Virtual constraints on Lie groups

18th Young Researchers Workshop in Geometry, Dynamics and Field Theory

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Virtual constraints

Virtual constraints are relations on the configuration variables of a control system which are imposed through feedback control and the action of actuators, instead of through physical connections such as gears or contact conditions with the the environment.

Virtual constraints

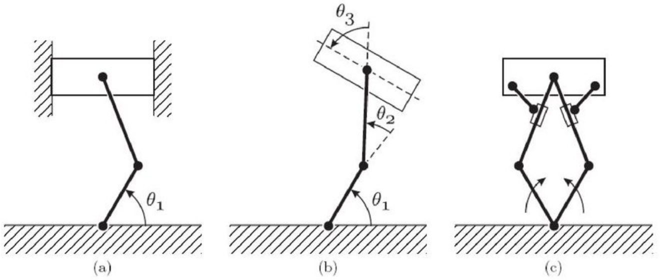


Figure: (a) Piston constrained to move in a cylinder. (b) Piston without constraints. (c) Piston constrained via additional links to evolve in the cylinder.

Virtual constraints

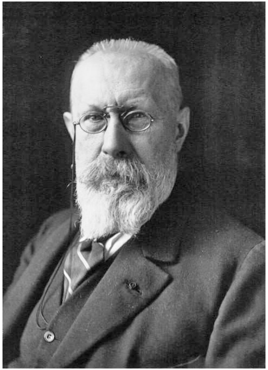
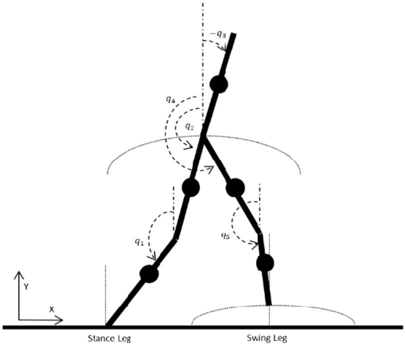


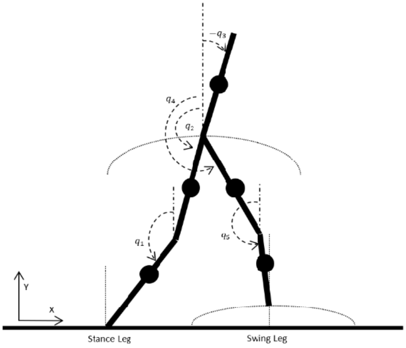
Figure: Paul M. Appell in *Exemple de mouvement d'un point assujetti à une liaison exprimèe par une relation non-linéaire entre les composantes de la vitesse*, (1911).

Virtual constraints



(A) Bipedal locomotion

Virtual constraints



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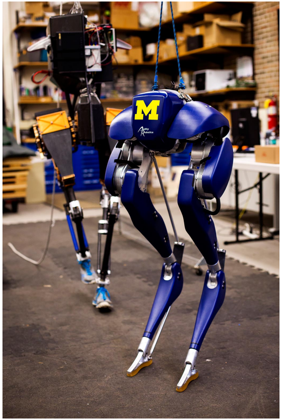
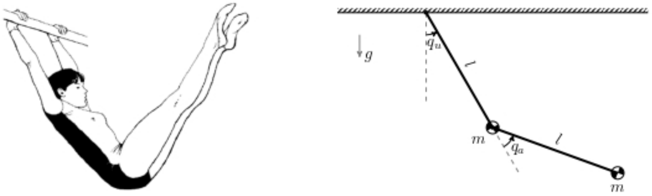


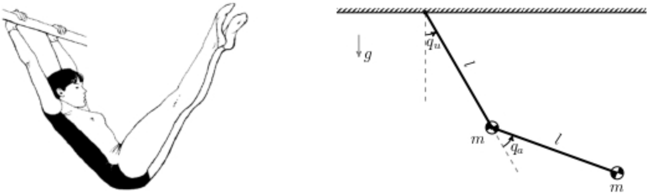
(B) University of Michigan's Cassie and MARLO robots.

Virtual constraints



(A) Gymnastics

Virtual constraints



(A) Gymnastics

(B) Prosthesis leg

(Physical) constraints

Nonholonomic constraints are determined by a non-integrable distribution $\mathcal{D} \subseteq TQ$.

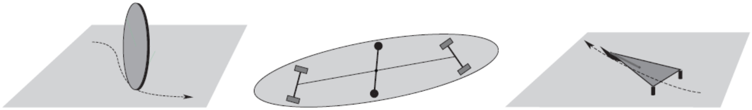


Figure: Examples of nonholonomic systems: Rolling disk, Snakeboard, and Sleigh

Riemannian Geometry & Lie Groups

Let $(G, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold where

- G is a smooth manifold
- $\langle\cdot,\cdot\rangle$ is a positive-definite symmetric covariant 2-tensor field called the *Riemannian metric*

For each point $g \in G$, we assign a positive-definite inner product $\langle\cdot,\cdot\rangle_g : T_gG \times T_gG \to \mathbb{R}$.

From this Riemannian metric we define the Levi-Civita affine connection, ∇, which

- the product rule $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,
- preserves the metric i.e. $\nabla\langle\cdot,\cdot\rangle = 0$ and
- it is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, for all vector fields X, Y on G and $f \in C^\infty(Q)$.

Riemannian Geometry & Lie Groups

Let G be a Lie group and $\mathfrak{g} = T_eG$ be its Lie algebra.
Consider the right-translation map by g denoted by $R_g : G \to G$ which provides a group action of G on itself under the relation $R_g h := hg$ for all $g, h \in G$.

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- A vector field, X, is called *right-invariant* if $(R_g)_* X = X$ for all $g \in G$ and for every Lie algebra element $\xi \in \mathfrak{g}$ we define a right-invariant vector field by $\xi_R(g) = (R_g)_* \xi$ for all $\xi \in \mathfrak{g}$ and $g \in G$.

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- Given any inner-product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, right-translation defines a *right-invariant* Riemannian metric $\langle\cdot,\cdot\rangle$ on G via the relation:

$$\langle u_g, v_g\rangle := \langle T_gR_{g^{-1}}u_g, T_gR_{g^{-1}}v_g\rangle_{\mathfrak{g}},$$

for all $g \in G$, $u_g, v_g \in T_gG$, where $T_gR_{g^{-1}}$ denotes the tangent map of $R_{g^{-1}}$ at the point g.

Riemannian Geometry & Lie Groups

With the Levi-Civita affine connection on G that corresponds to the right-invariant Riemannian metric above, we construct a bilinear map $\nabla^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\nabla^{\mathfrak{g}}_{\xi}\eta := \left(\nabla_{\xi_R}\eta_R\right)(e),$$

which is called *Riemannian* $\mathfrak{g}$*-connection*[1].
The equations of motion are

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = 0,$$

where $\xi = \dot{g}g^{-1}$.

[1] Goodman, Jacob R., Colombo, Leonardo J. Reduction by symmetry in obstacle avoidance problems on riemannian manifolds. SIAM Journal on Applied Algebra and Geometry, 2024, vol. 8, no 1, p. 26-53.

The Nonholonomic Constraints

The Nonholonomic Constraints

Consider constraints that defines a distribution, $\mathcal{D}$, given by a set of vectors that satisfy

$$\mu_i(g)\dot{g}^i = 0, \quad i = 1, \ldots, m, \text{ with } m < n.$$

$\mathcal{D}$ is a right invariant distribution if the fiber at each point is given by

$$\mathcal{D}_g = T_e R_g(\mathfrak{d}),$$

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Orthogonal projections $\mathfrak{P} : \mathfrak{g} \to \mathfrak{d}$ and $\mathfrak{Q} : \mathfrak{g} \to \mathfrak{d}^\perp$.
The nonholonomic $\mathfrak{d}$-connection $\nabla^{\mathfrak{d}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$\nabla^{\mathfrak{d}}_\xi \eta = \left(\nabla^{nh}_{\xi_R} \eta_R\right)(e) \quad \text{or equivalently} \quad \nabla^{\mathfrak{d}}_\xi \eta = \nabla^{\mathfrak{g}}_\xi \eta + (\nabla^{\mathfrak{g}}_\xi \mathfrak{Q})(\eta).$$

The equations of motion are

$$\dot{\xi} + \nabla^{\mathfrak{d}}_\xi \xi = 0.$$

The control system

Consider now the following control system

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = u^a f_a, \quad \dot{g} = T_e R_g(\xi),$$

where $\{f_a\}$ with $a = 1, \dots, m$ is the set of vectors which are called force vectors (or control forces) and define

$$\mathfrak{f} = \text{span}\{f_1, \dots, f_m\}$$

which is called the control input subspace.

A virtual nonholonomic constraint

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That is, there exist a control law $\hat{u} : \mathfrak{d} \to \mathbb{R}^m$ making the subspace $\mathfrak{d}$ invariant under the flow of the closed-loop system, i.e. $\xi(0) \in \mathfrak{d}$ and $\xi(t) \in \mathfrak{d}$, $\forall t \geq 0$.

Existence and Uniqueness of control

Theorem

If $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{d}$, then there exists a unique control law u^* making the subspace $\mathfrak{d}$ a virtual nonholonomic contraint associated with the controlled mechanical system above.

Example 1

Consider an homogeneous rigid body moving on the configuration space $G = SE(3) \simeq SO(3) \times \mathbb{R}^3$. Its Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \times \mathbb{R}^3$ is given by the base elements

$$e_1 = (\widehat{(1,0,0)}, \mathbf{0}), \; e_2 = (\widehat{(0,1,0)}, \mathbf{0}), \; e_3 = (\widehat{(0,0,1)}, \mathbf{0}),$$
$$e_4 = (\hat{\mathbf{0}}, 1, 0, 0), \; e_5 = (\hat{\mathbf{0}}, 0, 1, 0), \; e_6 = (\hat{\mathbf{0}}, 0, 0, 1),$$

where $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the hat map.
Equip $\mathfrak{g}$ with the following inner product

$$\langle \xi, \xi \rangle = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2)$$

for $\xi = (\omega_1, \omega_2, \omega_3, \dot{x}, \dot{y}, \dot{z}) \in \mathfrak{g}$ and $m, k \in \mathbb{R}$.

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Now suppose we would like to impose the constraints $\begin{cases} \omega_1 + \dot{y} = 0 \\ \omega_2 - \dot{x} = 0 \\ \omega_3 = 0 \\ \dot{z} = 0 \end{cases}$.

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A complementary vector space to $\mathfrak{d}$ is $\mathfrak{f} = \langle \{e_1 + e_5, e_2 - e_4, e_3, e_6\} \rangle$.

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Thus, by choosing any control vector from this space the above constraints are virtual nonholonomic constraints.

Induced constrained connection

Suppose that $\mathfrak{d}$ and $\mathfrak{f}$ are complementary so, the projections $\mathfrak{p}:\mathfrak{g}\to\mathfrak{d}$ and $\mathfrak{q}:\mathfrak{g}\to\mathfrak{f}$ associated to the direct sum are well-defined. Using the Riemannian $\mathfrak{g}$-connection we define the bilinear map $\nabla^{\mathfrak{c}}$

$$\nabla^{\mathfrak{c}}_{\xi}\eta = \nabla^{\mathfrak{g}}_{\xi}\eta + (\nabla^{\mathfrak{g}}_{\xi}\mathfrak{q})(\eta), \qquad (1)$$

where $\xi,\eta\in\mathfrak{g}$ which we call *induced constrained connection* associated to $\mathfrak{d}$ and $\mathfrak{f}$.

Remark

Note that if the control input subspace $\mathfrak{f}$ is orthogonal to the controlled invariant subspace $\mathfrak{d}$ then the constraint dynamics are precisely the nonholonomic dynamics. Indeed, since $\mathfrak{f}$ and $\mathfrak{d}$ are orthogonal then $\mathfrak{f}=\mathfrak{d}^{\perp}$ and $\mathfrak{P}=\mathfrak{p}$ (as well as $\mathfrak{Q}=\mathfrak{q}$). Thus, the induced constrained connection $\nabla^{\mathfrak{c}}$ is precisely the nonholonomic $\mathfrak{d}$-connection $\nabla^{\mathfrak{d}}$. Hence, the dynamics derived from the two connections coincide.

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2. $\dim V = \dim W_1 + \dim W_2$, i.e., the dimensions of W_1 and W_2 are complementary with respect to the ambient space dimension.

We consider that the dimension of an affine space is defined to be the dimension of its modelled space.

A virtual affine nonholonomic constraint

Consider, again, the control system

$$\dot{\xi} + \nabla^{\mathfrak{d}}_{\xi}\xi = u^a f_a, \quad \dot{g} = T_e R_g(\xi),$$

where the force vectors define an affine subspace $\mathfrak{a}$ modelled by the subspace $\mathfrak{d}$.

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Equivalently, as before, there exists a control law making $\mathfrak{a}$ invariant for the closed-loop system, i.e., $\xi(t) \in \mathfrak{a}$, whenever $\xi(0) \in \mathfrak{a}$.

Existence and Uniqueness of Control

Theorem

If the affine subspace $\mathfrak{a}$ and the control input subspace $\mathfrak{f}$ are transversal, then there exists a unique control function making the affine subspace, $\mathfrak{a}$, a virtual affine nonholonomic constraint associated with the control system.

Example 2

Consider a rigid body with an internal rotor. The configuration space is $G = \mathrm{SO}(3) \times \mathbb{S}^1$ with Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \times \mathbb{R}$.
Equip $\mathfrak{g}$ with the following inner product

$$\langle \xi, \xi \rangle = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2 + 2J\omega_3 \dot{\alpha} + J\dot{\alpha}^2,$$

where $\xi = (\omega, \dot{\alpha}) = (\omega_1, \omega_2, \omega_3, \dot{\alpha}) \in \mathfrak{g}$.

Example 2

Suppose, now, we would like to control the rigid rotor such as to satisfy the constraint

$$(J - k\lambda_3)\omega_3 + J(1-k)\dot{\alpha} = p,$$

for some $k \in \mathbb{R}$ and $p \in \mathbb{R}$.

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These constraints define an affine subspace, $\mathfrak{a}$, of $\mathfrak{g}$ whose model vector subspace is $\mathfrak{d}$ with

$$\mathfrak{d} = \text{span}\{e_1, e_2, e_4 + \frac{J(1-k)}{J - k\lambda_3} e_3\}.$$

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By choosing a complementary input force vector space $\mathfrak{f}$ defined by

$$\mathfrak{f} = \text{span}\{-\frac{J}{D} e_3 + \frac{\lambda_3}{D} e_4\},$$

where $D = J\lambda_3 - J^2$, we are sure that there is a unique control law making the affine constraint control invariant.

References

- A. Anahory Simoes, E. Stratoglou, A. Bloch, L. J. Colombo, Virtual nonholonomic constraints: A geometric approach, Automatica, Volume 155, 2023.
- E. Stratoglou, A. Anahory Simoes, A. Bloch, L. J. Colombo, Virtual Affine Nonholonomic Constraints , Proceedings of the Geometric Science of Information Conference, 2023.
- E. Stratoglou, A. Anahory Simoes, A. Bloch, L. J. Colombo, On the Geometry of Virtual Nonlinear Nonholonomic Constraints (under review).
- E. Stratoglou, A. Anahory Simoes, A. Bloch, L. J. Colombo, Virtual Constraints on Lie Groups (under review).

Future Work

For future work, we aim to

- extend the theory to homogeneous spaces and infinite dimensional Lie groups
- broaden the spectrum of application of virtual constraints
- implement the theory to robotic systems

Thank you!