MULTICONTACT FRAMEWORK FOR NON-CONSERVATIVE FIELD THEORIES

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ABSTRACT A new geometric structure inspired by multisymplectic and contact geometries, called *multicontact structure*, has been developed recently to describe non-conservative and action-dependent classical field theories [1]. We review the main features of this formulation, showing how it is applied to study some classical theories in theoretical physics which are modified in order to include action-dependence; namely: the modified Klein-Gordon equation and the action-dependent bosonic string.

MULTICONTACT LAGRANGIAN AND HAMILTONIAN FORMALISMS

MULTIVECTOR FIELDS

Let $\mathcal{M}$ be a manifold with $\dim\mathcal{M}=n$. The ***m-multivector fields*** on $\mathcal{M}$ are the contravariant skew-symmetric tensor fields of order m in $\mathcal{M}$. The set of m-multivector fields in $\mathcal{M}$ is denoted $\mathfrak{X}^m(\mathcal{M})$.
A multivector field $\mathbf{X}\in\mathfrak{X}^m(\mathcal{M})$ is ***locally decomposable*** if, for every $\mathrm{p}\in\mathcal{M}$, there exists an open neighbourhood $U_\mathrm{p}\subset\mathcal{M}$ such that

$$\mathbf{X}|_{U_\mathrm{p}} = X_1\wedge\cdots\wedge X_m \quad , \quad \text{for some } X_1,\dots X_m\in\mathfrak{X}(U_\mathrm{p}) \, .$$

The *contraction* of a locally decomposable multivector field $\mathbf{X}\in\mathfrak{X}^m(\mathcal{M})$ and a differentiable form $\Omega\in\Omega^k(\mathcal{M})$ is

$$\iota(\mathbf{X})\,\Omega|_{U_p} = \iota(X_1\wedge\cdots\wedge X_m)\,\Omega = \iota(X_m)\dots\iota(X_1)\,\Omega\,, \quad \text{if } k\geq m \quad ; \quad \iota(\mathbf{X})\,\Omega|_{U_p}=0, \quad \text{if } k<m$$

Let $\kappa\colon\mathcal{M}\to M$ be a fiber bundle with local coordinates (x^μ,z^I) on $\mathcal{M}$ (x^μ are coordinates on M and z^I are coordinates on the fibers).
A multivector field $\mathbf{X}\in\mathfrak{X}^m(\mathcal{M})$ is *κ-transverse* if $\iota(\mathbf{X})\,(\kappa^*\beta)|_\mathrm{p}\neq 0$, for $\mathrm{p}\in\mathcal{M}$ and $\beta\in\Omega^m(M)$. If M is an orientable manifold with volume form $\omega\in\Omega^m(M)$, then $\mathbf{X}\in\mathfrak{X}^m(\mathcal{M})$ is κ-transverse if, and only if, $\iota(\mathbf{X})\,(\kappa^*\omega)\neq0$. This condition can be fixed by taking $\iota(\mathbf{X})\,(\kappa^*\omega)=1$.
If $\mathbf{X}\in\mathfrak{X}^m(\mathcal{M})$ is locally decomposable and κ-transverse, a section $\psi(x^\mu)=(x^\mu,z^I(x^\nu))$ of κ is an ***integral section*** of $\mathbf{X}$ if $\dfrac{\partial z^I}{\partial x^\mu}=F^i_\mu$.
Then, $\mathbf{X}$ is ***integrable*** if, for $\mathrm{p}\in\mathcal{M}$, there exist $x\in M$ and an integral section ψ of $\mathbf{X}$ such that $\mathrm{p}=\psi(x)$.

MULTICONTACT LAGRANGIAN FORMALISM

For the Lagrangian formulation of non-conservative first-order field theories, the *configuration bundle* of a (first-order) Lagrangian field theory is $\pi\colon E\to M$ ($\dim M=m$, $\dim E=n+m$), where M is an orientable manifold with volume form $\omega\in\Omega^m(M)$, which usually represent space-time. The theory is developed on the bundle

$$\tau\colon\mathcal{P}=J^1\pi\times_M\Lambda^{m-1}(\mathrm{T}^*M)\to M\,,$$

where J^1 is the the first-order jet bundle of π and $\Lambda^{m-1}(\mathrm{T}^*M)$ is the bundle of $(m-1)$-forms on M, which can be identified with $\mathbb{R}^m$.
Natural coordinates in $\mathcal{P}$ are $(x^\mu,y^i,y^i_\mu,s^\mu)$ ($\mu=1,\dots,m$, $i=1,\dots,n$; $\dim\mathcal{P}=nm+n+2m$)), such that $\omega=\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^m\equiv\mathrm{d}^mx$.
A ***Lagrangian density*** on $\mathcal{P}$ as a m-form $\mathcal{L}\in\Omega^m(\mathcal{P})$, whose expression is $\mathcal{L}(x^\mu,y^i,y^i_\mu,s^\mu)=L(x^\mu,y^i,y^i_\mu,s^\mu)\,\mathrm{d}^mx$, where $L\in\mathscr{C}^\infty(\mathcal{P})$ is the *Lagrangian function* associated with $\mathcal{L}$. A Lagrangian L is ***regular*** if the matrix $\left(\dfrac{\partial^2L}{\partial y^i_\mu\partial y^j_\nu}\right)$ is regular everywhere; then $\Theta_\mathcal{L}$ is a *variational multicontact form* on $\mathcal{P}$ and $(\mathcal{P},\Theta_\mathcal{L},\omega)$ is a ***multicontact Lagrangian system***. Otherwise, L is a ***singular*** Lagrangian [1, 2].
The ***Lagrangian m-form*** associated with $\mathcal{L}$ is:

$$\Theta_\mathcal{L}=-\frac{\partial L}{\partial y^i_\mu}\mathrm{d}y^i\wedge\mathrm{d}^{m-1}x_\mu+\Big(\frac{\partial L}{\partial y^i_\mu}y^i_\mu-L\Big)\mathrm{d}^mx+\mathrm{d}s^\mu\wedge\mathrm{d}^{m-1}x_\mu \quad \Big(\text{where } \mathrm{d}^{m-1}x_\mu=\iota\Big(\frac{\partial}{\partial x^\mu}\Big)\mathrm{d}^mx=(-1)^{\mu-1}\mathrm{d}x^1\wedge\dots\wedge\widehat{\mathrm{d}x^\mu}\wedge\dots\wedge\mathrm{d}x^m\Big)\,. \tag{1}$$

The local function $E_\mathcal{L}=\dfrac{\partial L}{\partial y^i_\mu}y^i_\mu-L$ is the *energy Lagrangian function* associated with L. Then, the ***Lagrangian* $(m+1)$*-form*** is

$$\overline{\Omega}_\mathcal{L}:=\mathrm{d}\Theta_\mathcal{L}+\sigma_{\Theta_\mathcal{L}}\wedge\Theta_\mathcal{L}=\mathrm{d}\Big(-\frac{\partial L}{\partial y^i_\mu}\mathrm{d}y^i\wedge\mathrm{d}^{m-1}x_\mu+\Big(\frac{\partial L}{\partial y^i_\mu}y^i_\mu-L\Big)\mathrm{d}^mx\Big)-\Big(\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\mathrm{d}y^i-\frac{\partial L}{\partial s^\mu}\mathrm{d}s^\mu\Big)\wedge\mathrm{d}^mx\,,$$

where $\sigma_{\Theta_\mathcal{L}}=-\dfrac{\partial L}{\partial s^\mu}\,\mathrm{d}x^\mu$ is the so-called ***dissipation form***.
A section $\psi\colon M\to\mathcal{P}$ of the projection τ is a ***holonomic section*** on $\mathcal{P}$ if it is locally expressed as $\psi(x^\mu)=\big(x^\mu,y^i(x^\nu),y^i_\mu(x^\nu),s^\mu(x^\nu)\big)$.
Then $\mathbf{X}\in\mathfrak{X}^m(\mathcal{P})$ is a ***holonomic m-multivector field*** (a SOPDE) if it is τ-transverse, integrable, and has holonomic integral sections.
The *(pre)multicontact Lagrangian equations* can be derived from the *generalized Herglotz Principle* [3] and, for holonomic multivector fields, they can be stated as:

$$\iota(\mathbf{X}_\mathcal{L})\,\Theta_\mathcal{L}=0 \quad , \quad \iota(\mathbf{X}_\mathcal{L})\,\overline{\Omega}_\mathcal{L}=0 \quad , \quad \iota(\mathbf{X}_\mathcal{L})\,(\tau^*\omega)=1\,. \tag{2}$$

In a natural chart of coordinates of $\mathcal{P}$, a holonomic m-multivector field $\mathbf{X}_\mathcal{L}\in\mathfrak{X}^m(\mathcal{P})$ verifying the condition $\iota(\mathbf{X})\,(\tau^*\omega)=1$ is
$\mathbf{X}_\mathcal{L}=\displaystyle\bigwedge_{\mu=1}^m\Big(\frac{\partial}{\partial x^\mu}+y^i_\mu\frac{\partial}{\partial y^i}+(X_\mathcal{L})^i_{\mu\nu}\frac{\partial}{\partial y^i_\nu}+(X_\mathcal{L})^\nu_\mu\frac{\partial}{\partial s^\nu}\Big)$, and equations (2) lead to

$$(X_\mathcal{L})^\mu_\mu=L \quad ; \quad \frac{\partial L}{\partial y^i}-\frac{\partial^2L}{\partial x^\mu\partial y^i_\mu}-\frac{\partial^2L}{\partial y^j\partial y^i_\mu}y^j_\mu-\frac{\partial^2L}{\partial s^\nu\partial y^i_\mu}(X_\mathcal{L})^\nu_\mu-\frac{\partial^2L}{\partial y^j_\nu\partial y^i_\mu}(X_\mathcal{L})^j_{\mu\nu}=-\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\,. \tag{3}$$

For the holonomic integral sections $\psi(x^\nu)=\Big(x^\mu,y^i(x^\nu),\dfrac{\partial y^i}{\partial x^\mu}(x^\nu),s^\mu(x^\nu)\Big)$ of $\mathbf{X}_\mathcal{L}$ we have that $y^i_\mu=\dfrac{\partial y^i}{\partial x^\mu}$, $(X_\mathcal{L})^j_{\mu\nu}=\dfrac{\partial y^j_\mu}{\partial x^\nu}=\dfrac{\partial^2y^i}{\partial x^\mu\partial x^\nu}$, $(X_\mathcal{L})^\nu_\mu=\dfrac{\partial s^\mu}{\partial x^\nu}$, and these equations transform into the ***Herglotz–Euler– Lagrange field equations***:

$$\frac{\partial s^\mu}{\partial x^\mu}=L\circ\psi \quad ; \quad \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L}{\partial y^i_\mu}\circ\psi\Big)=\Big(\frac{\partial L}{\partial y^i}+\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\Big)\circ\psi\,. \tag{4}$$

For regular Lagrangians, these equations always have solution. When L is not regular, the field equations could have no solutions everywhere on $\mathcal{P}$. Hence, the final objective is, applying a constraint algorithm, to find the maximal submanifold $\mathcal{S}_f$ of $\mathcal{P}$ (if it exists) where there are holonomic Lagrangian multivector fields $\mathbf{X}_\mathcal{L}$ which are tangent solutions to the Lagrangian field equations on $\mathcal{S}_f$.

MULTICONTACT HAMILTONIAN FORMALISM

Consider the bundle

$$\widetilde{\tau}\colon\mathcal{P}^*:=J^{1*}\pi\times_M\Lambda^{m-1}(\mathrm{T}^*M)\to M\,,$$

which is identified with $J^{1*}\pi\times\mathbb{R}^m$; where $J^{1*}\pi$ is the *restricted multimomentum bundle*. Natural coordinates on $\mathcal{P}^*$ are $(x^\mu,y^i,p^\mu_i,s^\mu)$.
If $(\mathcal{P},\Theta_\mathcal{L},\omega)$ is a Lagrangian system, with $\mathcal{L}=L\omega$, the ***Legendre map*** associated with $\mathcal{L}$ is the map $\mathcal{FL}\colon\mathcal{P}\to\mathcal{P}^*$ locally given by

$$\mathcal{FL}^*x^\nu=x^\nu \quad , \quad \mathcal{FL}^*y^i=y^i \quad , \quad \mathcal{FL}^*p^\nu_i=\frac{\partial\mathrm{L}}{\partial y^i_\nu} \quad , \quad \mathcal{FL}^*s^\mu=s^\mu\,.$$

The Lagrangian L is regular if, and only if, $\mathcal{FL}$ is a local diffeomorphism, and L is ***hyperregular*** when $\mathcal{FL}$ is a global diffeomorphism.
In the hyperregular case (for the singular case and examples, see [2]), $\mathcal{FL}(\mathcal{P})=\mathcal{P}^*$, The form $\Theta_\mathcal{L}\in\Omega^m(\mathcal{P})$ projects to $\mathcal{P}^*$ by $\mathcal{FL}$ giving the ***Hamiltonian m-form*** $\Theta_\mathcal{H}\in\Omega^m(\mathcal{P}^*)$, $\Theta_\mathcal{L}=\mathcal{FL}^*\Theta_\mathcal{H}$, whose local expression is

$$\Theta_\mathcal{H}=-p^\mu_i\mathrm{d}y^i\wedge\mathrm{d}^{m-1}x_\mu+H\,\mathrm{d}^mx+\mathrm{d}s^\mu\wedge\mathrm{d}^{m-1}x_\mu\,, \tag{5}$$

where $H=p^\mu_i(\mathcal{FL}^{-1})^*y^i_\mu-(\mathcal{FL}^{-1})^*L\in\mathscr{C}^\infty(\mathcal{P}^*)$ is the *Hamiltonian function*. Then, $\Theta_\mathcal{H}$ is a variational multicontact form and $(\mathcal{P}^*,\Theta_\mathcal{H},\omega)$ is the ***multicontact Hamiltonian system*** associated with $(\mathcal{P},\Theta_\mathcal{L},\omega)$. Then, we define the ***Hamiltonian* $(m+1)$*-form***

$$\overline{\Omega}_\mathcal{H}:=\mathrm{d}\Theta_\mathcal{H}+\sigma_{\Theta_\mathcal{H}}\wedge\Theta_\mathcal{H}=\mathrm{d}(-p^\mu_i\mathrm{d}y^i\wedge\mathrm{d}^{m-1}x_\mu+H\,\mathrm{d}^mx)+\Big(\frac{\partial H}{\partial s^\mu}\,p^\mu_i\,\mathrm{d}y^i-\frac{\partial H}{\partial s^\mu}\mathrm{d}s^\mu\Big)\wedge\mathrm{d}^mx\,,$$

where $\sigma_\mathcal{H}=\dfrac{\partial H}{\partial s^\mu}\,\mathrm{d}x^\mu$ is the *dissipation form* in this formalism. We have that $\overline{\Omega}_\mathcal{L}=\mathcal{FL}^*\overline{\Omega}_\mathcal{H}$.
The *multicontact Hamilton–de Donder–Weyl equations* for $\widetilde{\tau}$-transverse and locally decomposable multivector fields are stated as:

$$\iota(\mathbf{X}_\mathcal{H})\,\Theta_\mathcal{H}=0 \quad , \quad \iota(\mathbf{X}_\mathcal{H})\,\overline{\Omega}_\mathcal{H}=0 \quad , \quad \iota(\mathbf{X}_\mathcal{H})\,(\widetilde{\tau}^*\omega)=1\,. \tag{6}$$

In natural coordinates, if $\mathbf{X}_\mathcal{H}=\displaystyle\bigwedge_{\mu=1}^m\Big(\frac{\partial}{\partial x^\mu}+(X_\mathcal{H})^i_\mu\frac{\partial}{\partial y^i}+(X_\mathcal{H})^\nu_{\mu i}\frac{\partial}{\partial p^\nu_i}+(X_\mathcal{H})^\nu_\mu\frac{\partial}{\partial s^\nu}\Big)\in\mathfrak{X}^m(\mathcal{P}^*)$ is a solution to the equations (6), then

$$(X_\mathcal{H})^\mu_\mu=p^\mu_i\,\frac{\partial H}{\partial p^\mu_i}-H \quad , \quad (X_\mathcal{H})^i_\mu=\frac{\partial H}{\partial p^\mu_i} \quad , \quad (X_\mathcal{H})^\mu_{\mu i}=-\left(\frac{\partial H}{\partial y^i}+p^\mu_i\,\frac{\partial H}{\partial s^\mu}\right)\,, \tag{7}$$

If $\psi(x^\nu)=(x^\mu,y^i(x^\nu),p^\mu_i(x^\nu),s^\mu(x^\nu))$ is an integral section of $\mathbf{X}_\mathcal{H}$, equations (6) lead to the ***Herglotz–Hamilton–de Donder– Weyl equations*** for ψ:

$$\frac{\partial s^\mu}{\partial x^\mu}=\Big(p^\mu_i\,\frac{\partial H}{\partial p^\mu_i}-H\Big)\circ\psi \quad , \quad \frac{\partial y^i}{\partial x^\mu}=\frac{\partial H}{\partial p^\mu_i}\circ\psi \quad , \quad \frac{\partial p^\mu_i}{\partial x^\mu}=-\Big(\frac{\partial H}{\partial y^i}+p^\mu_i\,\frac{\partial H}{\partial s^\mu}\Big)\circ\psi\,. \tag{8}$$

These equations are compatible in $\mathcal{P}^*$. As $\mathcal{FL}$ is a diffeomorphism, the solutions to the Lagrangian field equations for $(\mathcal{P},\Theta_\mathcal{L},\omega)$ are in one-to-one correspondence to those of the Hamilton-de Donder-Weyl field equations for $(\mathcal{P}^*,\Theta_\mathcal{H},\omega)$.

APPLICATION TO PHYSICAL THEORIES

The modified Klein–Gordon equation and the Telegrapher's equation

The *Klein–Gordon equation* in the Minkowski space-time $\mathbb{R}^4$ (with the metric signature $g_{\mu\nu}\equiv(-1,1,1,1)$) is

$$(\square+m^2)\phi\equiv\partial_\mu\partial^\mu\phi+m^2\phi=0\,,$$

where ϕ is a scalar field, m^2 is a constant, $\square$ denotes de D´Alembert operator in $\mathbb{R}^4$, and $\partial_\mu\equiv\dfrac{\partial}{\partial x^\mu}$, $\partial^\mu\equiv g^{\mu\nu}\partial_\nu$. It derives from the Lagrangian $L_0=\dfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\dfrac{1}{2}m^2\phi^2$, which can be modified to include a more generic potential, $\tilde{L}_0=\dfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-V(\phi)$.

LAGRANGIAN FORMALISM

Consider the bundle $\tau\colon\mathcal{P}=J^1\pi\times_M\Lambda^{m-1}(\mathrm{T}^*\mathbb{R}^4)\to\mathbb{R}^4$, with coordinates (x^μ,y,y_μ,s^μ) ($\mu=0,\dots,3$), where y denotes the field variable, and the volume form is $\omega=\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^3\equiv\mathrm{d}^4x$ on $\mathbb{R}^4$. Consider the contactified Lagrangian $L\in\mathscr{C}^\infty(\mathcal{P})$:

$$L(x^\mu,y,y_\mu,s^\mu)=L_0(x^\mu,y,y_\mu)+\gamma_\mu s^\mu=\frac{1}{2}\,y_\mu y^\mu-\frac{1}{2}m^2y^2+\gamma_\mu s^\mu\,,$$

where $\gamma\equiv(\gamma_\mu)\in\mathbb{R}^4$ is a constant vector, and $y^\mu=\partial^\mu y$. It is a quadratic hyperregular Lagrangian.
Using the Hodge star operator, $*$, the Lagrangian multicontact 4-form (1) is:

$$\Theta_\mathcal{L}=y^\mu\mathrm{d}y\wedge*\mathrm{d}x_\mu+E_\mathcal{L}\mathrm{d}^4x+\mathrm{d}s^\mu\wedge*\mathrm{d}x_\mu=y^\mu\mathrm{d}y\wedge*\mathrm{d}x_\mu+\Big(\frac{1}{2}y_\mu y^\mu+\frac{1}{2}m^2y^2-\gamma_\mu s^\mu\Big)\mathrm{d}^4x+\mathrm{d}s^\mu\wedge*\mathrm{d}x_\mu\,.$$

Then $\overline{\Omega}_\mathcal{L}=\mathrm{d}\Theta_\mathcal{L}+\sigma_{\Theta_\mathcal{L}}\wedge\Theta_\mathcal{L}$, where $\sigma_{\Theta_\mathcal{L}}=-\gamma_\mu\mathrm{d}x^\mu$.
For holonomic multivector fields $\mathbf{X}_\mathcal{L}=\displaystyle\bigwedge_\mu\Big(\frac{\partial}{\partial x^\mu}+y_\mu\frac{\partial}{\partial y}+F_{\mu\nu}\frac{\partial}{\partial y_\nu}+G^\nu_\mu\frac{\partial}{\partial s^\nu}\Big)\in\mathfrak{X}^4(\mathcal{P})$, the Lagrangian equations (3) are

$$G^\mu_\mu=L \quad , \quad m^2y+F^\mu_\mu=\gamma_\mu y^\mu\,. \tag{9}$$

For the integral holonomic sections $\psi(x^\nu)=\Big(x^\mu,y(x^\nu),\dfrac{\partial y}{\partial x^\mu}(x^\nu),s^\mu(x^\nu)\Big)$ of $\mathbf{X}_\mathcal{L}$, bearing in mind that $\dfrac{\partial y^\mu}{\partial x^\mu}=\dfrac{\partial^2y}{\partial x_\mu\partial x^\mu}$, equations (4) read,

$$\frac{\partial s^\mu}{\partial x^\mu}=L \quad , \quad \frac{\partial^2y}{\partial x_\mu\partial x^\mu}+m^2y=\gamma_\mu\frac{\partial y}{\partial x_\mu}=\gamma^\mu\frac{\partial y}{\partial x^\mu}\,. \tag{10}$$

where the last equation is the *Klein–Gordon equation with additional first-order terms*.
For simplicity, we have considered the Minkowski metric and γ_μ constants. However, a similar procedure can be performed for a generic metric $g_{\mu\nu}=g_{\mu\nu}(x^\nu)$ and functions $\gamma_\mu=\gamma_\mu(x^\nu)$, thus obtaining,

$$\frac{\partial s^\mu}{\partial x^\mu}=L \quad , \quad \frac{\partial^2y}{\partial x_\mu\partial x^\mu}+m^2y+\frac{\partial g_{\mu\nu}}{\partial x^\mu}\frac{\partial y}{\partial x^\nu}=\gamma^\mu\frac{\partial y}{\partial x^\mu}\,.$$

THE TELEGRAPHER'S EQUATION: As an interesting application of this modified Klein–Gordon equation, we can derive from it the so-called *telegrapher's equation* which describes the current and voltage on a uniform electrical transmission line:

$$\frac{\partial V}{\partial x}=-L\frac{\partial I}{\partial t}-RI \quad , \quad \frac{\partial I}{\partial x}=-C\frac{\partial V}{\partial t}-GV\,,$$

where V is the voltage, I is the current, R is the resistance, L is the inductance, C is the capacitance, and G is the conductance. This system can be uncoupled, obtaining the system

$$\frac{\partial^2V}{\partial x^2}=LC\frac{\partial^2V}{\partial t^2}+(LG+RC)\frac{\partial V}{\partial t}+RGV \quad , \quad \frac{\partial^2I}{\partial x^2}=LC\frac{\partial^2I}{\partial t^2}+(LG+RC)\frac{\partial I}{\partial t}+RGI\,.$$

Both equations above can be written as

$$\square y+\gamma\frac{\partial y}{\partial t}+m^2y=0\,, \tag{11}$$

where $\square$ is the d'Alembert operator in 1+1 dimensions, and γ and m^2 are adequate constants. Taking $\gamma_\mu=(-\gamma,0,0,0)$ in (10), we obtain the telegrapher's equation (11). In this way, we can see the telegrapher's equation as a modified Klein–Gordon equation.

HAMILTONIAN FORMALISM

The adapted coordinates of fiber bundle $\widetilde{\tau}\colon\mathcal{P}^*=J^{1*}\pi\times_M\Lambda^{m-1}(\mathrm{T}^*\mathbb{R}^4)\to\mathbb{R}^2$ are (x^μ,y,p^μ,s^μ). The Legendre map $\mathcal{FL}\colon\mathcal{P}\to\mathcal{P}^*$ is

$$\mathcal{FL}(x^\mu,y,y_\mu,s^\mu)=(x^\mu,y,p^\mu,s^\mu)\,,$$

with $p^\mu=y_\mu$. It is a diffeomorphism since the Lagrangian function is hyperregular. The contact Hamiltonian m-form (5) is,

$$\Theta_\mathcal{H}=p^\mu\mathrm{d}y\wedge*\mathrm{d}x_\mu+H\,\mathrm{d}^4x+\mathrm{d}s^\mu\wedge*\mathrm{d}x_\mu=p^\mu\mathrm{d}y\wedge*\mathrm{d}x_\mu+\Big(\frac{1}{2}p^\mu p_\mu+\frac{1}{2}m^2y^2-\gamma_\mu s^\mu\Big)\,\mathrm{d}^4x+\mathrm{d}s^\mu\wedge*\mathrm{d}x_\mu$$

and then $\overline{\Omega}_\mathcal{H}=\mathrm{d}\Theta_\mathcal{H}+\sigma_{\Theta_\mathcal{H}}\wedge\Theta_\mathcal{H}$, where $\sigma_{\Theta_\mathcal{H}}=-\gamma_\mu\mathrm{d}x^\mu$.
Equations (7) for $\widetilde{\tau}$-transverse 4-multivector fields $\mathbf{X}_\mathcal{H}=\displaystyle\bigwedge_\mu\Big(\frac{\partial}{\partial x^\mu}+f_\mu\frac{\partial}{\partial y}+F^\nu_\mu\frac{\partial}{\partial p^\nu}+G^\nu_\mu\frac{\partial}{\partial s^\nu}\Big)\in\mathfrak{X}^4(\mathcal{P}^*)$ are

$$G^\mu_\mu=\frac{1}{2}p^\mu p_\mu-\frac{1}{2}m^2y^2+\gamma_\mu s^\mu \quad , \quad f_\mu=p_\mu \quad , \quad F^\mu_\mu=-m^2y+\gamma_\mu p^\mu\,.$$

and using the Legendre map, these equations transform into (9) along with the holonomy condition. Thus, the Lagrangian and Hamiltonian formalisms are equivalent.
For the integral sections $\psi(x^\nu)=(x^\mu,y(x^\nu),p^\mu(x^\nu),s^\mu(x^\nu))$ of $\mathbf{X}_\mathcal{H}$, the Herglotz–Hamilton–De Donder–Weyl equations (8) read

$$\frac{\partial s^\mu}{\partial x^\mu}=\frac{1}{2}p^\mu p_\mu-\frac{1}{2}m^2y^2+\gamma_\mu s^\mu \quad , \quad \frac{\partial y}{\partial x^\mu}=p_\mu \quad , \quad \frac{\partial p^\mu}{\partial x^\mu}=-m^2y+\gamma_\mu p^\mu\,.$$

and, combining the last two equations above, we obtain the equation (10).

Action-dependent bosonic string theory

Spacetime is a $(d+1)$-dimensional manifold M, with local coordinates x^μ ($\mu=1,\dots,d$) and a metric $G_{\mu\nu}$ (signature $(-+\cdots+)$). The string worldsheet is a 2-dimensional manifold Σ, with local coordinates σ^i ($i=0,1$) and the volume form $\omega=\mathrm{d}^2\sigma$. The fields $x^\mu(\sigma)$ are scalar fields on Σ given by the embedding maps $\Sigma\to M:\sigma^a\mapsto x^\mu(\sigma)$. The configuration bundle is $\pi:E=\Sigma\times M\to\Sigma$. On $J^1\pi$ we also have a 2-form $g=\dfrac{1}{2}g_{ij}\mathrm{d}\sigma^i\wedge\mathrm{d}\sigma^j$, whose pullback by jet prolongations of sections $\phi\in\Gamma(\pi)$, $j^1\phi=\Big(\sigma^i,x^\mu(\sigma),\dfrac{\partial x^\mu}{\partial\sigma^i}(\sigma)\Big)$ gives the induced metric on Σ, $(j^1\phi)^*g=h\equiv\dfrac{1}{2}h_{ij}\mathrm{d}\sigma^i\wedge\mathrm{d}\sigma^j$, where $h_{ij}=G_{\mu\nu}\dfrac{\partial x^\mu}{\partial\sigma^i}\dfrac{\partial x^\nu}{\partial\sigma^j}$.

LAGRANGIAN FORMALISM

The bundle $\tau\colon\mathcal{P}\simeq J^1\pi\times\mathbb{R}^2\to\Sigma$ has adapted coordinates $(\sigma^i,x^\mu,x^\mu_i,s^i)$. Consider the contactified Lagrangian function

$$L(\sigma^i,x^\mu,x^\mu_i,s^i)=L_0(\sigma^i,x^\mu,x^\mu_i)+\gamma_is^i=-T\sqrt{-\det(G_{\mu\nu}x^\mu_ix^\nu_j)}\,\mathrm{d}^2\sigma+\gamma_is^i\in\mathscr{C}^\infty(\mathcal{P})\,,$$

where L_0 is the standard *Nambu–Goto Lagrangian*, T is a constant called the *string tension*, and $\gamma\equiv(\gamma_\mu)\in\mathbb{R}^2$ is a constant vector.
This is a regular Lagrangian since the following Hessian matrix is regular everywhere,

$$\frac{\partial^2L}{\partial x^\mu_i\partial x^\nu_j}=-T\sqrt{-\det g}\Big[G_{\mu\nu}g^{ji}-G_{\mu\alpha}G_{\rho\nu}x^\alpha_kx^\rho_\ell\,(g^{ji}g^{k\ell}+g^{kj}g^{i\ell}-g^{ki}g^{j\ell})\Big]\,.$$

The Lagrangian multicontact 2-form (1) is

$$\Theta_\mathcal{L}=\frac{\partial L}{\partial x^\mu_i}\mathrm{d}x^\mu\wedge\mathrm{d}^1\sigma_i-E_\mathcal{L}\wedge\mathrm{d}^2\sigma+\mathrm{d}s^i\wedge\mathrm{d}^1\sigma_i=-T\sqrt{-\det g}\,G_{\mu\nu}g^{ji}x^\nu_j\mathrm{d}x^\mu\wedge\mathrm{d}^1\sigma_i-\Big(T\sqrt{-\det g}+\gamma_is^i\Big)\mathrm{d}^2\sigma+\mathrm{d}s^i\wedge\mathrm{d}^1\sigma_i\,,$$

where $\mathrm{d}^1\sigma_i=\iota\left(\dfrac{\partial}{\partial\sigma^i}\right)\mathrm{d}^2\sigma$. Then, as usual, $\overline{\Omega}_\mathcal{L}=\mathrm{d}\Theta_\mathcal{L}+\sigma_{\Theta_\mathcal{L}}\wedge\Theta_\mathcal{L}$, where $\sigma_{\Theta_\mathcal{L}}=-\gamma_i\mathrm{d}\sigma^i$.
For a holonomic 2-multivector field $\mathbf{X}_\mathcal{L}=\displaystyle\bigwedge_i\Big(\frac{\partial}{\partial\sigma^i}+x^\mu_i\frac{\partial}{\partial x^\mu}+F^\mu_{ij}\frac{\partial}{\partial x^\mu_j}+f^j_i\frac{\partial}{\partial s^j}\Big)\in\mathfrak{X}^2(\mathcal{P})$ the Lagrangian equations (3) are

$$\begin{aligned}f^i_i=L \quad ; \quad T\sqrt{-\det g}\,G_{\mu\nu}g^{ji}x^\nu_j\gamma_i&=x^\rho_i\left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ji}x^\nu_j\right)-\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ji}x^\nu_j\right)\right]\\&+\sqrt{-\det g}\left[G_{\mu\nu}g^{ji}-G_{\mu\alpha}G_{\beta\nu}x^\alpha_kx^\beta_\ell\,(g^{ji}g^{k\ell}+g^{kj}g^{i\ell}-g^{ki}g^{j\ell})\right]F^\nu_{ij}\\&+\left[\frac{1}{2}\sqrt{-\det g}\,g^{ji}x^\alpha_ix^\beta_j\frac{\partial G_{\alpha\beta}}{\partial x^\mu}+\frac{\partial}{\partial\sigma^i}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ji}x^\nu_j\right)\right]\,.\end{aligned}$$

For the holonomic integral sections $\psi(\sigma)=\Big(\sigma^a,x^\mu(\sigma),\dfrac{\partial x^\mu}{\partial\sigma^a}\Big)$ of $\mathbf{X}_\mathscr{L}$, these equations become the Herglotz–Euler–Lagrange equations:

$$\begin{aligned}\frac{\partial s^i}{\partial\sigma^i}=L \quad ; \quad T\sqrt{-\det g}\,G_{\mu\nu}g^{ji}\gamma_i\frac{\partial x^\nu}{\partial\sigma^j}&=\frac{\partial x^\rho}{\partial\sigma^i}\left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ji}\frac{\partial x^\nu}{\partial\sigma^j}\right)-\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ji}\frac{\partial x^\nu}{\partial\sigma^j}\right)\right]\\&+\sqrt{-\det g}\left[G_{\mu\nu}g^{ji}-G_{\mu\alpha}G_{\beta\nu}\,(g^{ji}g^{k\ell}+g^{kj}g^{i\ell}-g^{ki}g^{j\ell})\,\frac{\partial x^\alpha}{\partial\sigma^k}\frac{\partial x^\beta}{\partial\sigma^\ell}\right]\frac{\partial^2x^\nu}{\partial\sigma^i\partial\sigma^j}\\&+\left[\frac{1}{2}\sqrt{-\det g}\,g^{ji}\frac{\partial G_{\alpha\beta}}{\partial x^\mu}\frac{\partial x^\alpha}{\partial\sigma^i}\frac{\partial x^\beta}{\partial\sigma^j}+\frac{\partial}{\partial\sigma^i}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ji}\frac{\partial x^\nu}{\partial\sigma^j}\right)\right]\,.\end{aligned}$$

HAMILTONIAN FORMALISM

The bundle $\tau\colon\mathcal{P}^*\simeq J^{1*}\pi\times\mathbb{R}^2\to\Sigma$ has adapted coordinates $(\sigma^i,x^\mu,p^i_\mu,s^i)$. The Legendre map $\mathcal{FL}:\mathcal{P}\to\mathcal{P}^*$ is

$$\mathcal{FL}^*\sigma^i=\sigma^i \quad , \quad \mathcal{FL}^*x^\mu=x^\mu \quad , \quad \mathcal{FL}^*p^i_\mu=-T\sqrt{-\det g}\;G_{\mu\nu}g^{ji}x^\nu_j \quad , \quad \mathcal{FL}^*s^\mu=s^\mu\,,$$

and is a diffeomorphism, since L is regular. Then, the 2-form g can be translated to $\mathcal{P}^*$ by the push-forward of the Legendre map.
Introducing $\Pi^{ij}\equiv G^{\mu\nu}p^i_\mu p^j_\nu$, the contact Hamiltonian 2–form can be written as

$$\Theta_\mathcal{H}=p^i_\mu\mathrm{d}x^\mu\wedge\mathrm{d}^1\sigma_i-H\wedge\mathrm{d}^2\sigma=p^i_\mu\mathrm{d}x^\mu\wedge\mathrm{d}^1\sigma_i+\left(\frac{1}{T}\sqrt{-\det\Pi}+\gamma_is^i\right)\mathrm{d}^2\sigma\,,$$

and then $\overline{\Omega}_\mathcal{H}=\mathrm{d}\Theta_\mathcal{H}+\sigma_{\Theta_\mathcal{H}}\wedge\Theta_\mathcal{H}$, where $\sigma_{\Theta_\mathcal{H}}=-\gamma_i\,\mathrm{d}\sigma^i$.
For $\widetilde{\tau}$-transverse 2-multivector fields $\mathbf{X}_\mathcal{H}=\displaystyle\bigwedge_i\Big(\frac{\partial}{\partial\sigma^i}+F^\mu_i\frac{\partial}{\partial x^\mu}+F^j_{i\mu}\frac{\partial}{\partial p^j_\mu}+f^j_i\frac{\partial}{\partial s^j}\Big)\in\mathfrak{X}^2(\mathcal{P}^*)$, equations (7) are

$$\begin{aligned}\frac{\partial s^i}{\partial\sigma^i}&=\frac{\sqrt{-\det\Pi}}{T}\,(1-\Pi_{ji}G^{\mu\nu}p^i_\mu p^j_\nu)+\gamma_is^i=\frac{\sqrt{-\det\Pi}}{T}\,(1-\Pi_{ji}\Pi^{ij})+\gamma_is^i\,,\\F^\mu_i&=-\frac{\sqrt{-\det\Pi}}{T}\Pi_{ji}G^{\mu\nu}p^j_\nu \quad , \quad F^i_{i\mu}=\frac{\sqrt{-\det\Pi}}{2T}\Pi_{ji}\frac{\partial G^{\rho\alpha}}{\partial x^\mu}p^i_\rho p^j_\alpha+\gamma_ip^i_\mu\,.\end{aligned}$$

For the integral sections $\psi(\sigma)=(\sigma^i,x^\mu(\sigma),p^i_\mu(\sigma),s^i(\sigma))$ of $\mathbf{X}_\mathcal{H}$, for which $F^\mu_i=\dfrac{\partial x^\mu}{\partial\sigma^i}$, $F^j_{i\mu}=\dfrac{\partial p^j_\mu}{\partial\sigma^i}$, and $f^j_i=\dfrac{\partial s^j}{\partial\sigma^i}$, the field equations become the Herglotz–Hamilton–De Donder–Weyl equations:

$$\begin{aligned}\frac{\partial s^i}{\partial\sigma^i}&=\frac{\sqrt{-\det\Pi}}{T}\,(1-\Pi_{ji}G^{\mu\nu}p^i_\mu p^j_\nu)+\gamma_is^i=\frac{\sqrt{-\det\Pi}}{T}\,(1-\Pi_{ji}\Pi^{ij})+\gamma_is^i\,,\\\frac{\partial x^\mu}{\partial\sigma^i}&=-\frac{\sqrt{-\det\Pi}}{T}\Pi_{ji}G^{\mu\nu}p^j_\nu \quad , \quad \frac{\partial p^i_\mu}{\partial\sigma^i}=\frac{\sqrt{-\det\Pi}}{2T}\Pi_{ji}\frac{\partial G^{\rho\alpha}}{\partial x^\mu}p^i_\rho p^j_\alpha+\gamma_ip^i_\mu\,.\end{aligned}$$

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