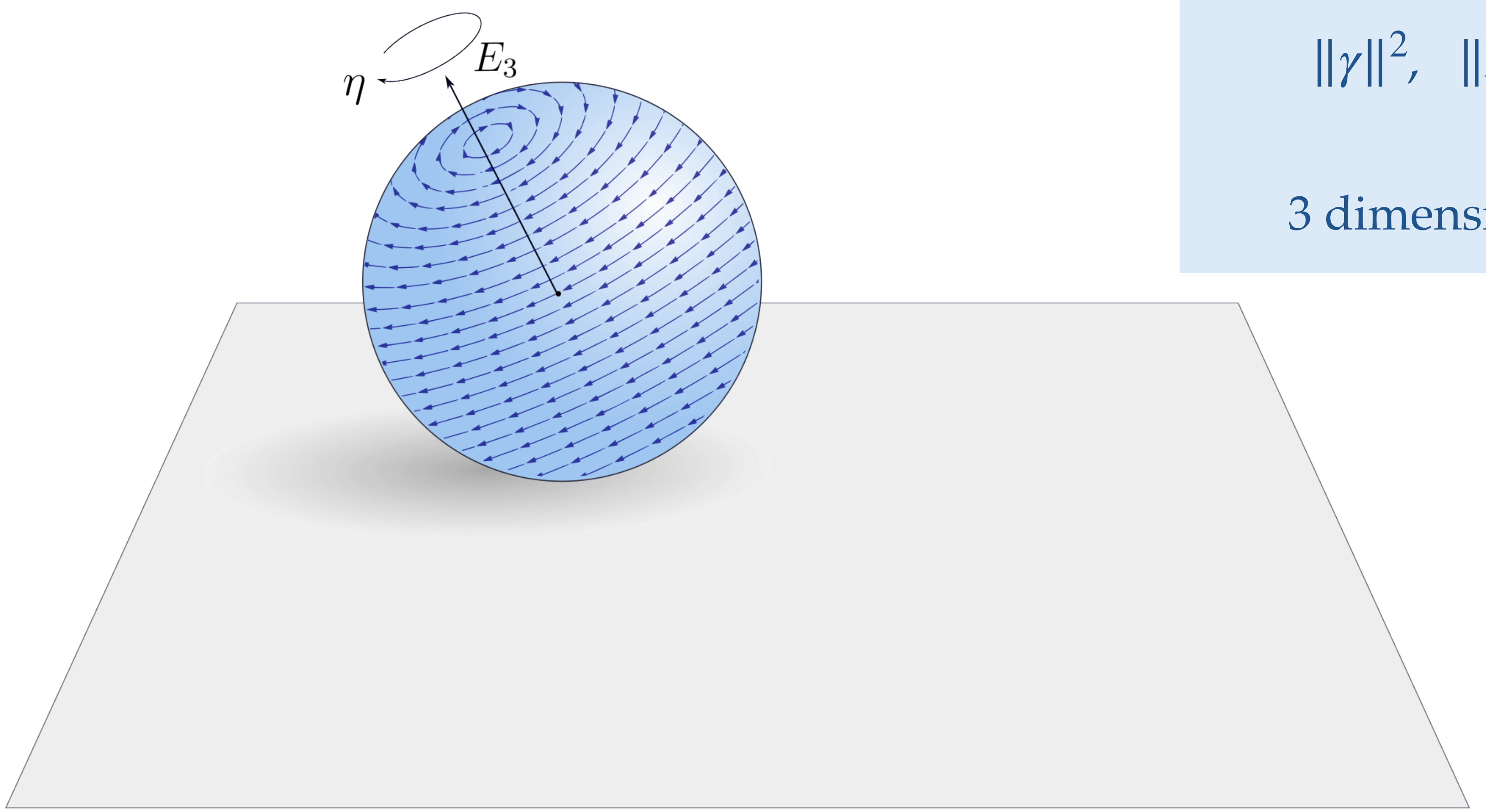
$$\dot{M} = M \times \Omega$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$M = \mathbb{I}\Omega + mr^2\gamma \times (\Omega \times \gamma) - m^2\eta\gamma \times (\gamma \times E_3)$$

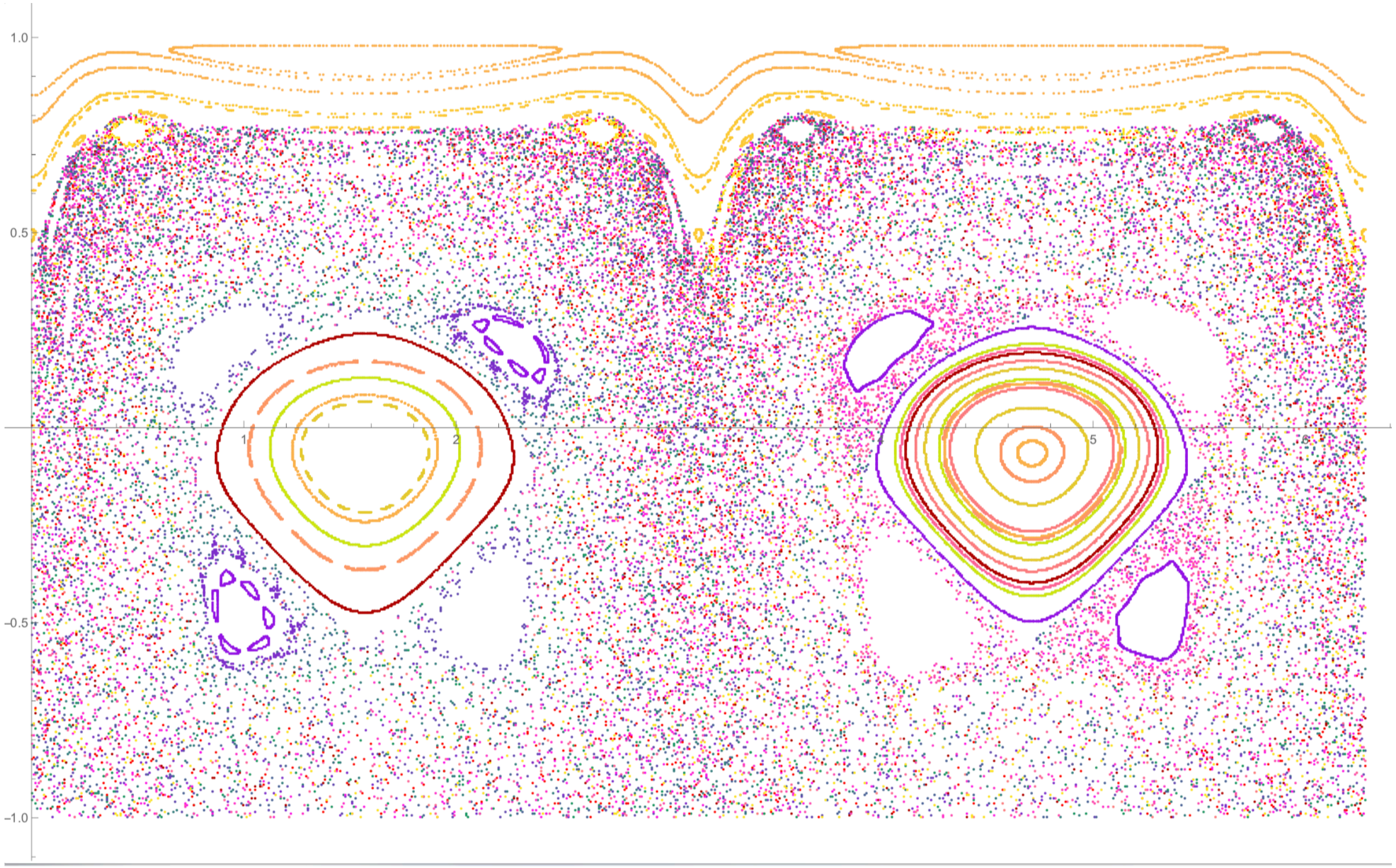
First integrals:

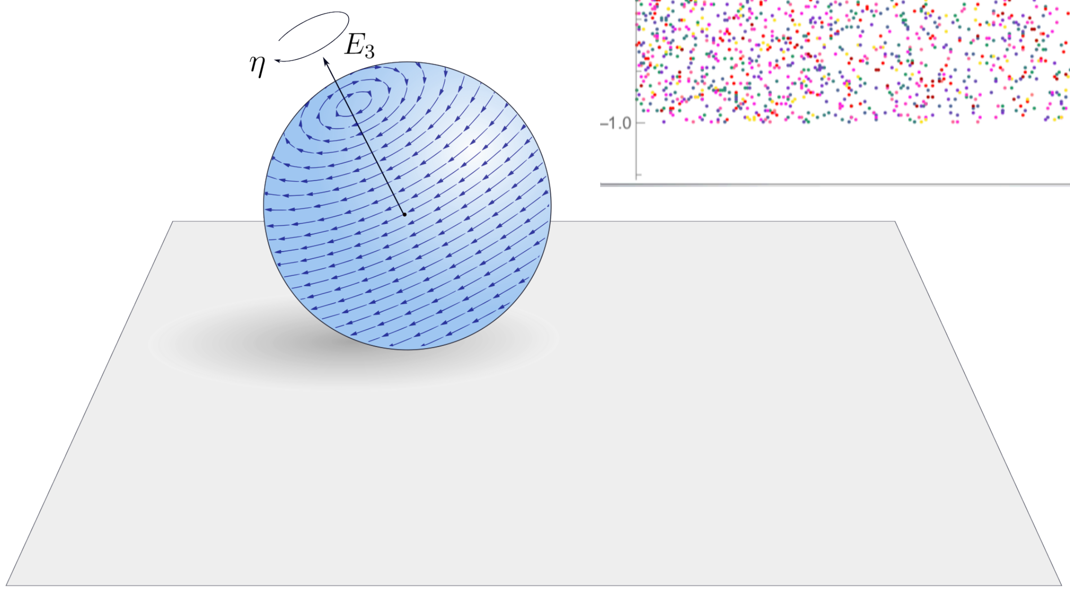
$$\|\gamma\|^2, \quad \|M\|^2, \quad \langle M, \gamma \rangle$$

3 dimensional level sets



Poincaré map





Poincaré map

Equations of motion:

$$\dot{M} = \varepsilon \left(kM \times \Omega_c \right) + M \times \Omega_a$$
$$\dot{\gamma} = \varepsilon(k\gamma \times \Omega_c) + \gamma \times \Omega_a$$

Poincaré map

$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

Equations of motion:

$$\dot{M} = \varepsilon \left(kM \times \Omega_c\right) + M \times \Omega_a$$
$$\dot{\gamma} = \varepsilon(k\gamma \times \Omega_c) + \gamma \times \Omega_a$$

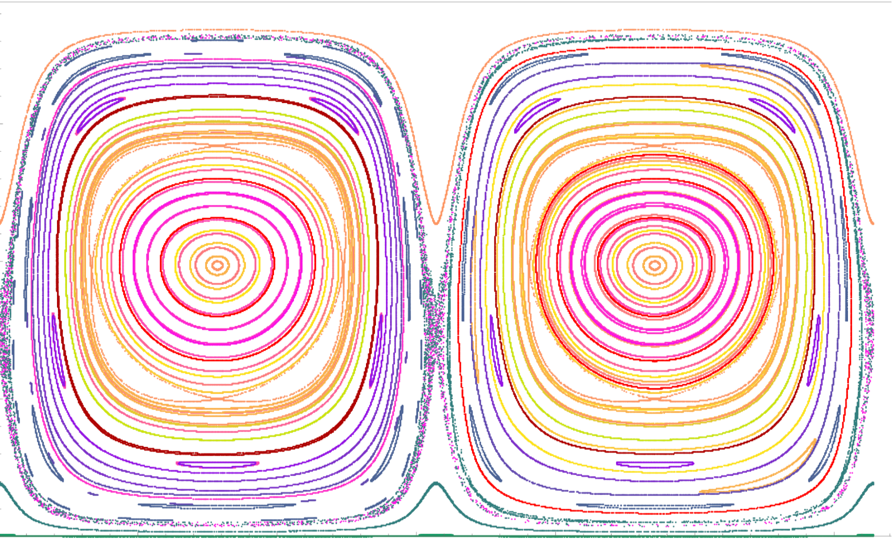
Poincaré map

Equations of motion:

$$\dot{M} = \varepsilon\left(kM \times \Omega_c\right) + M \times \Omega_a$$
$$\dot{\gamma} = \varepsilon(k\gamma \times \Omega_c) + \gamma \times \Omega_a$$

$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

$\varepsilon \gg 1$



Poincaré map

$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

Equations of motion:

$$\dot{M} = \varepsilon \left(kM \times \Omega_c\right) + M \times \Omega_a$$
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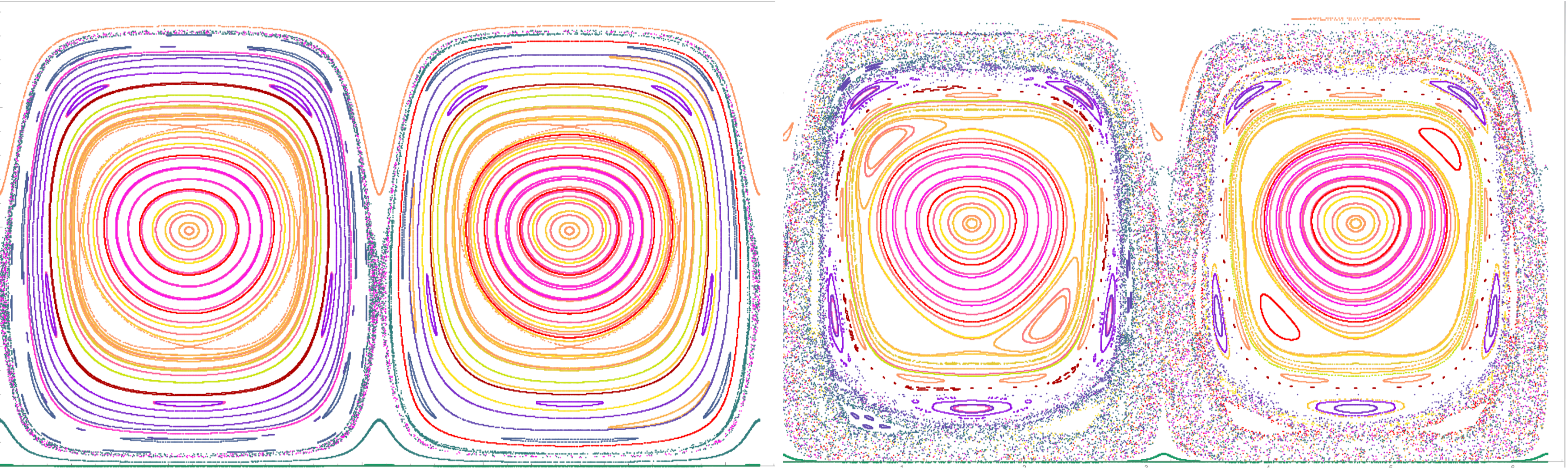

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$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

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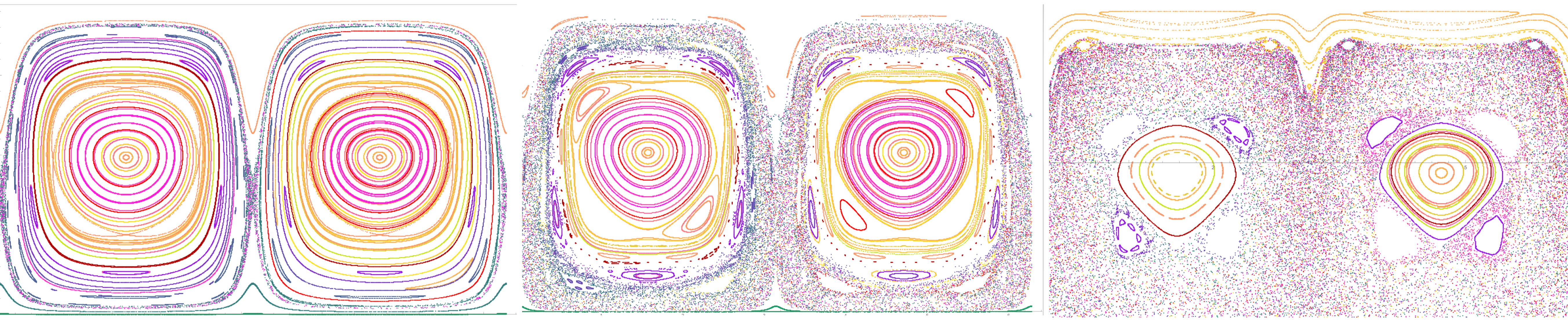

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$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

Equations of motion:

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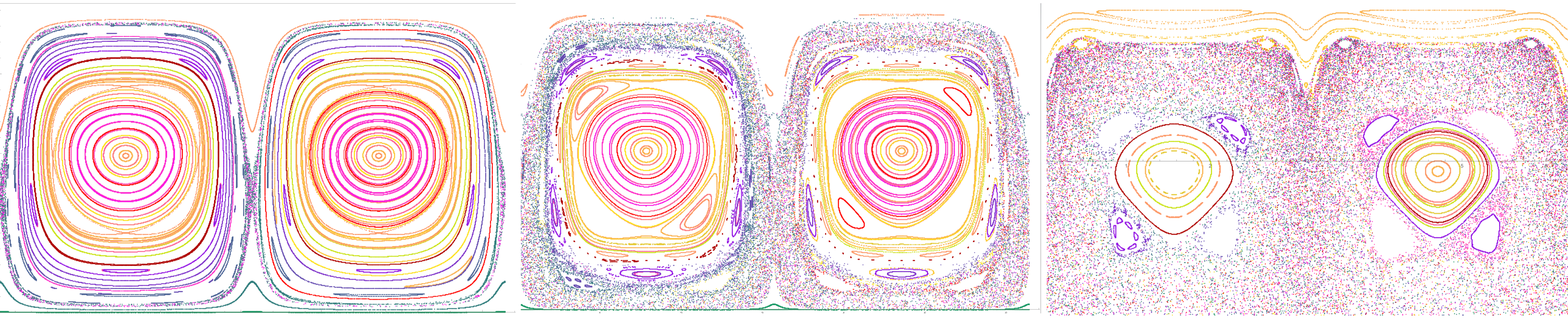

Poincaré map

$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

Equations of motion:

$$\dot{M} = \varepsilon\left(kM \times \Omega_c\right) + M \times \Omega_a$$
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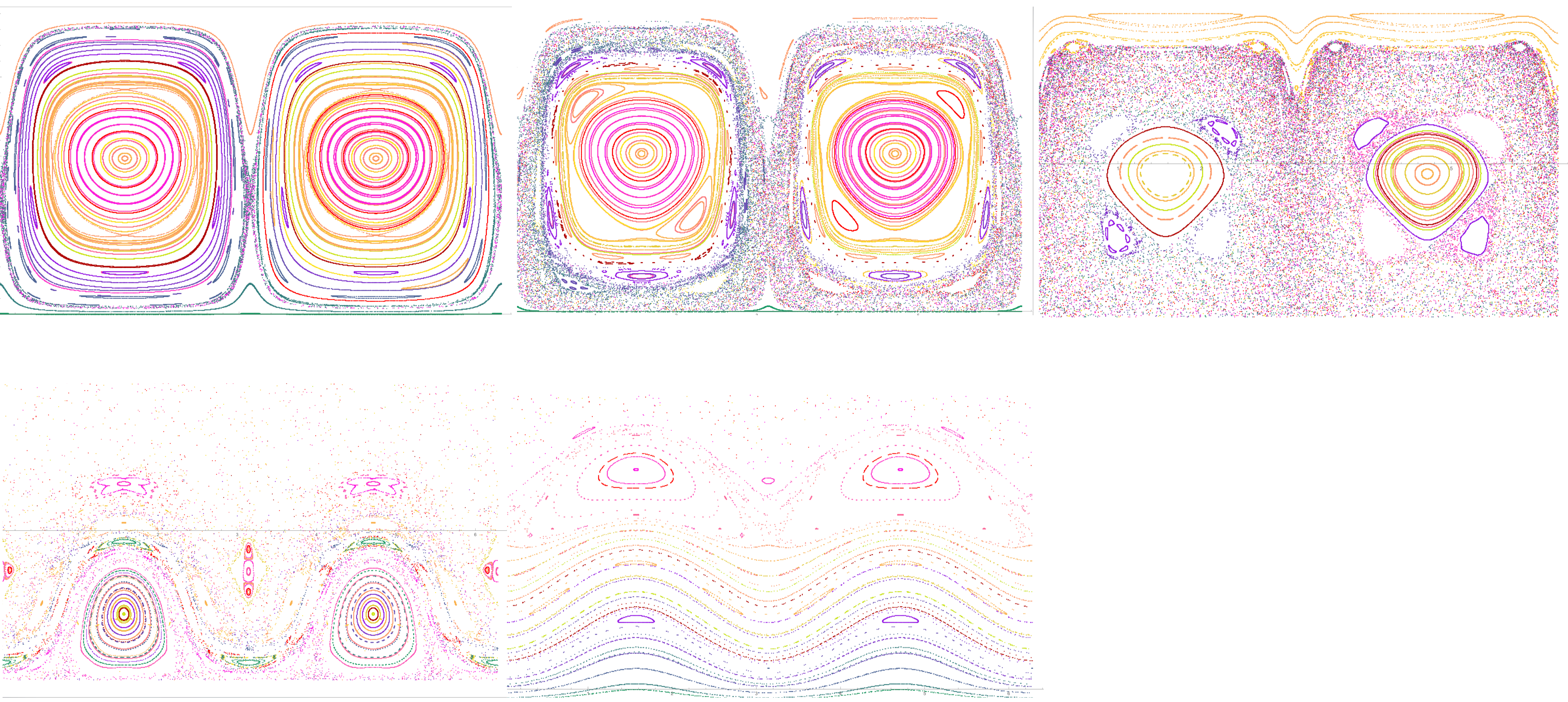
$\varepsilon \gg 1$



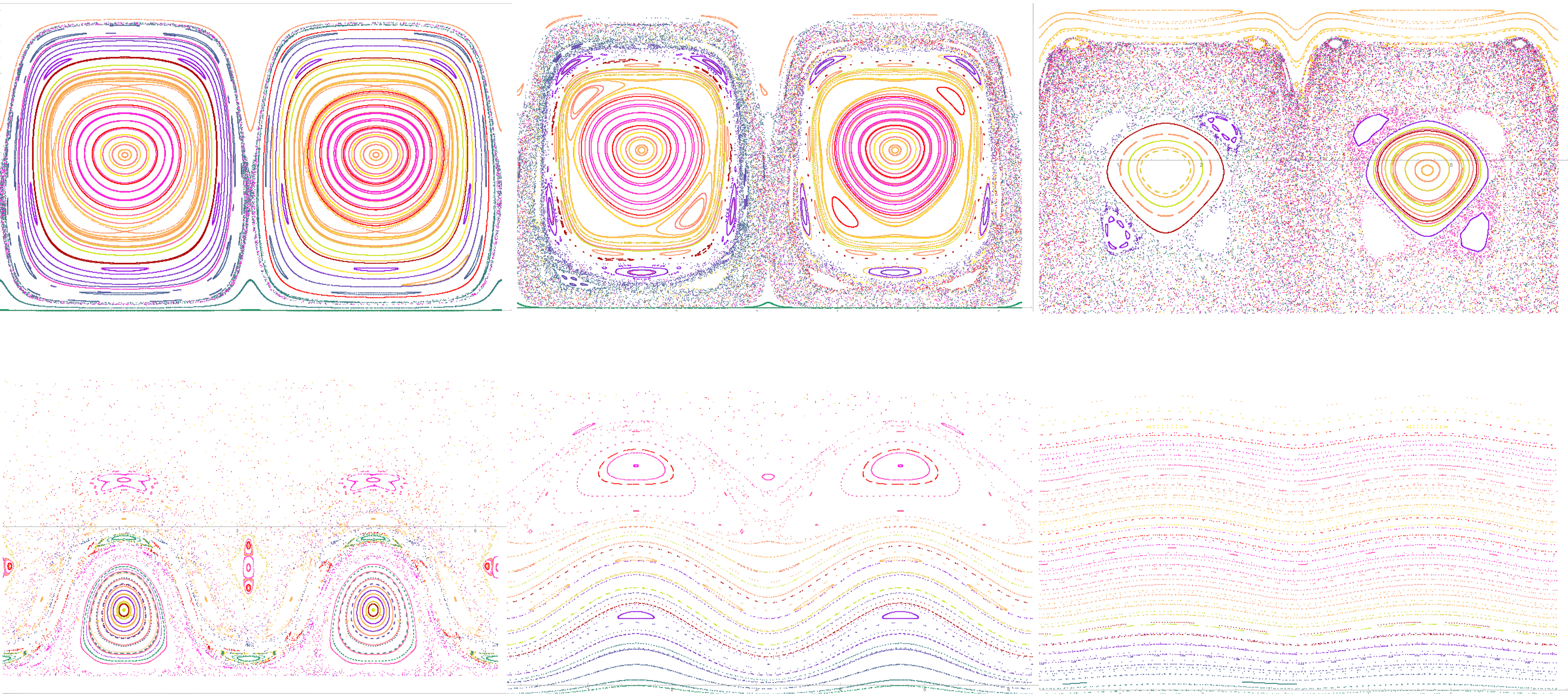
Poincaré map

Equations of motion:

$$\dot{M} = \varepsilon\left(kM \times \Omega_c\right) + M \times \Omega_a$$
$$\dot{\gamma} = \varepsilon(k\gamma \times \Omega_c) + \gamma \times \Omega_a$$

$$\varepsilon = \frac{\|M\|}{mr^2\eta}$$

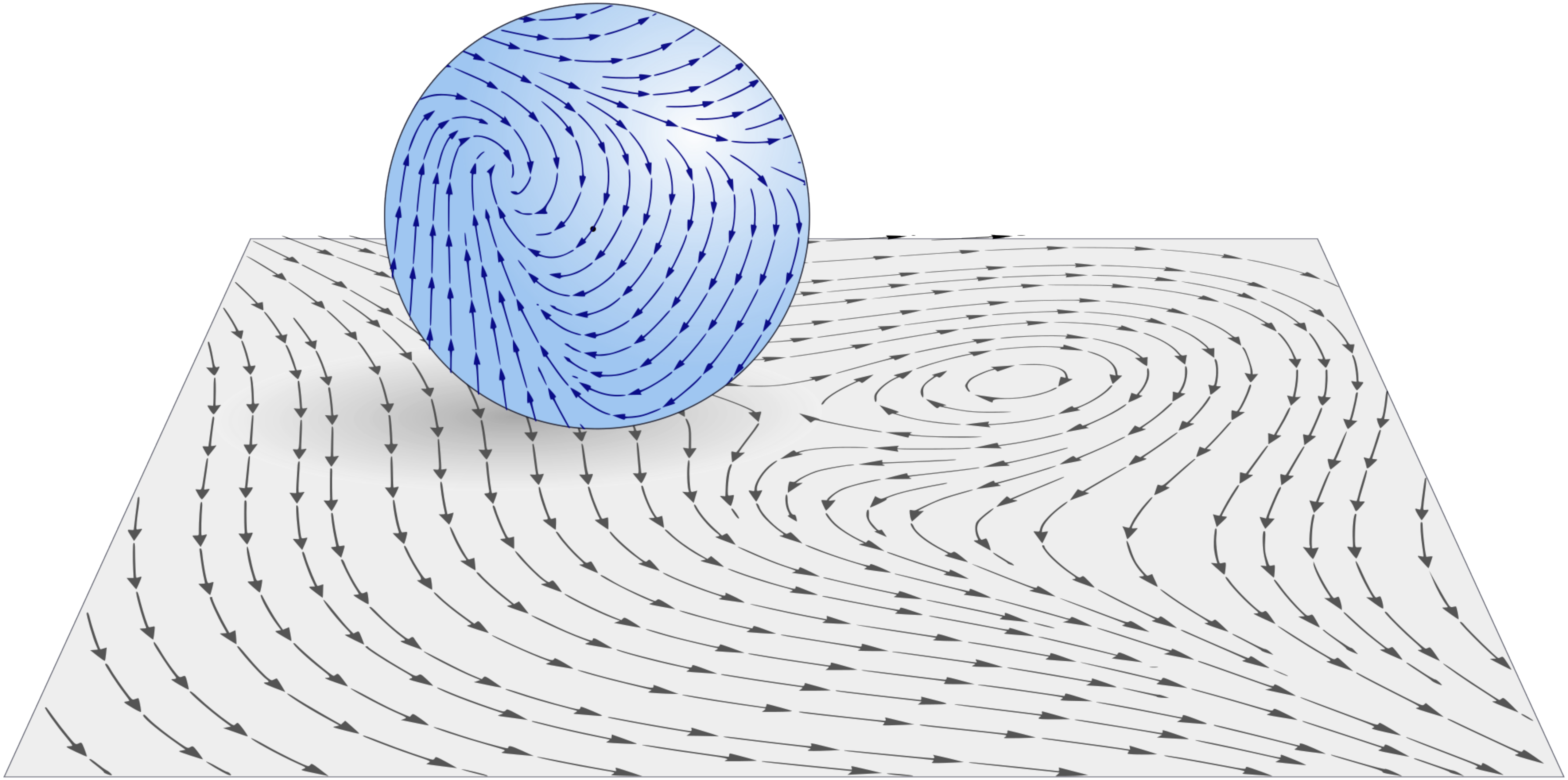
$\varepsilon \gg 1$



$\varepsilon \ll 1$

Homogeneous sphere

$I_1 = I_2 = I_3$



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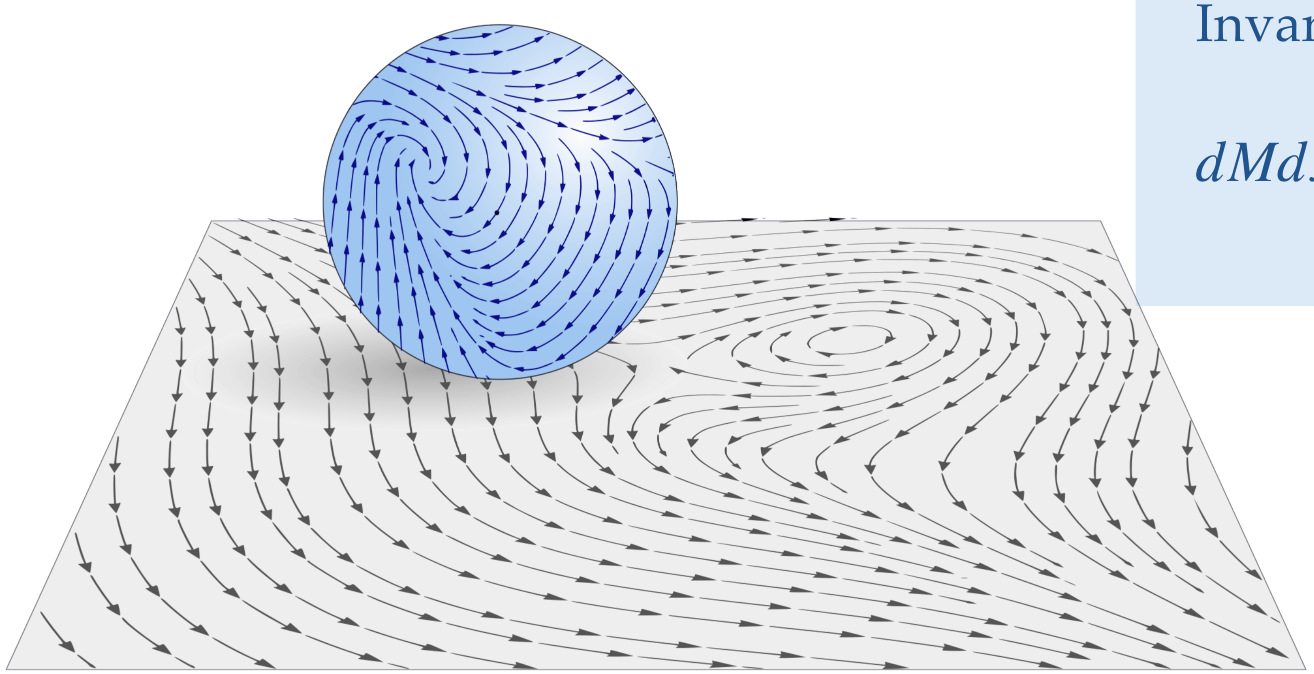


Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{u} = -rB(\gamma \times \Omega) + BV + W$$
$$\dot{B} = B\hat{\Omega}$$

$$M = I\Omega + mr^2\gamma \times (\Omega \times \gamma) + mr\gamma \times (V + B^{-1}W)$$

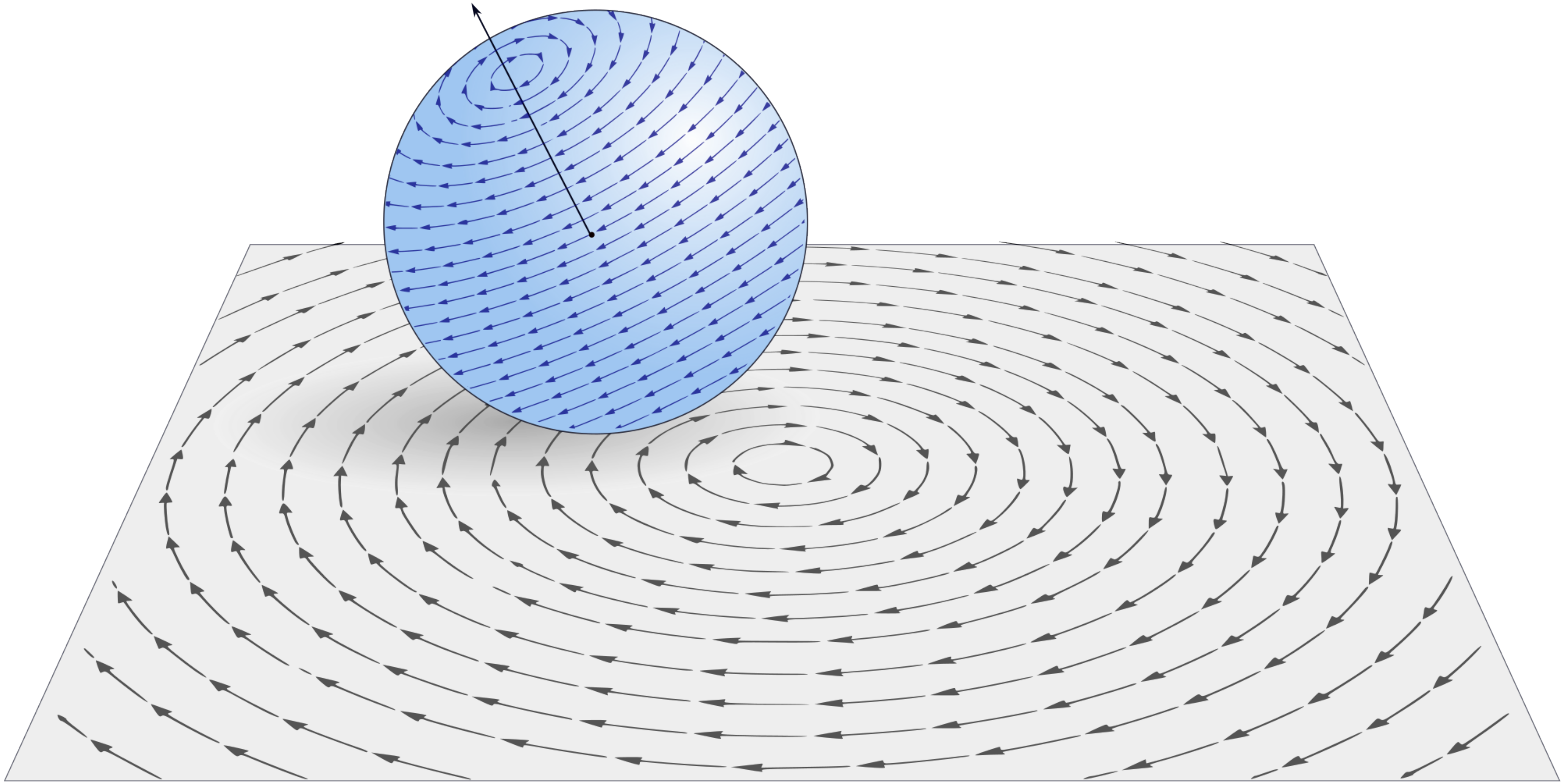
First integrals:

$$\|M\|^2, \quad \langle M, \gamma \rangle, \quad \|\gamma\|^2$$

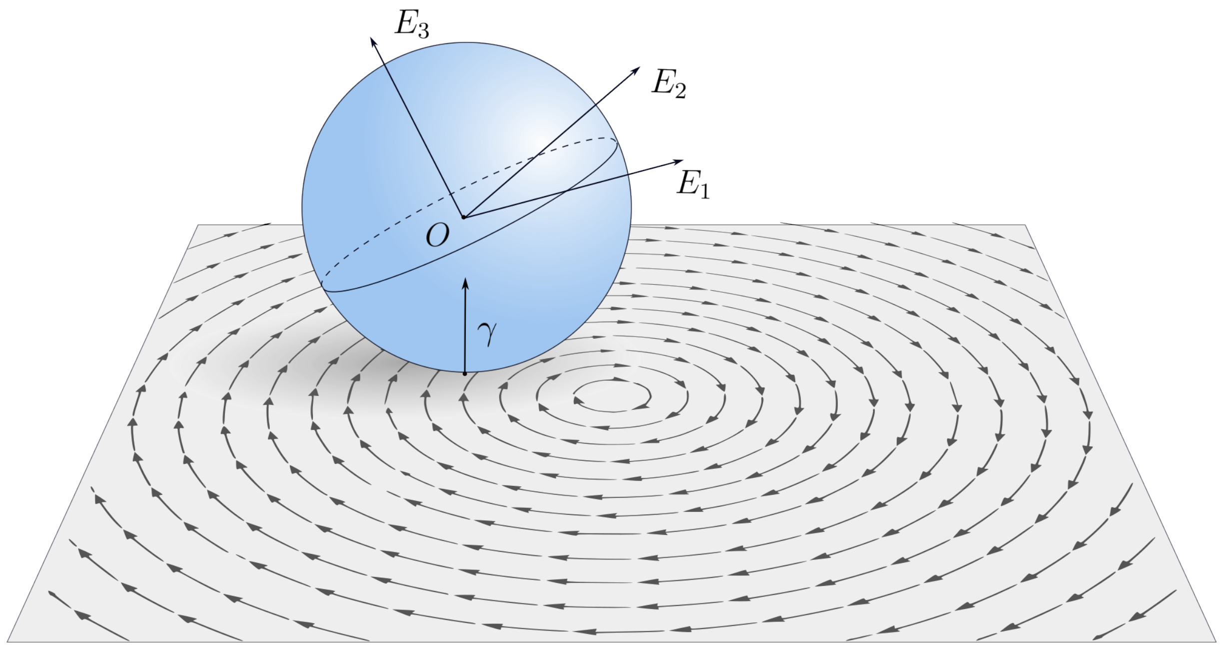
Theorem (C., García Naranjo, 2023)
Invariant measure:

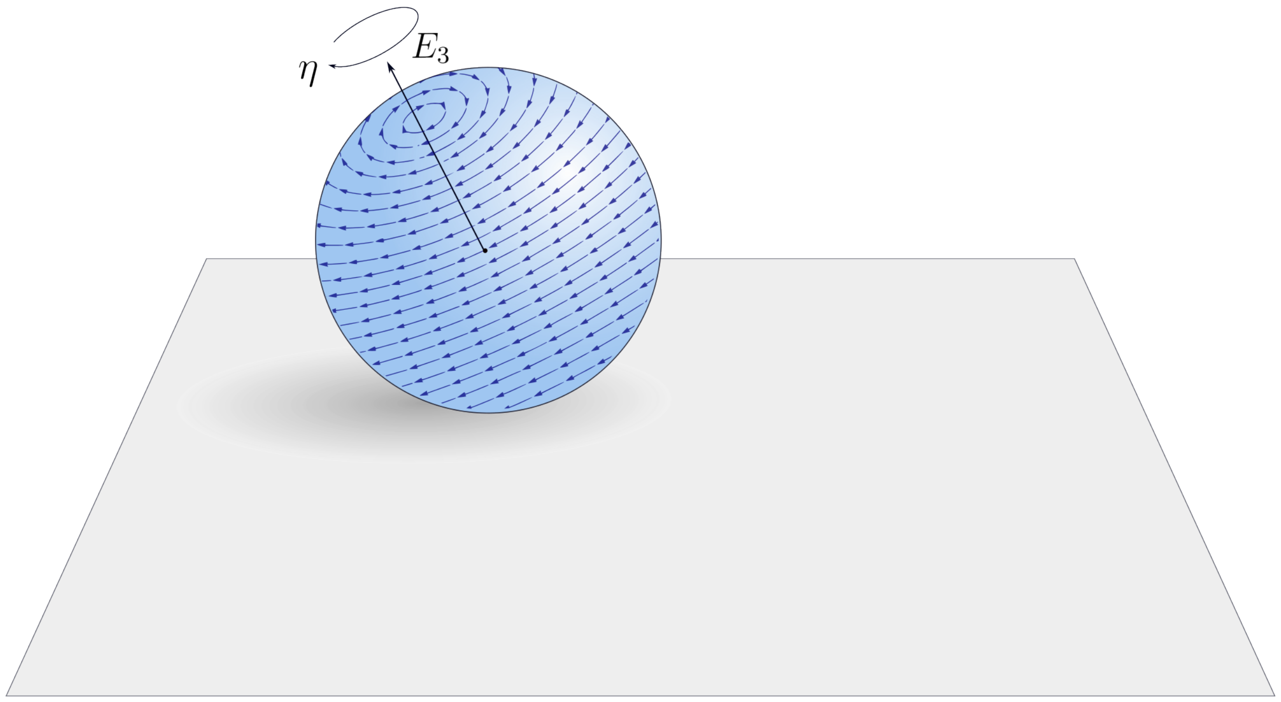
$dMdxdyd\alpha d\beta d\gamma$ if $\mathrm{div}_{\mathbb{R}^2} \mathrm{W} = 0$ and $\mathrm{div}_{S^2} \mathrm{V} = 0$

Homogeneous sphere



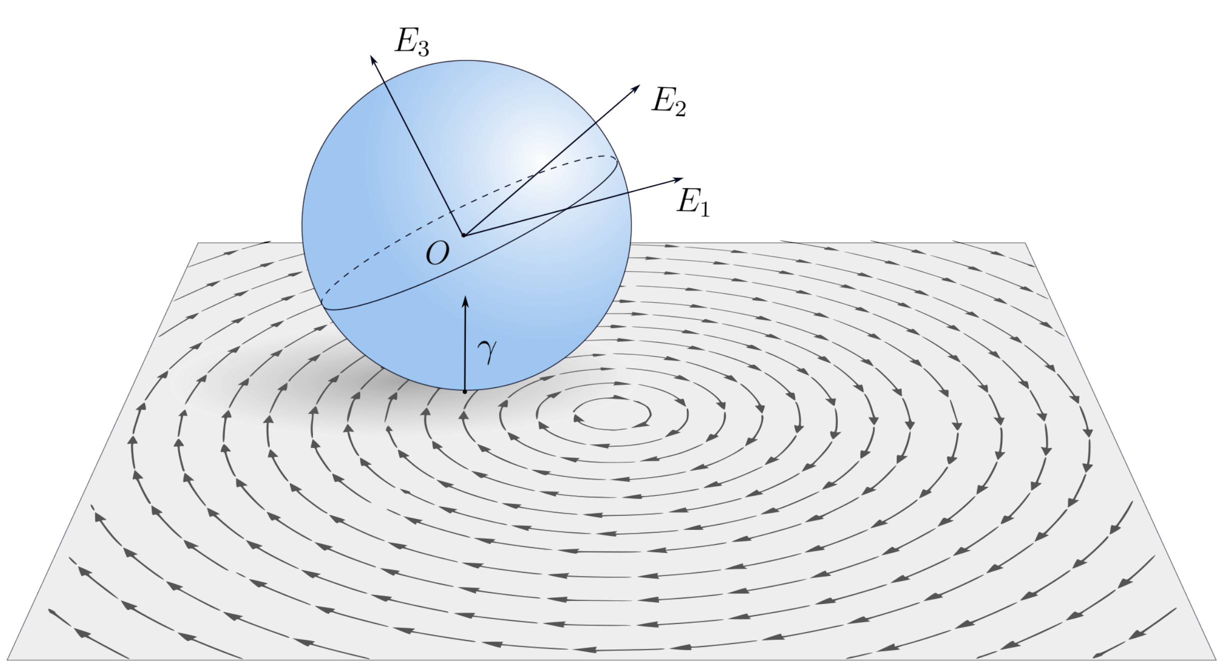
Homogeneous sphere

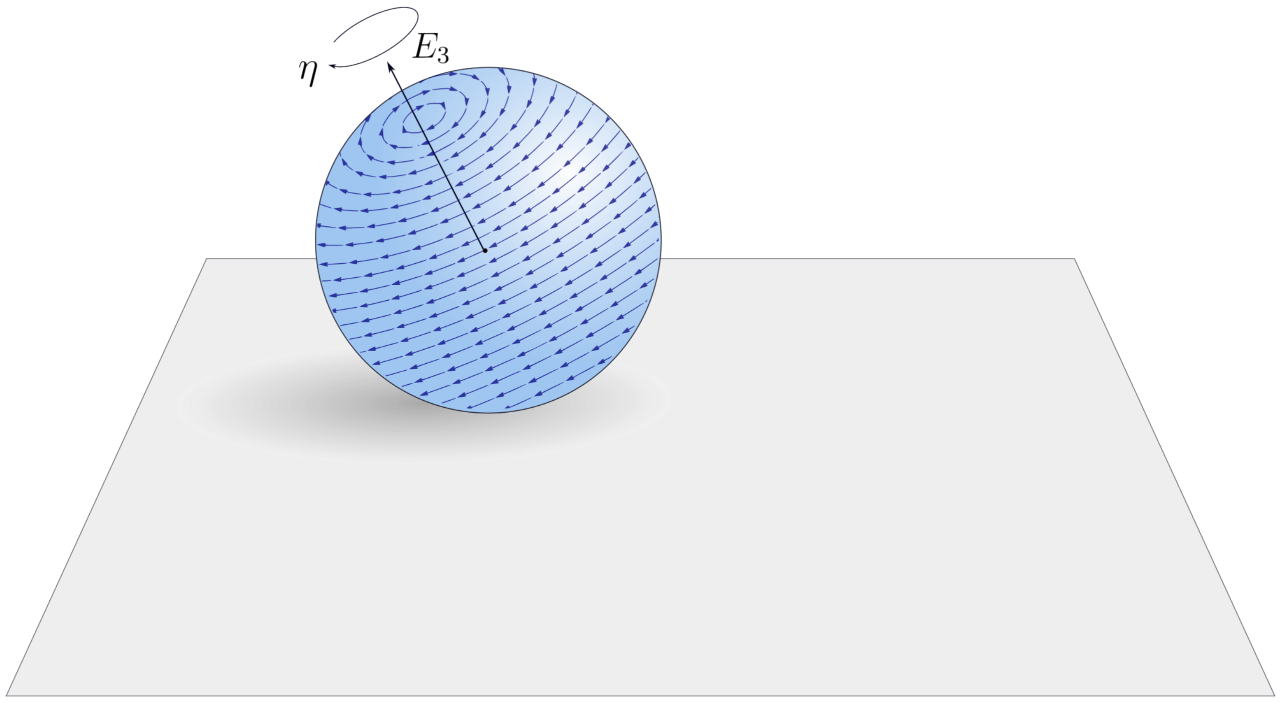




Homogeneous sphere

Integrable

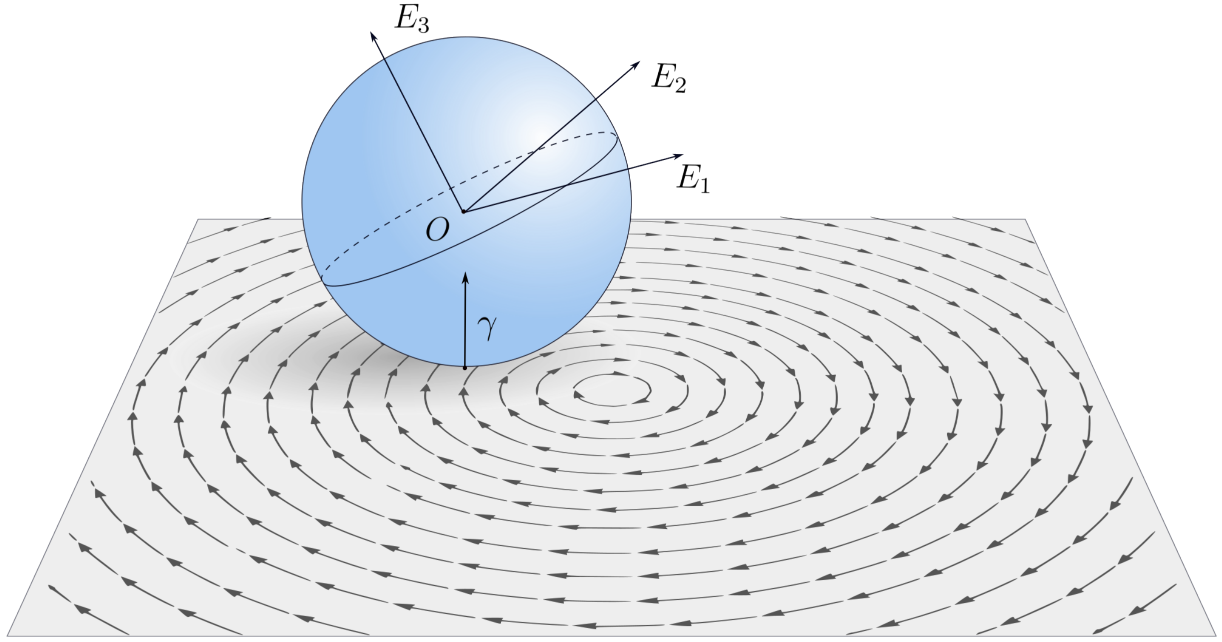


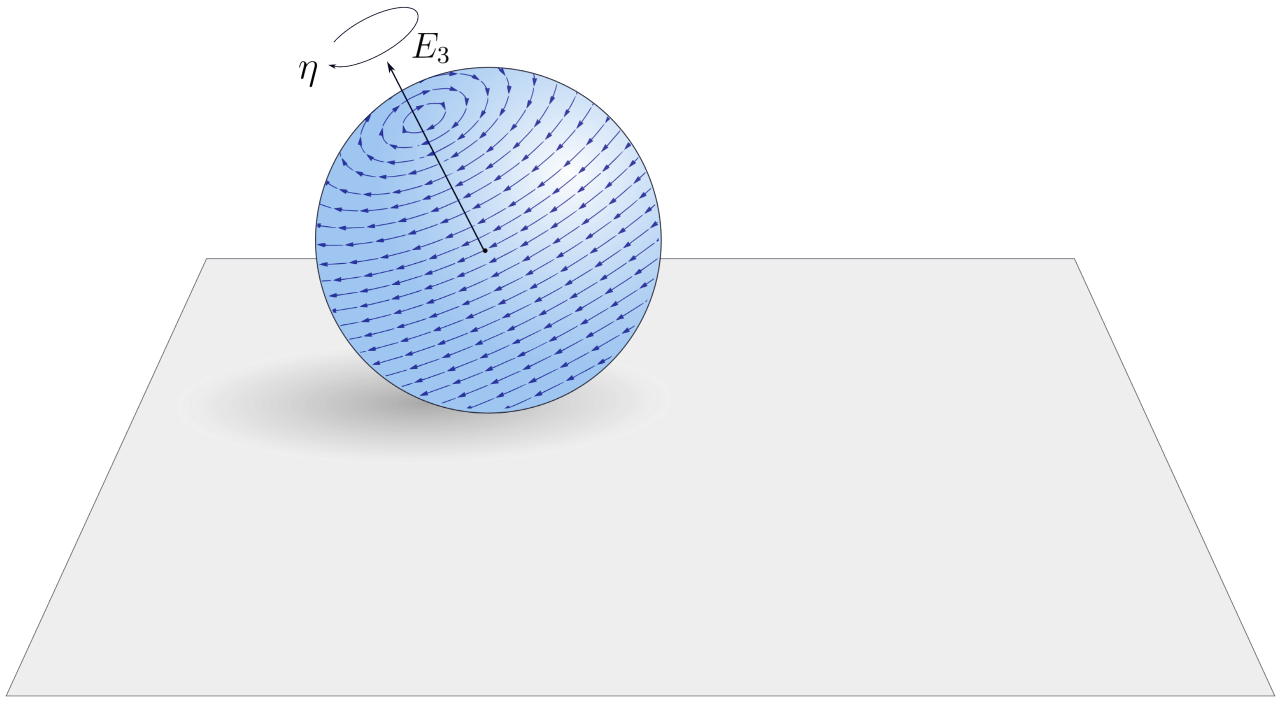


Homogeneous sphere

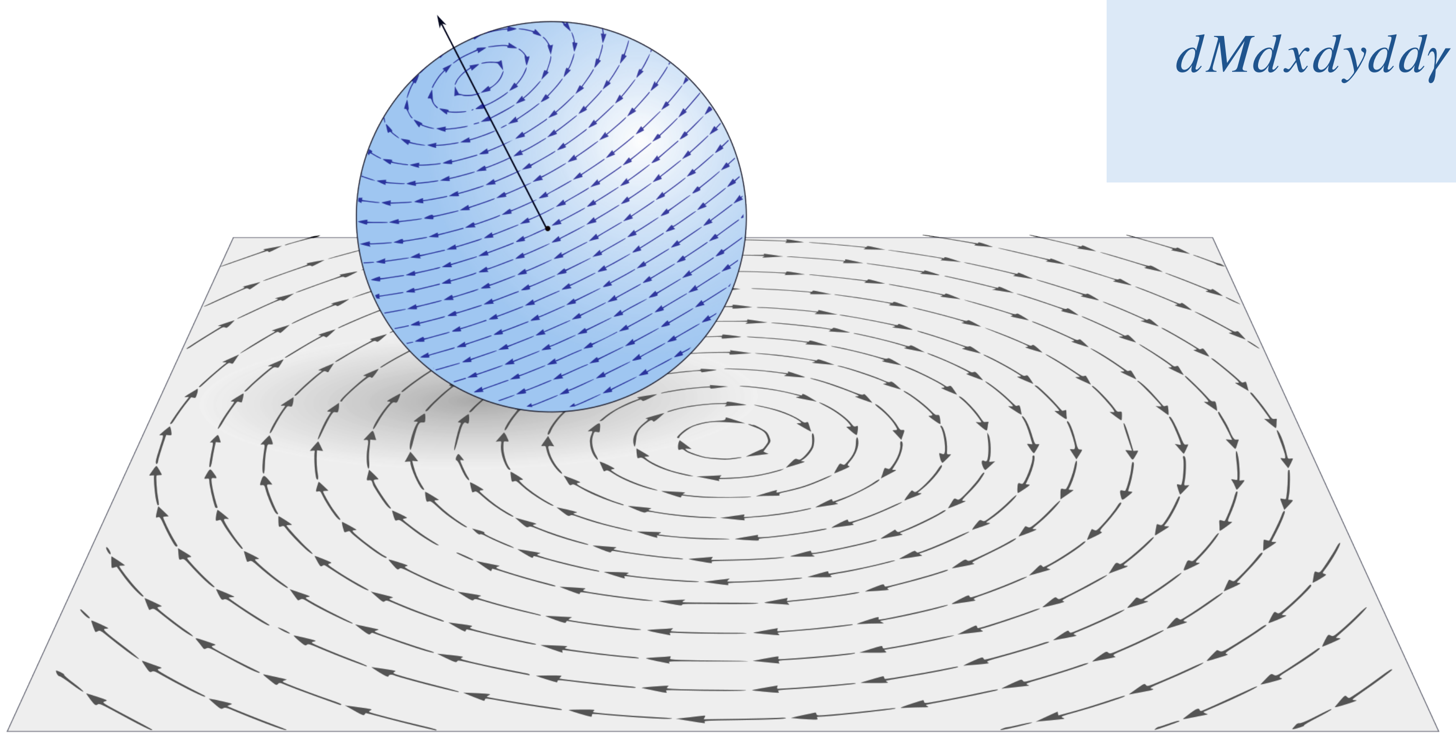
Integrable

Integrable





Homogeneous sphere



Equations of motion:

$$\dot{M} = M \times \Omega$$
$$\dot{U} = -r(\gamma \times \Omega) + U \times \Omega - r\eta_1(\gamma \times E_3) - \eta_2\gamma \times U$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$M = I\Omega + mr^2\gamma \times (\Omega \times \gamma) - mr^2\eta_1\gamma \times (\gamma \times E_3) + \eta_2\gamma \times (\gamma \times U)$$

First integrals

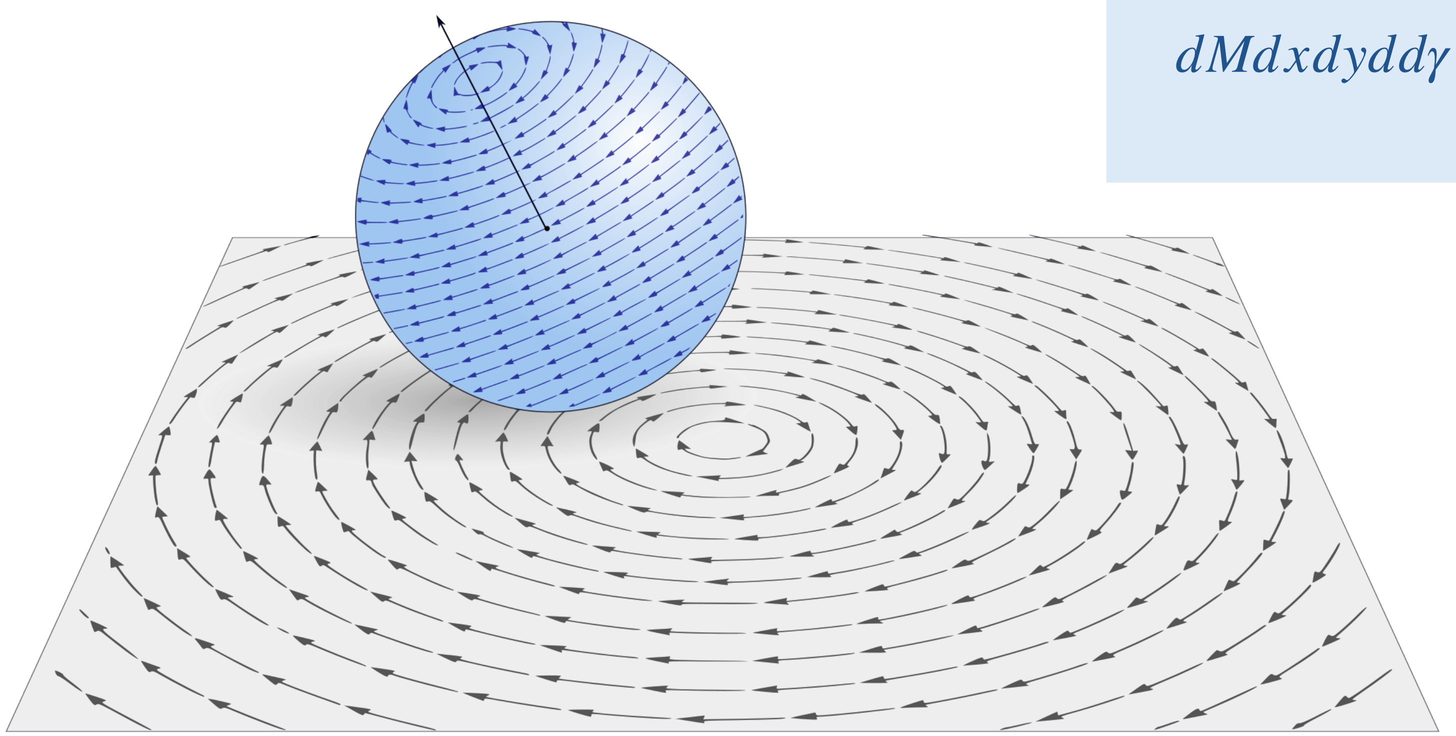
$$\|M\|^2, \quad \langle M, \gamma \rangle, \qquad \|\gamma\|^2, \quad \langle U, \gamma \rangle = 0, \qquad E_{mov}$$

Invariant measure

$$dMdxdyd\gamma$$

Homogeneous sphere

Current work



Equations of motion:

$$\dot{M} = M \times \Omega$$
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$$\dot{\gamma} = \gamma \times \Omega$$

$$M = I\Omega + mr^2\gamma \times (\Omega \times \gamma) - mr^2\eta_1\gamma \times (\gamma \times E_3) + \eta_2\gamma \times (\gamma \times U)$$

First integrals

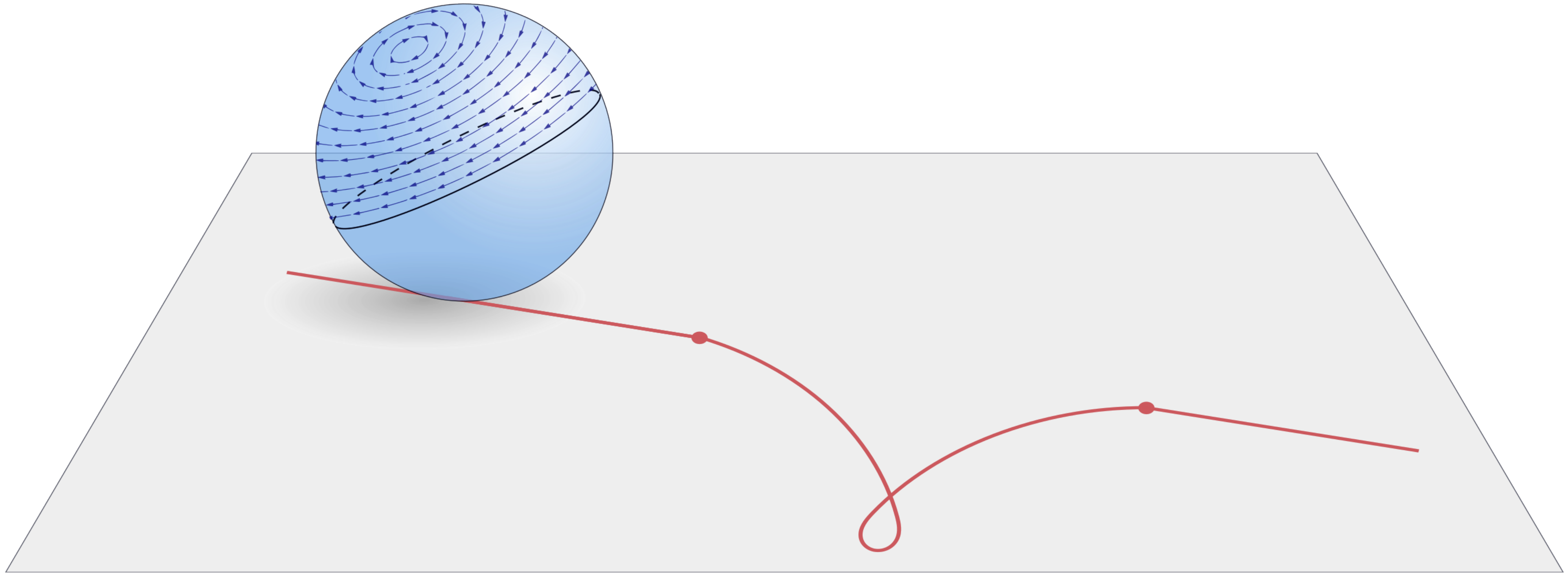
$$\|M\|^2, \quad \langle M, \gamma \rangle, \quad \|\gamma\|^2, \quad \langle U, \gamma \rangle = 0, \quad E_{mov}$$

Invariant measure

$$dMdxdyd d\gamma$$

Homogeneous sphere with rotating shell

We proved an analogous to the
Anais-billiard phenomenon



Conclusions

Conclusions

- Identified a rich class of examples to analyze dynamical properties of affine nonholonomic systems
- Recognized a robust mechanism leading to existence of first integrals (momenta)
- Found new instances of moving energy
- Observed integrable and chaotic behavior and transition as function of problem parameters

Future work

Future work

- Role of reversibility in affine nonholonomic sistems
- Momentum first integrals for affine nonholonomic systems
- Develop a theory to frame these examples:
  Integrability, perturbation theory and chaos of nonholonomic systems

Thank you!