k-contact Lie systems

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1. Abstract

This poster introduces k-contact geometry, it relates it to the so-called k-symplectic geometry, and extends to the k-contact case some properties known in contact geometry. In general, k-contact geometry is mainly used to study field theories, but we here develop a new approach to study systems of ordinary differential equations.
Then, we use our results to analyse a particular type of systems of first-order differential equations, the so-called *Lie systems*, whose properties are determined by a finite-dimensional Lie algebra of vector fields of Hamiltonian vector fields relative to a k-contact structure. The obtained systems, the hereafter called *k-contact Lie systems*, are analysed and applied to analyse physical and mathematical problems.

2. Fundamentals on k-contact geometry

From now on $\{e_1, \dots, e_k\}$ stands for a basis of $\mathbb{R}^k$.

Definition 1. A *k-contact form* on a manifold M is a vector valued one-form on M given by $\boldsymbol{\eta} = \eta^1 \otimes e_1 + \dots + \eta^k \otimes e_k$, where $\eta^1, \dots, \eta^k \in \Omega^1(M)$ such that

(1) $\mathrm{T}M \supset \mathcal{D}^{\mathrm{C}} = \bigcap_{\alpha=1}^k \ker \eta^\alpha =: \ker \boldsymbol{\eta}$ is a regular distribution of corank k,

(2) $\mathrm{T}M \supset \mathcal{D}^{\mathrm{R}} = \bigcap_{\alpha=1}^k \ker \mathrm{d}\eta^\alpha =: \ker \mathrm{d}\boldsymbol{\eta}$ is a regular distribution of rank k,

(3) $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} = \{0\}$.

A pair $(M, \boldsymbol{\eta})$ given by a manifold M and a k-contact form $\boldsymbol{\eta} \in \Omega^1(M, \mathbb{R}^k)$ is called a *k-contact manifold*. If, in addition, $\dim M = n + nk + k$ and M is endowed with an integrable subdistribution $\mathcal{V}$ of $\mathcal{D}^{\mathrm{C}}$ with $\operatorname{rank} \mathcal{V} = nk$, we say that $(M, \boldsymbol{\eta}, \mathcal{V})$ is *polarized k-contact manifold*. We call the distribution $\mathcal{V}$ a *polarization*.

We call $\mathcal{C}^{\mathrm{C}}$ the *contact codistribution*, $\mathcal{D}^{\mathrm{C}}$ the *contact distribution*, $\mathcal{D}^{\mathrm{R}}$ the *Reeb distribution* and $\mathcal{C}^{\mathrm{R}}$ the *Reeb codistribution*. Note that $\mathcal{D}$ stands for "distribution" and $\mathcal{C}$ means "codistribution". Meanwhile, C and R represent "contact" and "Reeb", respectively. This helps to recall the terminology. Let us study k-contact manifolds.

Definition 2. Given a k-contact manifold $(M, \boldsymbol{\eta})$, its *Reeb k-vector field* is the unique k-vector field on M, let us say $\boldsymbol{R} = \sum_{\alpha=1}^k R_\alpha \otimes e_\alpha$, such that

$$\iota_{R_\alpha} \eta^\beta = \delta_\alpha^\beta, \qquad \iota_{R_\alpha} \mathrm{d}\eta^\beta = 0, \qquad \alpha, \beta = 1, \dots, k.$$

Theorem 3 (Darboux theorem for k-contact manifolds). *Let $(M, \boldsymbol{\eta}, \mathcal{V})$ be a polarized k-contact manifold. Then, around every point of M, there exists a local chart of coordinates $(U; q^i, p_i^\alpha, s^\alpha)$, with $1 \le \alpha \le k$, $1 \le i \le n$, called* ***Darboux coordinates****, such that*

$$\eta^\alpha|_U = \mathrm{d}s^\alpha - p_i^\alpha \mathrm{d}q^i, \qquad \mathcal{D}^{\mathrm{R}}|_U = \left\langle \mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha} \right\rangle, \qquad \mathcal{V}|_U = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle.$$

Example 1. *A jet space $J^1(E, \pi, M)$, $\operatorname{rk} E = n$, $\dim M = m$, with local adapted coordinates (x^i, y^j, y_i^j), $i = 1, \dots, m$, $j = 1, \dots, n$, induces an n-contact manifold*

$$\theta^j = \mathrm{d}y^j - \sum_{i=1}^m y_i^j \mathrm{d}x^i, \qquad \mathrm{d}\theta_j = \sum_{\alpha=1}^m \mathrm{d}x^\alpha \wedge \mathrm{d}y_\alpha^j.$$

Distributions spanned by joint kernels of above forms and their derivatives are, respectively:

$$\mathcal{D}^C = \left\langle \partial_{x^i} + \sum_{\alpha=1}^n y^{\alpha i} \partial_{y^\alpha}, \partial_{y_i^j} \right\rangle, \quad \mathcal{D}^R = \langle \partial_{y^j} \rangle.$$

Example 2. Consider $\mathbb{R}^6$ with linear global coordinates $\{x, y, p, q, z, t\}$. Then,

$$\eta^1 = \mathrm{d}z - \frac{1}{2}(y\mathrm{d}x - x\mathrm{d}y), \quad \eta^2 = \mathrm{d}t - p\mathrm{d}x - q\mathrm{d}y$$

define a 2-contact form on M given by $\boldsymbol{\eta} = \eta^1 \otimes e_1 + \eta^2 \otimes e_2$. Let us show that $\boldsymbol{\eta}$ is a two-contact form on M by verifying the conditions in Definition 1. Firstly $\eta^1 \wedge \eta^2 \neq 0$ and $\mathcal{D}^{\mathrm{C}} = \ker \boldsymbol{\eta}$ is a distribution of rank 2. Moreover,

$$\mathrm{d}\eta^1 = \mathrm{d}x \wedge \mathrm{d}y, \quad \mathrm{d}\eta^2 = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q \implies \mathcal{D}^{\mathrm{R}} = \left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\rangle,$$

and $\mathcal{D}^{\mathrm{R}}$ has rank two. Moreover $\mathcal{D}^{\mathrm{R}} \cap \mathcal{D}^{\mathrm{C}} = 0$, which is the third condition in Definition 1. The Reeb vector fields are

$$R_1 = \frac{\partial}{\partial z}, \quad R_2 = \frac{\partial}{\partial t}.$$

Noteworthy, the coordinates (x, y, p, q, z, t) are not Darboux coordinates.

3. On k-contact geometry and k-symplectic manifolds

As shown next, it is possible to relate a k-contact form with a k-symplectic form (i.e. a closed non-degenerate form $\boldsymbol{\omega} \in \Omega^2(\mathbb{R}_\times^l \times M, \mathbb{R}^k)$ for different values of l. Although this method works for analysing k-contact forms, it may not be useful to study related notions, as k-contact Hamiltonian vector fields to be described next.

Theorem 4. *Let $(M, \boldsymbol{\eta})$ be a k-contact manifold. Then, $(\widetilde{M} = M \times \mathbb{R}_\times^k, \boldsymbol{\omega} = \sum_\alpha \mathrm{d}(z^\alpha \eta^\alpha) \otimes e_\alpha)$, with a coordinate z^α on each copy of $\mathbb{R}_\times = \mathbb{R} \setminus \{0\}$, is a k-symplectic manifold. Conversely, if $(M \times \mathbb{R}_\times^k, \boldsymbol{\omega} = \sum_\alpha (z^\alpha \mathrm{d}\eta^\alpha + \mathrm{d}z^\alpha \wedge \eta^\alpha) \otimes e_\alpha)$ is a k-symplectic manifold, then $\ker \mathrm{d}\boldsymbol{\eta} \cap \ker \boldsymbol{\eta} = 0$. If $\ker \mathrm{d}\boldsymbol{\eta}$ has rank k, then $(M, \boldsymbol{\eta})$ is a k-contact manifold.*

It turns out that last requirement is necessary to ensure k-contact structure on M. Without it, it is impossible to prove that $\ker \mathrm{d}\boldsymbol{\eta}$ and $\ker \boldsymbol{\eta}$ have constant rank.
It is worth that an analogue of Theorem 4 can be enunciated to extend a k-contact form on M to a k-symplectic manifold in $\mathbb{R}_\times \times M$, but this relation suggests a notion of k-contact Hamiltonian vector field that is too restrictive for applications.

4. $\boldsymbol{\eta}$-Hamiltonian vector fields

Definition 5. Let (M, η) be a k-contact manifold with Reeb vector fields $R_1, \dots, R_k$. A vector field $X \in \mathfrak{X}(M)$ is *$\boldsymbol{\eta}$-Hamiltonian* if

$$\iota_X \mathrm{d}\eta^\alpha = \mathrm{d}h^\alpha - (R_\alpha h^\alpha)\eta^\alpha, \qquad \iota_X \eta^\alpha = -h^\alpha, \qquad \alpha = 1, \dots, k \tag{1}$$

for some vector-valued function $\boldsymbol{h} = \sum_\alpha h^\alpha \otimes e_\alpha \in \mathscr{C}^\infty(M, \mathbb{R}^k)$. We denote by $\mathfrak{X}_{\boldsymbol{\eta}}(M)$ the set of all the $\boldsymbol{\eta}$-Hamiltonian vector fields. A vector-valued function $\boldsymbol{h} = \sum_\alpha h^\alpha \otimes e_\alpha \in \mathscr{C}^\infty(M, \mathbb{R}^k)$ is *$\boldsymbol{\eta}$-Hamiltonian* if it induces an $\boldsymbol{\eta}$-Hamiltonian vector field as above. We denote by $\mathscr{C}_{\boldsymbol{\eta}}^\infty(M)$ the set of all $\boldsymbol{\eta}$-Hamiltonian functions.

Proposition 6. *Let X be an $\boldsymbol{\eta}$-Hamiltonian vector field on M with $\boldsymbol{\eta}$-Hamiltonian function $\boldsymbol{h}$. Then,*

$$\widetilde{X} = \sum_\alpha z^\alpha R_\alpha h^\alpha \frac{\partial}{\partial z^\alpha} + X$$

is an $\boldsymbol{\omega}$-Hamiltonian vector field on $\widetilde{M}$ (following the notations of Theorem 4) with $\boldsymbol{\omega}$-Hamiltonian function $\widetilde{\boldsymbol{h}} = \sum_\alpha z^\alpha h^\alpha \otimes e_\alpha$, i.e. $\iota_{\widetilde{X}} \boldsymbol{\omega} = \mathrm{d}\widetilde{\boldsymbol{h}}$.

Proposition 7. *Let $\boldsymbol{\eta}$ be a k-contact form on a manifold M. Then, every $X \in \mathfrak{X}_{\boldsymbol{\eta}}(M)$ is associated to a unique $\boldsymbol{f} \in \mathscr{C}_{\boldsymbol{\eta}}^\infty(M)$. Conversely, every $\boldsymbol{f} \in \mathscr{C}_{\boldsymbol{\eta}}^\infty(M)$ induces a unique $X \in \mathfrak{X}_{\boldsymbol{\eta}}(M)$. Moreover, $\mathscr{C}_{\boldsymbol{\eta}}^\infty(M)$ is a real vector space.*

Proposition 8. *Let $X_{\boldsymbol{f}}$ be the Hamiltonian vector field of $\boldsymbol{f} \in \mathscr{C}^\infty(M, \mathbb{R}^k)$ relative to a k-contact manifold $(M, \boldsymbol{\eta})$. Then, $\mathscr{L}_{X_{\boldsymbol{f}}} \boldsymbol{\eta} = -\sum_\alpha (R_\alpha f^\alpha) \eta^\alpha \otimes e_\alpha$; $X_{\boldsymbol{f}} \boldsymbol{f} = -\sum_\alpha (R_\alpha f^\alpha) f^\alpha \otimes e_\alpha$.*

Theorem 9. *Let $(M, \boldsymbol{\eta})$ be a k-contact manifold. Then, $\mathscr{C}_{\boldsymbol{\eta}}^\infty(M)$ is a Lie algebra relative to the Lie bracket given by*

$$\{\textstyle\sum_\alpha f^\alpha \otimes e_\alpha, \sum_\alpha g^\alpha \otimes e_\alpha\} = \sum_\alpha \{f^\alpha, g^\alpha\}_{\boldsymbol{\eta}} \otimes e_\alpha, \quad \alpha = 1, \dots, k,$$

where $\{f^\alpha, g^\alpha\}_{\boldsymbol{\eta}} = X_{\boldsymbol{f}} g^\alpha + g^\alpha R_\alpha f^\alpha$, with $\boldsymbol{f} = \sum_\alpha f^\alpha \otimes e_\alpha$ and $\boldsymbol{g} = \sum_\alpha g^\alpha \otimes e_\alpha$. Moreover, the mapping $\boldsymbol{f} \in \mathscr{C}_{\boldsymbol{\eta}}^\infty(M) \mapsto X_{\boldsymbol{f}} \in \mathfrak{X}_{\boldsymbol{\eta}}(M)$ is a Lie algebra isomorphism.

5. k-Contact Lie systems

Definition 10. A *k-contact Lie system* is a triple $(M, \boldsymbol{\eta}, X)$, where $\boldsymbol{\eta}$ is a k-contact form on M and X is a t-dependent vector field on M of the form $X = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha$, where $V = \langle X_1, \dots, X_r \rangle$ is an r-dimensional real Lie algebra of $\boldsymbol{\eta}$-Hamiltonian vector fields relative to $\boldsymbol{\eta}$. A *k-contact Lie system* is called *conservative* if the Hamiltonian functions associated to the vector fields in V are first integrals of all the Reeb vector fields of $(M, \boldsymbol{\eta})$.

Definition 11. A *k-contact Hamiltonian structure* is a triple $(M, \boldsymbol{\eta}, \boldsymbol{h} : \mathbb{R} \times M \to \mathbb{R})$, where $\boldsymbol{\eta}$ is a k-contact form on M and $\boldsymbol{h}$ is a t-dependent function on M taking values in $\mathbb{R}^k$ so that so that $\{\boldsymbol{h}_t\}_{t \in \mathbb{R}}$ are contained in a finite-dimensional Lie algebra of $\boldsymbol{\eta}$-Hamiltonian functions. A *k-contact Lie system admits a k-contact Hamiltonian structure* if X_t is the Hamiltonian vector field of $\boldsymbol{h}_t : x \in M \mapsto \boldsymbol{h}(t, x) \in \mathbb{R}^k$ for every $t \in \mathbb{R}$.

Proposition 12. *Every k-contact Lie system admits a unique k-contact Hamiltonian structure and vice versa.*

Note that given a k-contact Lie system, with k-contact Hamiltonian structure given by $\boldsymbol{h} = \sum_\alpha h^\alpha \otimes e_\alpha : \mathbb{R} \times M \to \mathbb{R}$, if

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X_{\boldsymbol{h}}^i(x) \frac{\partial}{\partial x^i}, \qquad i = 1, \dots, n,$$

one has that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \{h_t^\alpha, g\}_{\boldsymbol{\eta}} - g R_\alpha h_t^\alpha, \qquad \alpha = 1, \dots, k \quad \forall t \in \mathbb{R}.$$

In other words, if one of the $R_\alpha h_t^\alpha$ vanishes for every t, then a Casimir of the abstract Lie algebra $\langle \boldsymbol{h}_1, \dots, \boldsymbol{h}_r \rangle$ gives rise to constants of motion of the k-contact Lie system.

(**Riccati equation**) Consider the system of Riccati equations on $\mathbb{R}^4$ of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = a_0(t) + a_1(t) x_i + a_2(t) x_i^2, \qquad i = 1, 2, 3, 4.$$

There are three vector fields

$$X_0 = \sum_{i=1}^4 \frac{\partial}{\partial x_i} \qquad X_1 = \sum_{i=1}^4 x_i \frac{\partial}{x_i}, \qquad X_2 = \sum_{i=1}^4 x_i^2 \frac{\partial}{\partial x_i}.$$

The vector fields X_0, X_1, X_2 span a distribution of rank three almost everywhere and there exist functions h^0, h^1, h^2 such that $\mathrm{d}h^0 \wedge \mathrm{d}h^1 \wedge \mathrm{d}h^2 \neq 0$ and $X_0 = h^1 \partial_{h^2} - h^2 \partial_{h^1}$, $X_1 = h^2 \partial_{h^1} - h^1 \partial_{h^2}$, when one has a non-constant function h^3 that is a first integral of X_0, X_1, X_2. Then, $X_3 = \partial_{h^3}$ commutes with X_0, X_1, X_2 and $X_0 \wedge \dots \wedge X_3 \neq 0$. The vector fields X_0, X_1, X_2, X_3 are the fundamental vector fields of a locally transitive Lie group action of $GL(2, \mathbb{R})$ on $\mathbb{R}^4$, which is locally diffeomorphic to $GL(2, \mathbb{R})$. It can be proved that there exists a Lie algebra of Lie symmetries $Y_0, \dots, Y_3$ isomorphic to $\mathfrak{gl}(2, \mathbb{R})$, i.e. $[Y_i, X_j] = 0$ for $i, j = 1, \dots, 4$ with $Y_0 \wedge \dots \wedge Y_3 \neq 0$. Let $\Upsilon_0, \dots, \Upsilon_3$ be their dual forms. Then,

$$\mathrm{d}\Upsilon_1 = 2\Upsilon_0 \wedge \Upsilon_2, \qquad \mathrm{d}\Upsilon_3 = 0.$$

Hence, $\boldsymbol{\Upsilon} = \Upsilon_1 \otimes e_1 + \Upsilon_3 \otimes e_2$ define a two-contact form that is invariant relative to $X_0, \dots, X_3$. This implies that

$$0 = \mathscr{L}_{X_i} \Upsilon^j = \iota_{X_i} \mathrm{d}\Upsilon^j + \mathrm{d}\iota_{X_i} \Upsilon^j = 0, \qquad i = 1, 2, 3, \ j = 1, 3.$$

Define $h_i^j = -\iota_{X_i} \Upsilon^j$. Since $Y_1 = R_1, Y_3 = R_3$, one has that $R_k h_i^j = -R_k \iota_{X_i} \Upsilon^j = 0$ and $X_0, \dots, X_3$ are locally Hamiltonian relative to $\boldsymbol{\Upsilon}$. The above method can be generalised to construct k-contact Lie systems and study $\boldsymbol{\eta}$-Hamiltonian vector fields.

6. Applications

(**Control system**) Let us consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{\alpha=1}^5 b_\alpha(t) X_\alpha, \tag{2}$$

where $b_1(t), \dots, b_5(t)$ are arbitrary t-dependent functions and

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1^2 \frac{\partial}{\partial x_4} + 2x_1 x_2 \frac{\partial}{\partial x_5},$$
$$X_3 = \frac{\partial}{\partial x_3} + 2x_1 \frac{\partial}{\partial x_4} + 2x_2 \frac{\partial}{\partial x_5}, \qquad X_4 = \frac{\partial}{\partial x_4}, \qquad X_5 = \frac{\partial}{\partial x_5}.$$

The above vector fields span a nilpotent Lie algebra V of vector fields whose non-vanishing commutation relations read $[X_1, X_2] = X_3$, $[X_1, X_3] = 2X_4$, $[X_2, X_3] = 2X_5$. This makes (2) into a Lie system.
Let us consider the Lie algebra of symmetries of V, i.e. $[Y_i, X_j] = 0$ for $i, j = 1, \dots, 5$, given by the vector fields

$$Y_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + 2x_3 \frac{\partial}{\partial x_4} + x_2^2 \frac{\partial}{\partial x_5}, \qquad Y_2 = \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_5}, \qquad Y_k = \frac{\partial}{\partial x_k}, \ k = 3, 4, 5.$$

Since $Y_1 \wedge \dots \wedge Y_5 \neq 0$, the vector fields $Y_1, \dots, Y_5$ admit the corresponding dual forms given by

$$\Upsilon_1 = \mathrm{d}x_1, \qquad \Upsilon_2 = \mathrm{d}x_2, \qquad \Upsilon_3 = -x_2 \mathrm{d}x_1 + \mathrm{d}x_3,$$
$$\Upsilon_4 = -2x_3 \mathrm{d}x_1 + \mathrm{d}x_4, \qquad \Upsilon_5 = -x_2^2 \mathrm{d}x_1 - 2x_3 \mathrm{d}x_2 + \mathrm{d}x_5,$$

with

$$\mathrm{d}\Upsilon_1 = 0, \quad \mathrm{d}\Upsilon_2 = 0, \quad \mathrm{d}\Upsilon_3 = \Upsilon_1 \wedge \Upsilon_2, \quad \mathrm{d}\Upsilon_4 = 2\Upsilon_1 \wedge \Upsilon_3, \quad \mathrm{d}\Upsilon_5 = 2\Upsilon_2 \wedge \Upsilon_3.$$

We will show that $(\Upsilon_4, \Upsilon_5, \mathrm{d}\Upsilon_4, \mathrm{d}\Upsilon_5)$ give rise to 2-contact structure. Indeed, joint kernel of both one-forms and their derivatives are distributions $\mathcal{D}^C = \langle Y_1, Y_2, Y_3 \rangle$, $\mathcal{D}^R = \langle Y_4, Y_5 \rangle$, respectively. Clearly $\mathcal{D}^C \cap \mathcal{D}^R = \{0\}$, therefore we constructed a 2-contact structure. By verifying the conditions (1) for $X_1, \dots, X_5$, we obtain their $\boldsymbol{\eta}$-Hamiltonians

$$\boldsymbol{h}_1 = 2x_3 \otimes e_4 + x_2^2 \otimes e_5, \qquad \boldsymbol{h}_2 = -x_1^2 \otimes e_4 + (2x_3 - 2x_1 x_2) \otimes e_5,$$
$$\boldsymbol{h}_3 = -2x_1 \otimes e_4 - 2x_2 \otimes e_5, \qquad \boldsymbol{h}_4 = -1 \otimes e_4, \qquad \boldsymbol{h}_5 = -1 \otimes e_5.$$

(**Schwarz equation**) The Schwarz derivative is particularly related to the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = a, \quad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b(t)v, \qquad z, v, a \in \mathbb{C}, \tag{3}$$

for a certain complex t-dependent function $b(t)$. Note that (3) is a differential equation on $\mathcal{O} = \{(z, v, a) \in \mathrm{T}^2\mathbb{C} : v \neq 0\}$. In real coordinates

$$v_1 = \mathfrak{Re}(z), \quad v_2 = \mathfrak{Im}(z), \quad v_3 = \mathfrak{Re}(v), \quad v_4 = \mathfrak{Im}(v), \quad v_5 = \mathfrak{Re}(a), \quad v_6 = \mathfrak{Im}(a),$$

system (3) is associated with the t-dependent vector field

$$X = X_1 + 2b_R(t) X_2 + 2b_I(t) X_3,$$

where $b_R(t) = \mathfrak{Re}(b(t))$, $b_I(t) = \mathfrak{Im}(b(t))$, and

$$X_1 = \sum_{\alpha=1}^4 v_{\alpha+2} \frac{\partial}{\partial v_\alpha} + \frac{3}{2} \frac{2v_4 v_5 v_6 + (v_5^2 - v_6^2) v_3}{v_3^2 + v_4^2} \frac{\partial}{\partial v_5} + \frac{3}{2} \frac{2v_3 v_5 v_6 - v_4(v_5^2 - v_6^2)}{v_3^2 + v_4^2} \frac{\partial}{\partial v_6},$$
$$X_2 = v_3 \frac{\partial}{\partial v_5} + v_4 \frac{\partial}{\partial v_6}, \quad X_3 = -v_4 \frac{\partial}{\partial v_5} + v_3 \frac{\partial}{\partial v_6},$$
$$X_4 = -v_3 \frac{\partial}{\partial v_3} - v_4 \frac{\partial}{\partial v_4} - 2v_5 \frac{\partial}{\partial v_5} - 2v_6 \frac{\partial}{\partial v_6}, \quad X_5 = v_4 \frac{\partial}{\partial v_3} - v_3 \frac{\partial}{\partial v_4} + 2v_6 \frac{\partial}{\partial v_5} - 2v_5 \frac{\partial}{\partial v_6},$$
$$X_6 = \sum_{\alpha=1}^4 (-1)^\alpha v_{\alpha+2} \frac{\partial}{\partial v_\alpha} - \frac{3}{2} \frac{2v_3 v_5 v_6 - v_4(v_5^2 - v_6^2)}{2(v_3^2 + v_4^2)} \frac{\partial}{\partial v_5} + \frac{3}{2} \frac{2v_4 v_5 v_6 + v_3(v_5^2 - v_6^2)}{2(v_3^2 + v_4^2)} \frac{\partial}{\partial v_6}$$

span the Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C} \otimes \mathfrak{sl}_2$ as a real vector space. Following the procedure as in previous example, we obtain dual forms $\eta_1, \dots, \eta_6$ to symmetries $Y_1, \dots, Y_6$ of $X_1, \dots, X_6$, fulfilling

$$\mathrm{d}\eta_1 = -\eta_5 \wedge \eta_6 - \eta_1 \wedge \eta_4, \quad \mathrm{d}\eta_2 = -\eta_3 \wedge \eta_5 - \eta_4 \wedge \eta_2, \quad \mathrm{d}\eta_3 = -\eta_4 \wedge \eta_3 - \eta_5 \wedge \eta_2,$$
$$\mathrm{d}\eta_4 = -\eta_1 \wedge \eta_2 - \eta_3 \wedge \eta_6, \quad \mathrm{d}\eta_5 = -\eta_1 \wedge \eta_3 - \eta_6 \wedge \eta_2, \quad \mathrm{d}\eta_6 = -\eta_1 \wedge \eta_5 - \eta_6 \wedge \eta_4.$$

It can be shown, similarly as before, that $(\eta_4, \eta_5, \mathrm{d}\eta_4, \mathrm{d}\eta_5)$ give rise to the 2-contact structure, with $\mathcal{D}^C = \langle Y_1, Y_2, Y_3, Y_6 \rangle$, $\mathcal{D}^R = \langle Y_4, Y_5 \rangle$.

7. Conclusions and outlook

Lie systems have appear in many mathematical studies, like in the investigation of foliations, generalised distributions, Lie group actions, finite-dimensional Lie algebras, and in physics. Here, Lie systems are analysed throughout k-contact geometry, which leads to interesting k-contact and k-symplectic geometric results applicable in some physical and mathematical problems. The results so far encourage further development of this approach and searches for another examples.

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