Canonical Lifts in Multisymplectic De Donder-Weyl Hamiltonian Field Theories

Arnoldo Guerra IV
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Definition

Given a differentiable manifold $\mathscr{M}$, a differential form $\Omega \in \Omega^m(\mathscr{M})$ is 1-**nondegenerate** if, for every $\mathrm{p} \in \mathscr{M}$ and $Y \in \mathfrak{X}(\mathscr{M})$, it follows that $i(Y)\Omega|_{\mathrm{p}} = 0 \iff Y|_{\mathrm{p}} = 0$.

Definition

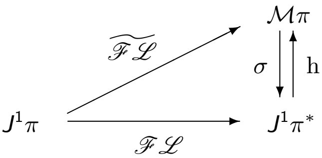
If $\Omega \in \Omega^m(\mathscr{M})$ is closed and 1-nondegenerate, then Ω is a **multisymplectic form** and $(\mathscr{M}, \Omega)$ is a **multisymplectic manifold**.

Definition

If $\Omega \in \Omega^m(\mathscr{M})$ is closed and 1-degenerate, then Ω is a **pre-multisymplectic form** and $(\mathscr{M}, \Omega)$ is a **pre-multisymplectic manifold**.

Setting for field theories

- **Spacetime:**
  M with coordinates x^μ and Lorentzian metric $g_{\mu\nu}$ where $\mu = \{0, 1, \ldots, m-1\}$, i.e. $\dim(M) = m$
- **Configuration bundle:**
  $\pi : E \to M : (x^\mu, y^A) \mapsto x^\mu$ where $A = \{1, 2, \ldots, n\}$, i.e. $\dim(E) = m + n$.
- **Local sections and fields:**
  $\phi : M \to E : x^\mu \mapsto (x^\mu, y^A(x))$
  The $y^A(x)$ are the *fields* of the field theory under investigation.
- (pre)multisymplectic phase spaces:



Definition

$\mathcal{M}\pi \equiv \Lambda_2^m T^* E$ is called **the extended multimomentum bundle**, the bundle of *m*-forms on E vanishing by the action of two π-vertical vector fields.

- $\mathcal{M}\pi$ has natural coordinates (x^μ, y^A, p_A^μ, p) and the p_A^μ are called the *multimomenta*
- $\dim(\mathcal{M}\pi) = m + n + mn + 1$

$\mathcal{M}\pi$ comes equipped with the following canonical *Cartan forms*:

$$\Theta = p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + p\,\mathrm{d}^m x \quad , \quad \Omega = -\mathrm{d}\Theta = -\mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \mathrm{d}p \wedge \mathrm{d}^m x \ .$$

$\Omega \in \Omega^{m+1}(\mathcal{M}\pi)$ is, in general, multisymplectic.

Note: $\mathrm{d}^{m-1}x_\mu \equiv i(\partial_\mu)\mathrm{d}^m x = \dfrac{1}{(m-1)!}\epsilon_{\mu\mu_2\cdots\mu_m}\mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_m}$

Definition

The quotient bundle $J^1\pi^* := \mathcal{M}\pi/\Lambda^m_1({}^*\mathrm{E})$, where $\Lambda^m_1({}^*\mathrm{E})$ denotes the bundle of π-semibasic m-forms on E is called the **multimomentum bundle**.

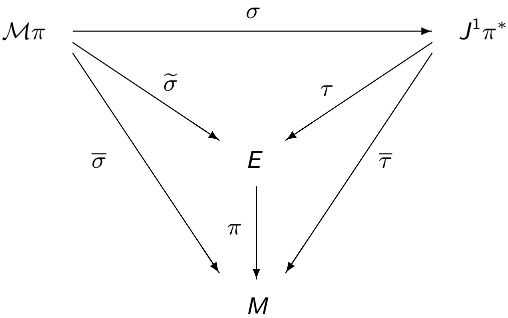
- $J^1\pi^*$ has natural coordinates (x^μ, y^A, p^μ_A)
- $\mathcal{M}\pi$ is a fiber bundle over $J^1\pi^*$ with projection map $\sigma : \mathcal{M}\pi \to J^1\pi^* : (x^\mu, y^A, p^\mu_A, p) \mapsto (x^\mu, y^A, p^\mu_A)$

Definition

Sections $\mathrm{h}\colon J^1\pi^* \to \mathcal{M}\pi : (x^\mu, y^A, p^\mu_A) \mapsto \left(x^\mu, y^A, p^\mu_A, p = -\mathscr{H}(x^\nu, y^B, p^\nu_B)\right)$ are called **Hamiltonian sections** and $\mathscr{H}(x^\nu, y^B, p^\nu_B) \in C^\infty(J^1\pi^*)$ is called the **De Donder–Weyl Hamiltonian**.

- The De Donder–Weyl Hamitonian formulation of classical field theories takes place on $J^1\pi^*$

Note that both $\mathcal{M}\pi$ and $J^1\pi^*$ are fiber bundles over E and M. So far:



The *Hamilton–Cartan forms* on $J^1\pi^*$ are obtained by pulling back the forms on $\mathcal{M}\pi$ by Hamiltonian sections:

$$\Theta_{\mathscr{H}} := \mathrm{h}^*\Theta \in \Omega^m(J^1\pi^*) \ , \qquad \Omega_{\mathscr{H}} := -\mathrm{d}\Theta_{\mathscr{H}} = \mathrm{h}^*\Omega \in \Omega^{m+1}(J^1\pi^*) \ .$$

The local expressions for the Hamilton–Cartan forms are:

$$\Theta_{\mathscr{H}} = p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \mathscr{H}\mathrm{d}^m x \quad , \quad \Omega_{\mathscr{H}} = -\mathrm{d}p_A^\mu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\mathscr{H} \wedge \mathrm{d}^m x \ .$$

If the field theory under investigation is **regular**, then $\Omega_{\mathscr{H}}$ **is multisymplectic**.

The *action* is a functional on the set of sections $\Gamma(\overline{\tau})$ given as $\Gamma(M, J^1\pi^*) \to \mathbb{R} : \psi \mapsto \int_M \psi^*\Theta_{\mathscr{H}}$. Need to find critical (stationary) sections

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_M \psi_s^*\Theta_{\mathscr{H}} = 0 \ ,$$

for variations $\psi_s = \eta_s \circ \psi$ of ψ, where η_s is the flow of a $\overline{\tau}$-vertical vector field on E compactly supported on M.

Critical sections are characterized in the following equivalent ways:

- ψ is an integral section of a class of integrable, locally decomposable, $\bar{\tau}$-transverse multivector fields $\{X_{\mathscr{H}}\} \subset \mathfrak{X}^m(J^1\pi^*)$ which satisfy

$$i(X_{\mathscr{H}})\Omega_{\mathscr{H}} = 0 \quad , \quad \text{for every } X_{\mathscr{H}} \in \{X_{\mathscr{H}}\} \ .$$

- The section ψ satisfies the **Hamilton–De Donder–Weyl equations**:

$$\frac{\partial(y^A \circ \psi)}{\partial x^\mu} = \frac{\partial \mathscr{H}}{\partial p_A^\mu} \circ \psi \quad , \quad \frac{\partial(p_A^\mu \circ \psi)}{\partial x^\mu} = -\frac{\partial \mathscr{H}}{\partial y^A} \circ \psi \ .$$

Multivector field solutions

- Locally decomposable:

$$X_{\mathscr{H}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + D_\mu^A \frac{\partial}{\partial y^A} + H_{\mu A}^\nu \frac{\partial}{\partial p_A^\nu} \right) \in \mathfrak{X}^m(J^1\pi^*)$$

- We choose a representative of the class $\{X_{\mathscr{L}}\}$ which is a normalized, $\overline{\tau}$-transverse, multivector field:

$$i(X_{\mathscr{H}})\mathrm{d}^m x = 1 \ .$$

- Integral sections satisfy:

$$D_\mu^A = \partial_\mu y^A(x) \quad , \quad H_{\mu A}^\nu = \partial_\mu p_A^\nu(x) \quad .$$

Singular Hamiltonian field theories come with the following:

- *primary constraint submanifold* $P_\circ \subset J^1\pi^*$ which is a τ-transverse embedded submanifold $\jmath_\circ : P_\circ \hookrightarrow J^1\pi^*$
- $P_\circ$ specified by some *primary constraint equations* $\varphi_I(y^A, p^\mu_A) = 0$
- restricted projection maps $\tau_\circ : P_\circ \to E$ and $\overline{\tau}_\circ : P_\circ \to M$
- *Restricted Hamiltonian sections* $\mathrm{h}_\circ = \mathrm{h} \circ \jmath_\circ \colon P_\circ \to \mathcal{M}\pi$

Then, the Hamilton–Cartan forms are obtained using the restricted Hamiltonian sections:

$$\Theta^\circ_{\mathscr{H}} = \mathrm{h}_\circ^*\Theta \in \Omega^m(P_\circ) \quad , \quad \Omega^\circ_{\mathscr{H}} = -\mathrm{d}\Theta^\circ_{\mathscr{H}} = \mathrm{h}_\circ^*\Omega \in \Omega^{m+1}(P_\circ) \ ,$$

and now $\Omega^\circ_{\mathscr{H}}$ is **pre-multisymplectic**.

De Donder–Weyl Hamiltonian constraint analysis

The variational problem is formulated similarly as before, except now we need a constraint analysis...

- Looking now for critical sections of decomposable $\overline{\tau}_\circ$-transverse m-multivector field on $P_\circ$ which satisfies the field equations: $i(X^\circ_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0$
- The field equations may give compatibility constraints $\varphi^c(y^A, p^\mu_A) = 0$ which gives the compatibility constraint submanifold $C \hookrightarrow P_\circ \hookrightarrow J^1\pi^*$
- Impose tangency (stability) of multivector field solution iterativley to all constraint submanifolds

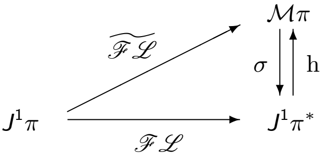
$$\varphi_1 \equiv L(X^\circ_{\mathscr{H}})\varphi^c \underset{C}{=} 0$$
$$\varphi_2 \equiv L(X^\circ_{\mathscr{H}})\varphi_1 \underset{C_1}{=} 0$$
$$\vdots$$
$$\varphi_f \equiv L(X^\circ_{\mathscr{H}})\varphi_{f-1} \underset{C_{f-1}}{=} 0$$

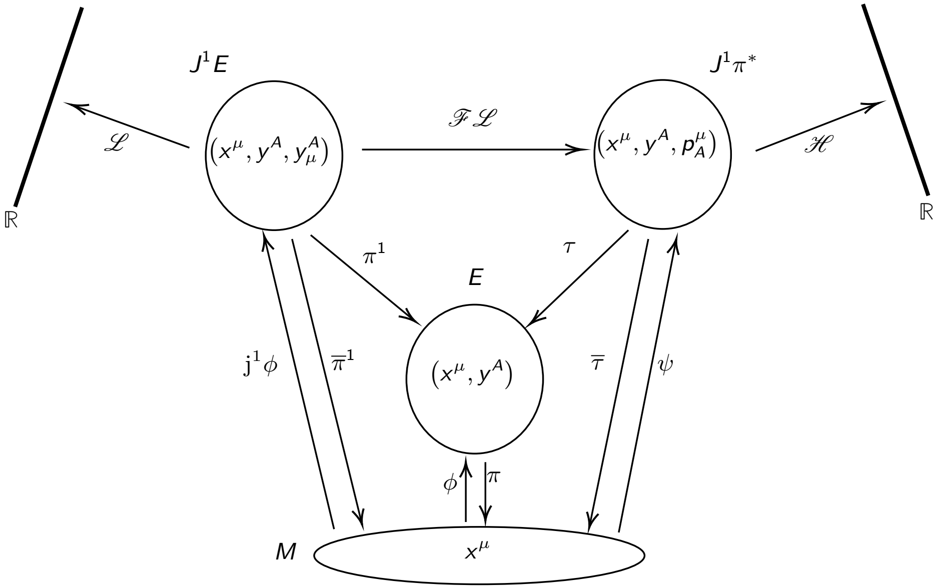
First jet bundle

Definition

Consider a section $\phi : M \to E : x^\mu \mapsto (x^\mu, y^A(x))$ of the bundle $\pi : E \to M : (x^\mu, y^A) \to x^\mu$. Given a point $x \in M$, two sections $\phi, \widetilde{\phi}$ of π are **(1*st*-order) equivalent** at x if $\phi(x) = \widetilde{\phi}(x)$ and $\partial_\mu \phi|_x = \partial_\mu \widetilde{\phi}|_x$ (i.e. $T_x\phi = T_x\widetilde{\phi}$). The corresponding equivalence classes are called the 1-**jets** of ϕ at x, denoted $j^1_x\phi$. Then, the **first-order jet bundle** $J^1\pi$ of π is defined as $J^1\pi = \{j^1_x\phi : x \in M, \phi \in \Gamma(\pi)\}$.

- The first-order-jet bundle $J^1\pi$ has natural coordinates (x^μ, y^A, y^A_μ) and hence $\dim(J^1\pi) = \dim(J^1\pi^*) = m + n + mn$
- $y^A \circ j^1\phi = y^A(x)$ (fields), $y^A_\mu \circ j^1\phi = \partial_\mu y^A(x)$
- $\mathscr{L}(x^\mu, y^A, y^A_\mu) \in C^\infty(J^1\pi)$, $E_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial y^A_\mu} y^A_\mu - \mathscr{L} \in C^\infty(J^1\pi)$
- Legendre transform $\mathscr{F}\mathscr{L} : J^1\pi \to J^1\pi^*$
  $\mathscr{F}\mathscr{L}^* p^\mu_A = \frac{\partial \mathscr{L}}{\partial y^A_\mu} \quad , \quad \mathscr{F}\mathscr{L}^* \mathscr{H} = E_{\mathscr{L}}$
- Extended Legendre transform $\widetilde{\mathscr{F}\mathscr{L}} : J^1\pi \to \mathcal{M}\pi^*$
  $\widetilde{\mathscr{F}\mathscr{L}}^* p^\mu_A = \frac{\partial \mathscr{L}}{\partial y^A_\mu} \quad , \quad \widetilde{\mathscr{F}\mathscr{L}}^* p = -E_{\mathscr{L}}$





$J^1 E$
$J^1 \pi^*$
$\mathscr{L}$
$\mathscr{F}\mathscr{L}$
$\mathscr{H}$
(x^μ, y^A, y^A_μ)
(x^μ, y^A, p^μ_A)
$\mathbb{R}$
$\mathbb{R}$
π^1
τ
E
(x^μ, y^A)
$\mathrm{j}^1\phi$
$\overline{\pi}^1$
$\overline{\tau}$
ψ
ϕ
π
M
x^μ

The Cartan forms are defined by pulling back from $\mathcal{M}\pi$:

$$\Theta_{\mathscr{L}} \equiv \widetilde{\mathscr{F}\mathscr{L}}^{*}\Theta = \frac{\partial \mathscr{L}}{\partial y^A_\mu} \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - E_{\mathscr{L}} \mathrm{d}^m x \quad , \quad \Omega_{\mathscr{L}} \equiv \widetilde{\mathscr{F}\mathscr{L}}^{*}\Omega = -\mathrm{d}\Theta_{\mathscr{L}}$$

- $\Omega_{\mathscr{L}}$ is degenerate and hence pre-multisymplectic
- The multi-Hessian is degenerate:

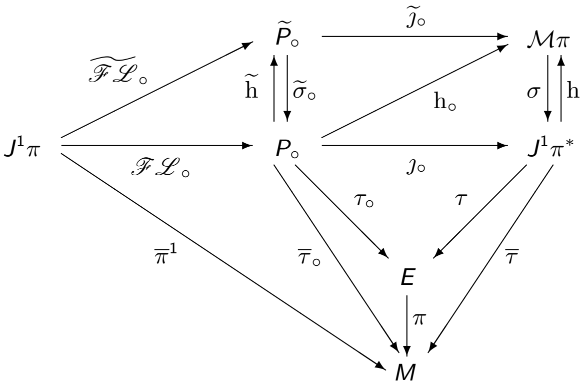
$$H^{\mu\nu}_{AB} = \frac{\partial^2 \mathscr{L}}{\partial y^A_\mu \partial y^B_\nu} \quad , \quad \text{nullvectors:} \quad H^{\mu\nu}_{AB} V^A_\mu = 0$$

- Given primary constraints ($\varphi_i = 0$), the null vectors are given as $(V_i)^A_\mu = \mathscr{F}\mathscr{L}^* \left(\frac{\partial \varphi_i}{\partial p^\mu_A}\right)$
- $\mathrm{Im}(\mathscr{F}\mathscr{L}) = P_\circ \subset J^1\pi^*$
- $\jmath_\circ : P_\circ \hookrightarrow J^1\pi^* \quad , \quad \mathscr{F}\mathscr{L}_\circ : J^1\pi \to P_\circ \quad , \quad \mathscr{F}\mathscr{L} = \jmath_\circ \circ \mathscr{F}\mathscr{L}_\circ$
- A function $f \in C^\infty(J^1\pi)$ is $\mathscr{F}\mathscr{L}$-projectable if

$$L(\Gamma_i)f = 0$$

where,

$$\Gamma_i = (V_i)^A_\mu \frac{\partial}{\partial y^A_\mu}$$


$\widetilde{P}_\circ$
$\widetilde{\jmath}_\circ$
$\mathcal{M}\pi$
$\widetilde{\mathscr{FL}}_\circ$
$\widetilde{\mathrm{h}}$
$\widetilde{\sigma}_\circ$
$\mathrm{h}_\circ$
σ
h
$J^1\pi$
$P_\circ$
$J^1\pi^*$
$\mathscr{FL}_\circ$
$\jmath_\circ$
$\tau_\circ$
τ
$\overline{\pi}^1$
$\overline{\tau}_\circ$
$\overline{\tau}$
E
π
M

The variational problem is formulated similarly as before, except now we need a constraint analysis...

- Looking now for critical sections of decomposable $\overline{\pi}^1$-transverse m-multivector field on $J^1\pi$ which satisfies the field equations: $i(X^\circ_{\mathscr{L}})\Omega^\circ_{\mathscr{L}} = 0$
- The field equations may give compatibility constraints $\varphi^c(y^A, y^A_\mu) = 0$ which gives the compatibility constraint submanifold $C \hookrightarrow J^1\pi$
- Impose SOPDE condition on $X^\circ_{\mathscr{L}}$; this may produce SOPDE constraints
- Impose tangency (stability) of multivector field solution iterativley to all constraint submanifolds

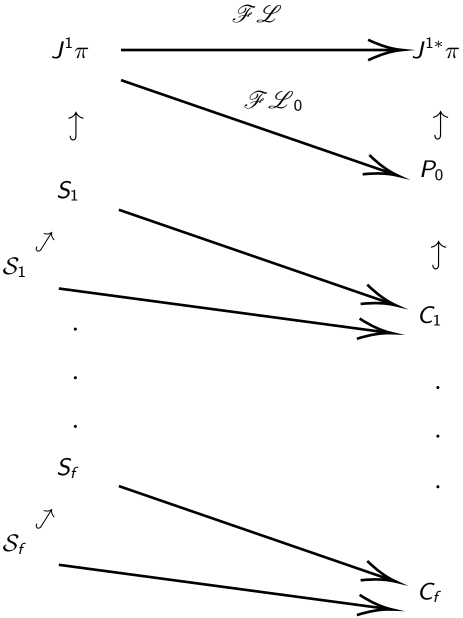
$$\varphi_1 \equiv L(X^\circ_{\mathscr{L}})\varphi^c \underset{S_1}{=} 0$$

$$\varphi_2 \equiv L(X^\circ_{\mathscr{L}})\varphi_1 \underset{\mathcal{S}_1}{=} 0$$

$$\vdots$$

$$\varphi_f \equiv L(X^\circ_{\mathscr{L}})\varphi_{f-1} \underset{\mathcal{S}_{f-1}}{=} 0$$

Geometric constraints



Lifted group actions on E

Consider spacetime diffeomorphisms

$$\Phi_M : M \to M$$

generated (infinitesimally) by

$$\xi_M = -\xi(x)^\mu \frac{\partial}{\partial x^\mu}$$

so that

$$x^\mu \to x^\mu + \xi^\mu(x) \ .$$

The lift of this group action to configuration manifold is a diffeomorphism:

$$\Phi_E : E \to E$$

satisfies

$$\Phi_M \circ \pi = \pi \circ \Phi_E$$

as shown below:

$$\begin{array}{ccc} E & \xrightarrow{\quad \Phi_E \quad} & E \\ \downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\downarrow \\ M & \xrightarrow{\quad \Phi_M \quad} & M \end{array}$$

Generating vector field and field variations

Φ_E is generated (infinitesimally) by π-projectable vector field:

$$\xi_E = -\xi(x)^\mu \frac{\partial}{\partial x^\mu} - \xi^A(x,y)\frac{\partial}{\partial y^A} \ .$$

Field variations:

$$\delta y^A(x) = L(\xi_M) y^A(x) = -\xi^\mu \partial_\mu y^A + \xi^A$$

For guage transformations $\xi^\mu = 0$ so,

$$\delta y^A(x) = \xi^A$$

and are generated on E by

$$\xi_E = -\xi^A(x,y)\frac{\partial}{\partial y^A} \ .$$

We think of these as a special case...

Lifting to Lagrangian phase space $J^1\pi$

Given a point $\gamma = (x^\mu, y^A, y^A_\mu) \in J^1\pi$:

Definition

$$\Phi_{J^1\pi}(\gamma) \equiv T\Phi_E \circ \gamma \circ T\Phi_M^{-1}$$

Then, the flow of $\Phi_{J^1\pi}(\gamma)$ is generated (infinitesimally) by

$$X_\xi = -\xi(x)^\mu \frac{\partial}{\partial x^\mu} - \xi^A(x,y)\frac{\partial}{\partial y^A} - \left(\partial_\mu \xi^A - y^A_\nu \partial_\mu \xi^\nu + y^B_\mu \frac{\partial \xi^A}{\partial y^B}\right)\frac{\partial}{\partial y^A_\mu}$$

(well-known).

$$\begin{array}{ccc}
J^1\pi & \xrightarrow{\Phi_{J^1\pi}} & J^1\pi \\
\downarrow{\scriptstyle \pi^1} & & {\scriptstyle \pi^1}\downarrow \\
E & \xrightarrow{\Phi_E} & E \\
\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\downarrow \\
M & \xrightarrow{\Phi_M} & M
\end{array}$$

Lifting to $\mathcal{M}\pi$

Recall $\mathcal{M}\pi \equiv \Lambda^m_2 T^* E$. Given a point $y = (x^\mu, y^A)$ and a point $p = (y, \zeta) \in \mathcal{M}\pi$ so that $\zeta \in \Lambda^m_2 T^*_y E$

Definition

$$\Phi_{\mathcal{M}\pi} \equiv \left(\Phi_E(y), (\Phi_E^{-1})^*_y \zeta \right)$$

This is generated (infinitesimally) by

$$Z_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^\mu}{\partial x^\nu} p^\nu_A - \frac{\partial \xi^\nu}{\partial x^\nu} p^\mu_A - \frac{\partial \xi^B}{\partial y^A} p^\mu_B \right) \frac{\partial}{\partial p^\mu_A} + \left(\frac{\partial \xi^A}{\partial x^\mu} p^\mu_A + \frac{\partial \xi^\mu}{\partial x^\mu} p \right) \frac{\partial}{\partial p}$$

$$\begin{array}{ccc}
\mathcal{M}\pi & \xrightarrow{\Phi_{\mathcal{M}\pi}} & \mathcal{M}\pi \\
\downarrow & & \downarrow \\
E & \xrightarrow{\Phi_E} & E \\
\downarrow \pi & & \pi \downarrow \\
M & \xrightarrow{\Phi_M} & M
\end{array}$$

Lifting to $J^1\pi^*$

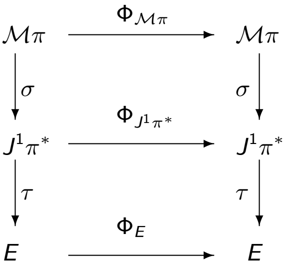
Definition

The canonical lift $\Phi_{J^1\pi^*}$ to $J^1\pi^*$ is the diffeomorphism $\Phi_{J^1\pi^*} : J^1\pi^* \to J^1\pi^*$ induced by $\Phi_{\mathcal{M}\pi} : \mathcal{M}\pi \to \mathcal{M}\pi$; that is

$$\Phi_{J^1\pi^*} \circ \sigma = \sigma \circ \Phi_{\mathcal{M}\pi} \ .$$

This is generated (infinitesimally) by

$$Y_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^\mu}{\partial x^\nu} p_A^\nu - \frac{\partial \xi^\nu}{\partial x^\nu} p_A^\mu - \frac{\partial \xi^B}{\partial y^A} p_B^\mu \right) \frac{\partial}{\partial p_A^\mu}$$



Definition

The canonical lift $\Phi_{P_\circ}$ to $P_\circ$ is the diffeomorphism $\Phi_{P_\circ} : P_\circ \to P_\circ$ induced by $\Phi_{\mathcal{M}\pi} : \mathcal{M}\pi \to \mathcal{M}\pi$; that is

$$\Phi_{P_\circ} \circ \sigma_\circ = \sigma_\circ \circ \Phi_{\mathcal{M}\pi} \ .$$

This is generated infinitesimally by

$$Y_\xi^\circ \equiv Y_\xi|_{P_\circ}$$

The analogous commuting diagraim is:

$$\begin{array}{ccc}
\mathcal{M}\pi & \xrightarrow{\ \Phi_{\mathcal{M}\pi}\ } & \mathcal{M}\pi \\
\Big\downarrow \sigma_\circ & & \sigma_\circ \Big\downarrow \\
P_\circ & \xrightarrow{\ \Phi_{P_\circ}\ } & P_\circ \\
\Big\downarrow \tau & & \tau \Big\downarrow \\
E & \xrightarrow{\ \Phi_E\ } & E
\end{array}$$

Lagrangian lifts and Hamiltonian lifts are $\mathscr{F}\mathscr{L}$-related

We were forced to start on $J^1\pi$ and then project to $J^1\pi^*$ by calculating

$$\mathscr{F}\mathscr{L}_* X_\xi$$

via

$$\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ \dfrac{\partial^2 \mathscr{L}}{\partial x^\mu \partial y^B_\nu} & \dfrac{\partial^2 \mathscr{L}}{\partial y^A \partial y^B_\nu} & \dfrac{\partial^2 \mathscr{L}}{\partial y^A_\mu \partial y^B_\nu} \end{pmatrix}$$

Now we can go directly to $\mathcal{M}\pi$ and $J^1\pi^*$ (or $P_\circ \subset J^1\pi^*$). In fact, we have proven

$$\mathscr{F}\mathscr{L}_* X_\xi = Y_\xi$$

when $L(X_\xi)\mathbf{L} = 0$, as one would expect.

Noether symmetries

$$
\begin{aligned}
\Phi^*_{\mathcal{M}\pi}\Omega = \Omega \quad &\Leftrightarrow \quad L(Z_\xi)\Omega = 0 \\
\Phi^*_{J^1\pi^*}\Omega_{\mathscr{H}} = \Omega_{\mathscr{H}} \quad &\Leftrightarrow \quad L(Y_\xi)\Omega_{\mathscr{H}} = 0 \\
\Phi^*_{J^1\pi}\Omega_{\mathscr{L}} = \Omega_{\mathscr{L}} \quad &\Leftrightarrow \quad L(X_\xi)\Omega_{\mathscr{L}} = 0 \\
\Phi^*_{P_\circ}\Omega^\circ_{\mathscr{H}} = \Omega^\circ_{\mathscr{H}} \quad &\Leftrightarrow \quad L(Y^\circ_\xi)\Omega^\circ_{\mathscr{H}} = 0
\end{aligned}
$$

Thank you!

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