A Review on Reduction and Reconstruction of Dynamics in Symplectic Geometry

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Introduction

Reduction

General procedure in mechanics by which symmetries of a mechanical system allow to remove ignorable degrees of freedom and reduce the problem to a smaller one.

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Finally, the procedure by which the unreduced dynamics is recovered from the reduced dynamics is called reconstruction.

Marsden-Weinstein reduction

Theorem (Marsden, Weinstein)

Let (P, ω) be a symplectic manifold on which a Lie group G acts symplectically and let $J : P \to \mathfrak{g}^*$ be an Ad^*-equivariant momentum mapping for this action. Assume $\mu \in \mathfrak{g}^*$ is a regular value of J and that the isotropy group $G_\mu = \mathrm{Stab}_G(\mu)$ acts freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ with the property

$$\pi_\mu^* \omega_\mu = \iota_\mu^* \omega$$

where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection and $\iota_\mu : J^{-1}(\mu) \hookrightarrow P$ the inclusion.

Marsden-Weinstein reduction and Hamiltonian dynamics

Theorem

Under the assumptions of the previous theorem, let $H : P \to \mathbb{R}$ *be invariant under the action of* G. *Then the flow* φ^H *of* X_H *leaves* $J^{-1}(\mu)$ *invariant and commutes with the action of* G_μ *on* $J^{-1}(\mu)$*, so it induces caninocally a flow* φ^{H_μ} *on* P_μ *satisfying* $\pi_\mu \circ \varphi^H = \varphi^{H_\mu} \circ \pi_\mu$. *This flow is Hamiltonian on* P_μ *with Hamiltonian the unique* H_μ *satisfying* $H_\mu \circ \pi_\mu = H \circ \iota_\mu$. *This Hamiltonian is called the reduced Hamiltonian.*

Hamiltonian systems

General setting

Configuration space Finite-dimensional smooth manifold Q which models the set of possible positions of a mechanical system.

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This induces a **momentum mapping** $J : \mathrm{T}^*Q \to \mathfrak{g}$ which is Ad^*-equivariant, i.e., such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{T}^*Q & \xrightarrow{(\varphi_{g^{-1}})^*} & \mathrm{T}^*Q \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathfrak{g}^* & \xrightarrow[\mathrm{Ad}^*_{g^{-1}}]{} & \mathfrak{g}^* \end{array}$$

Reduction and embedding

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- Apply Marsden-Weinstein reduction to obtain $((\mathrm{T}^* Q)_\mu, \omega_\mu)$.
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Our goal is to realize $(\mathrm{T}^* Q)_\mu$ as a symplectically embedded subbundle of $\mathrm{T}^*(Q_\mu)$ with a suitable symplectic structure. This symplectic structure can be constructed using a connection γ in the principal bundle ρ_μ, call it Ω_μ.

Embedding theorem

Theorem (Marsden)

There exists a symplectic embedding $j : ((\mathrm{T}^*Q)_\mu, \omega_\mu) \hookrightarrow (\mathrm{T}^*Q_\mu, \Omega_\mu)$ *whose image is a vector subbundle with base* Q_μ*. This embedding is onto if and only if* $\mathfrak{g} = \mathfrak{g}_\mu$*.*

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Idea of proof.

Consider $J_\mu : T^*Q \to \mathfrak{g}_\mu^*$, $p_q \mapsto J(p_q)|_{\mathfrak{g}_\mu}$.

$$\begin{array}{ccccc}
J^{-1}(\mu) & \hookrightarrow & J_\mu^{-1}(\mu) & \xrightarrow{t_\mu} & J_\mu^{-1}(0) \\
\downarrow \rho_\mu & & \downarrow \overline{\rho}_\mu & & \downarrow \overline{\rho}_0 \\
J^{-1}(\mu)/G_\mu & \hookrightarrow & J_\mu^{-1}(\mu)/G_\mu & \xrightarrow{\overline{t}_\mu} & J_\mu^{-1}(0)/G_\mu \simeq \mathrm{T}^*Q_\mu
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$$[p_q] \longmapsto [p_q] \longmapsto [p_q - \langle \mu, \gamma_q(\bullet) \rangle]$$

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Trivial cases: G abeilan, $\mu = 0$.

What about the Hamiltonian dynamics?

If $H : \mathrm{T}^*Q \to \mathbb{R}$, it may be reduced to $H_\mu : (\mathrm{T}^*Q)_\mu \to \mathbb{R}$ and $\tilde{H}_\mu : J_\mu^{-1}(\mu)/G_\mu \to \mathbb{R}$, with flows behaving appropriately.

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When translated to T^*Q_μ via $\bar{t}_\mu$, we get a Hamiltonian $H_0 = \tilde{H}_\mu \circ \bar{t}_\mu^{-1} : \mathrm{T}^*Q_\mu \to \mathbb{R}$ for which

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In conclusion, not only is $(\mathrm{T}^*Q)_\mu$ symplectically embedded in T^*Q_μ, but also the Hamiltonian dynamics in $(\mathrm{T}^*Q)_\mu$ comes from a Hamiltonian dynamics in T^*Q_μ.

Brief comment about Lagrangian systems

Hyperregular Lagrangian systems

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In this case, we can explore further questions about the nature of the dynamics in $\mathrm{T}Q_\mu$.

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Questions?