Coisotropic reduction in different phase spaces

XVIII International Young Researchers Workshop in Geometry, Dynamics and Field Theory

Rubén Izquierdo, Manuel De León

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Outline

Coisotropic reduction in non-dissipative mechanics

The canonical phase spaces

- If Q is the configuration space of a mechanical system, the phase space $M := T^*Q$ inherits a canonical symplectic structure (M, ω),

$$\omega := \omega_Q = -d\lambda_Q = dq^i \wedge dp_i.$$

- The phase space $M := T^*Q \times \mathbb{R}$ inherits a canonical cosymplectic structure, (M, ω, θ),

$$\omega = \omega_Q = dq^i \wedge dp_i, \quad \theta = dt.$$

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The Poisson bracket

Symplectic and cosympelctic manifolds are Poisson manifolds with the bracket

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

In each of these cases, the bracket is induced by the Poisson bivector

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \quad \{f, g\} = \Lambda(df, dg).$$

We have an induced map

$$\sharp_\Lambda : T^*M \to TM, \quad \alpha \mapsto \iota_\alpha \Lambda.$$

Denote

$$\mathcal{H} := \operatorname{im} \sharp_\Lambda = \langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i} \rangle.$$

In symplectic manifolds, $\mathcal{H} = TM$.

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Coisotropic and Lagrangian submanifolds

If $\Delta \subseteq T_x M$, we define the orthogonal as

$$\Delta^{\perp_\Lambda} := \sharp_\Lambda(\Delta^0),$$

where $\Delta^0 \subseteq T_x^* M$ is the annihilator of Δ. We say that Δ is

- Coisotropic, if

$$\Delta^{\perp_\Lambda} \subseteq \Delta,$$

- Lagrangian, if

$$\Delta^{\perp_\Lambda} = \Delta \cap \mathcal{H}.$$

These definitions apply to submanifolds $N \hookrightarrow M$ as well.

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Coisotropic reduction in symplectic geometry

Let (M, ω) be a symplectic manifold and $i : N \hookrightarrow M$ be a coisotropic submanifold.

Proposition

$(TN)^{\perp_\Lambda} \subseteq TN$ is an involutive distribution.

Define $\mathcal{F}$ to be the maximal foliation associated to $(TN)^{\perp_\Lambda}$. We will assume that $N/\mathcal{F}$ admits a manifold structure such that the canonical projection $\pi : N \to N/\mathcal{F}$ is a summersion.

Theorem (Weinstein)

There exists an unique symplectic form ω_N defined on $N/\mathcal{F}$ such that

$$\pi^* \omega_N = i^* \omega.$$

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Suppose $N/\mathcal{F}$ admits a manifold structure such that $\pi : N \to N/\mathcal{F}$ defines a summersion.

Theorem (RIL-MLR)

- *If $TN \subseteq \mathcal{H}$, $N/\mathcal{F}$ admits an unique symplectic structure compatible with the strcuture defined on M.*
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Coisotropic reduction in dissipative mechanics

The canonical phase spaces

- The phase space of an autonomous dissipative system is $T^*Q \times \mathbb{R}$, with its canonical contact structure

$$\eta = dz - p_i dq^i.$$

- If we want to study time-dependent dissipative mechanics, the phase space is $T^*Q \times \mathbb{R} \times \mathbb{R}$ endowed with its cocontact structure

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The Jacobi bracket

In both of these phase spaces there is a Jacobi bracket which is locally given by

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} + p_i\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial z}\right) + g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}.$$

This Jacobi bracket is defined through the Jacobi bivector and a vector field

$$\Lambda = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial z},$$

$$E = -\frac{\partial}{\partial z},$$

as

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If $N \hookrightarrow M$ is a coisotropic submanifold, then $(TN)^{\perp_\Lambda}$ is involutive and thus arises from a maximal foliation $\mathcal{F}$.

We assume that $\frac{\partial}{\partial z} \in TN$.

Theorem

If M is a contact manifold, $N/\mathcal{F}$ admitis an unique contact structure compatible with the one on M.

If M is a cocontact manifold:

- *If $\frac{\partial}{\partial t} \in TN$, $N/\mathcal{F}$ inherits an unique cocontact structure from M.*
- *If $TN \subseteq \operatorname{im} \sharp_\Lambda \oplus \langle \frac{\partial}{\partial z} \rangle$, $N/\mathcal{F}$ inherits an unique contact structure from M.*

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