Poisson cohomology of b^m-Poisson manifolds

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b^m-Poisson geometry

We say a Poisson structure Π is *b^m-Poisson* if, locally,

$$\Pi = z^m \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

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A b^m-symplectic form ω and the space of b^m-vector fields are, respectively, locally described by

$$\omega = \frac{\mathrm{d}z}{z^m} \wedge \mathrm{d}t + \sum_{i=1}^{n-1} \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

$$\Gamma({}^{b^m}TM) = z^m \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial y_{n-1}}.$$

b^m-Poisson cohomology

Lemma — Let M be a b^m-symplectic manifold with Poisson structure Π. The Poisson differential d_Π admits a restriction to the sub-complex ob b^m-vector fields ${}^{b^m}\mathfrak{X}^\bullet(M) \subseteq \mathfrak{X}^\bullet(M)$.

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Theorem — *The b^m-Poisson cohomology ${}^{b^m}\mathrm{H}^\bullet_\Pi(M)$ is isomorphic to the b^m-de Rham cohomology ${}^{b^m}\mathrm{H}^\bullet(M)$.*

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The Poisson cohomology is trapped inside the short exact sequence

$$0 \longrightarrow {}^{b^m}\mathfrak{X}^\bullet(M) \xrightarrow{i^\bullet} \mathfrak{X}^\bullet(M) \xrightarrow{\pi^\bullet} \mathfrak{X}^\bullet_Q(M) \longrightarrow 0.$$

A semi-local computation

Lemma — If $\mathfrak{U}$ is the poset of tubular neighbourhoods of Z, the cohomology of $\mathfrak{X}^\bullet_{\mathcal{Q}}(M)$ is

$$\mathrm{H}^\bullet(\mathfrak{X}_{\mathcal{Q}}(M)) = \varinjlim_{U\in\mathfrak{U}} \mathrm{H}^\bullet(\mathfrak{X}_{\mathcal{Q}}(U)).$$

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- For any class $[X] \in \mathrm{H}^k(\mathfrak{X}^k_{\mathcal{Q}}(M))$, the restriction $Y = X|_U$ determines a class in $\mathrm{H}^k_\Pi(U)$ because d_Π is local.
- Given a class $[X] \in \mathrm{H}^k_\Pi(U)$, we extend it through a bump function ψ with $\psi = 1$ at $Z \subset K \subseteq U$ and $\operatorname{supp}\psi \subseteq U$.

The cohomology of the quotient complex

Theorem — *In the previous notation,*

$$\mathrm{H}^k(\mathfrak{X}_{\mathcal{Q}}(M)) \simeq \left(\mathrm{H}^{k-1}_{\Lambda}(\mathcal{F}_Z)\right)^{m-1} \oplus \left(\mathrm{H}^{k-2}_{\Lambda}(\mathcal{F}_Z)\right)^{m-1}.$$

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Theorem — In the previous assumptions,

$$\ker \delta^k \simeq \left(\mathrm{H}_\Lambda^{k-2}(\mathcal{F}_Z)\right)^{m-1},$$

$$\operatorname{coker} \delta^{k-1} \simeq \mathrm{H}^k(M) \oplus \left(\mathrm{H}^{k-1}(\mathcal{F}_Z)_R\right)^m \oplus \mathrm{H}^{k-2}(\mathcal{F}_Z).$$

Corollary — The Poisson cohomology of a b^m-symplectic manifold is computed by

$$\mathrm{H}^k_\Pi(M) \simeq \left(\mathrm{H}^{k-2}_\Lambda(\mathcal{F}_Z)\right)^{m-1} \oplus \mathrm{H}^k(M) \oplus \left(\mathrm{H}^{k-1}(\mathcal{F}_Z)_R\right)^m \oplus \mathrm{H}^{k-2}(\mathcal{F}_Z).$$

Spectral sequence for b^m-Poisson

In the notation of the a splitting $\varphi\colon U \to (-\varepsilon,\varepsilon) \times Z$, we set

$$
\begin{aligned}
F_0 \cap \mathfrak{X}^q(U) &= \mathfrak{X}^q(U), \\
F_1 \cap \mathfrak{X}^q(U) &= R \wedge \mathfrak{X}^{q-1}(U) + f^m \partial_f \wedge \mathfrak{X}^{q-1}(U), \\
F_2 \cap \mathfrak{X}^q(U) &= f^m \partial_f \wedge R \wedge \mathfrak{X}^q(U), \\
F_p \cap \mathfrak{X}^q(U) &= \{0\} && \text{for } p \geqslant 3.
\end{aligned}
$$

$$
\begin{aligned}
F_0 \cap \mathfrak{X}^q / F_1 \cap \mathfrak{X}^q &\simeq \mathfrak{X}^q(\mathcal{F}_U) \oplus \left(\mathfrak{X}^{q-1}(\mathcal{F}_Z)\right)^m, \\
F_1 \cap \mathfrak{X}^q / F_2 \cap \mathfrak{X}^q &\simeq \mathfrak{X}^{q-1}(\mathcal{F}_U) \oplus \mathfrak{X}^{q-1}(\mathcal{F}_U) \oplus \left(\mathfrak{X}^{q-2}(\mathcal{F}_Z)\right)^m, \\
F_2 \cap \mathfrak{X}^q / F_3 \cap \mathfrak{X}^q &\simeq \mathfrak{X}^{q-2}(\mathcal{F}_U).
\end{aligned}
$$

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$$\begin{aligned}\mathrm{d}_0\colon A'' + \partial_f \wedge (B_0'' + fB_1'' + \cdots + f^{m-1}B_{m-1}'') \longmapsto \\ \mathrm{d}_\Lambda A'' - \partial_f \wedge (\mathrm{d}_\Lambda B_0'' + f\,\mathrm{d}_\Lambda B_1'' + \cdots + f^{m-1}\,\mathrm{d}_\Lambda B_{m-1}''),\end{aligned}$$

$$\begin{aligned}\mathrm{d}_0\colon R \wedge A' + f^m\partial_f \wedge B_m'' + \partial_f \wedge R \wedge (B_0' + fB_1' + \cdots + f^{m-1}B_{m-1}') \longmapsto \\ -R \wedge \mathrm{d}_\Lambda A' - f^m\partial_f \wedge \mathrm{d}_\Lambda B_m'' + \partial_f \wedge R \wedge (\mathrm{d}_\Lambda B_0' + f\,\mathrm{d}_\Lambda B_1' \\ + \cdots + f^{m-1}\,\mathrm{d}_\Lambda B_{m-1}'),\end{aligned}$$

$$\mathrm{d}_0\colon f^m\partial_f \wedge R \wedge B_m' \longmapsto f^m\partial_f \wedge R \wedge \mathrm{d}_\Lambda B_m'.$$

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$$\mathrm{d}_0 : A'' + \partial_f \wedge (B'' \ldots f^{m-1} B'' \ldots) \ldots$$
$$\mathrm{d}_\Lambda A'' - \partial_f \wedge (\ldots \mathrm{d}_\Lambda B''_{m-1}),$$

$$\mathrm{H}(E_0^{0,q}) \simeq \mathrm{H}^q_\Lambda(\mathcal{F}_U) \oplus \left(\mathrm{H}^{q-1}_\Lambda(\mathcal{F}_Z)\right)^m$$

$$\begin{aligned}\mathrm{d}_0 : R \wedge A' + f^m \partial_f \wedge B''_m + \partial_f \wedge R \wedge (B'_0 + fB'_1 + \cdots + f^{m-1} B'_{m-1}) \longmapsto \\ - R \wedge \mathrm{d}_\Lambda A' - f^m \partial_f \wedge \mathrm{d}_\Lambda B''_m + \partial_f \wedge R \wedge (\mathrm{d}_\Lambda B'_0 + f\, \mathrm{d}_\Lambda B'_1 \\ + \cdots + f^{m-1}\, \mathrm{d}_\Lambda B'_{m-1}),\end{aligned}$$

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$$- R \ldots$$

$$\mathrm{H}(E_0^{1,q-1}) \simeq \mathrm{H}^{q-1}_\Lambda(\mathcal{F}_U) \oplus \mathrm{H}^{q-1}_\Lambda(\mathcal{F}_U) \oplus \left(\mathrm{H}^{q-2}(\mathcal{F}_Z)\right)^m$$

$$+ \cdots + f^{m-1} \mathrm{d}_\Lambda B'_{m-1}),$$

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$$+ \cdots + f^{m-1} d_\Lambda B'_{m-1}),$$

$$d_0 : f^m \partial_f \wedge R \wedge B'_m \longmapsto \ldots$$

$$\mathrm{H}(E_0^{2,q-2}) \simeq \mathrm{H}^{q-2}_\Lambda(\mathcal{F}_U)$$

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$$\mathrm{d}_1\colon A'' + \partial_f \wedge (B''_0 + fB''_1 + \cdots + f^{m-1}B''_{m-1}) \longmapsto$$
$$- f^m R \wedge \mathcal{L}_{\partial_f} A'' + f^m \partial_f \wedge \mathcal{L}_R A'' - m f^{m-1} \partial_f \wedge R \wedge B''_0,$$

$$\mathrm{d}_1\colon R \wedge A' + f^m \partial_f \wedge B''_m + \partial_f \wedge R \wedge (B'_0 + fB'_1 + \cdots + f^{m-1}B'_{m-1})$$
$$\longmapsto \partial_f \wedge R \wedge (f^m \mathcal{L}_R A' + f^{2m} \mathcal{L}_{\partial_f} B''_m),$$

$$\mathrm{d}_1\colon f^m \partial_f \wedge R \wedge B'_m \longmapsto 0.$$

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$$\mathrm{d}_1 : A'' + \partial_f \wedge (B''_0 + fB''_1 + \cdots + f^{m-1}B''_{m-1}) \ldots$$

$$- f^m R \wedge \ldots \wedge B''_0,$$

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$$\mathrm{H}(E_1^{2,q-2}) \simeq \left(\mathrm{H}^{q-2}_\Lambda(\mathcal{F}_Z)\right)^m$$

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$$\mathrm{d}_2\colon A'' + \partial_f \wedge (fB''_1 + \cdots + f^{m-1}B''_{m-1}) \longmapsto$$
$$- f^m \partial_f \wedge R \wedge \big((m-1)B''_1 + (m-2)fB''_2 + \cdots + f^{m-2}B''_{m-1}\big),$$

$$\mathrm{d}_2\colon R \wedge A' + f^m \partial_f \wedge B''_m + \partial_f \wedge R \wedge (B'_0 + fB'_1 + \cdots + f^{m-1}B'_{m-1})$$
$$\longmapsto 0,$$

$$\mathrm{d}_2\colon f^m \partial_f \wedge R \wedge B'_m \longmapsto 0.$$

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$$\mathrm{H}(E_2^{0,q}) \simeq \mathrm{H}^q_\Lambda(\mathcal{F}_Z)_R$$

$$\mathrm{d}_2\colon R \wedge A' + f^m \partial_f \wedge B''_m + \partial_f \wedge R \wedge \big(B'_0 + fB'_1 + \cdots + f^{m-1}B'_{m-1}\big) \longmapsto 0,$$

$$\mathrm{d}_2\colon f^m \partial_f \wedge R \wedge B'_m \longmapsto 0.$$

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$$\mathrm{d}_2\colon A'' + \partial_f \wedge (fB_1'' + \cdots$$

$$- f^m \partial_f \wedge R \wedge ((m \ldots B_2'' + \cdots + f^{m-2} B_{m-1}''),$$

$$\mathrm{H}(E_2^{0,q}) \simeq \mathrm{H}_\Lambda^q(\mathcal{F}_Z)_R$$

$$\mathrm{d}_2 \ldots {}_{-1})$$

$$\mathrm{H}(E_2^{1,q-1}) \simeq \left(\mathrm{H}_\Lambda^{q-1}(\mathcal{F}_Z)_R\right)^m \oplus \mathrm{H}_\Lambda^{q-1}(\mathcal{F}_Z) \oplus \left(\mathrm{H}_\Lambda^{q-2}(\mathcal{F}_Z)\right)^{m-1}$$

$$\mathrm{d}_2\colon f^m \partial_f \wedge R \wedge B_m' \longmapsto 0.$$

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$$- f^m \partial_f \wedge R \wedge \big((\cdots + \cdots + f^{m-2} B''_{m-1}\big),$$

$$\boxed{\mathrm{H}(E_2^{0,q}) \simeq \mathrm{H}^q_\Lambda(\mathcal{F}_Z)_R}$$

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$$\mathrm{d}_2 : f^m \partial_f \wedge R \wedge B'_m \longmapsto \boxed{\mathrm{H}(E_1^{2,q-2}) \simeq \{0\}}$$

Theorem — The Poisson cohomology groups of a b^m-Poisson manifold are

$$\mathrm{H}^k_\Pi(M) = \mathrm{H}^k_\Lambda(\mathcal{F}_Z)_R \oplus \mathrm{H}^{k-1}_\Lambda(\mathcal{F}_Z) \oplus \left(\mathrm{H}^{k-1}_\Lambda(\mathcal{F}_Z)_R\right)^{m-1} \oplus \left(\mathrm{H}^{k-2}_\Lambda(\mathcal{F}_Z)\right)^{m-1}.$$

Thank you for your attention!