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Hamiltonian and Lagrangian perspectives on integrable systems

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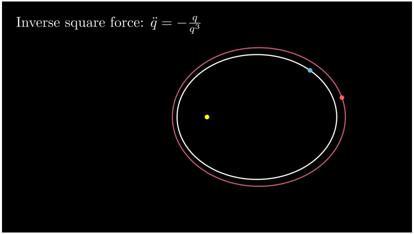
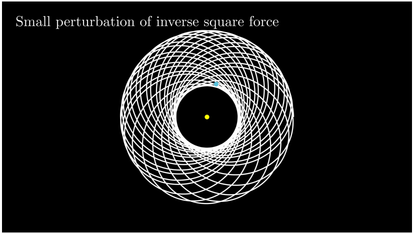
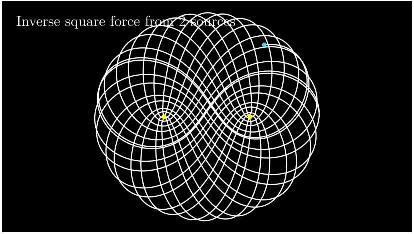
Introduction



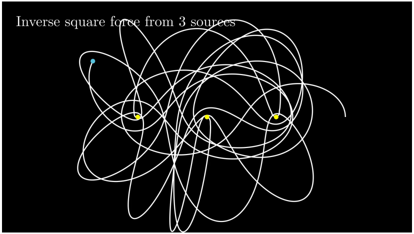
Kepler problem: (super)integrable



Central force problem: integrable



2 gravitating centres: integrable



3 gravitating centres: nonintegrable

Integrable systems

Most nonlinear differential equations are impossible to solve explicitly.

Integrable systems are the exception. They have some underlying structure which helps. Often, this structure consists of a number of symmetries:

An equation is integrable if has sufficiently many symmetries.

Each symmetry, in it infinitesimal form, defines a differential equation. Hence:

An equation is integrable if it is part of a sufficiently large family of compatible equations.

A common interpretation of "compatible" is given in terms of Hamiltonian mechanics.

Hamiltonian mechanics

Hamilton function

$$H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R} : (q,p) \mapsto H(q,p)$$

Dynamics given by canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Flow consists of symplectic maps and preserves H.

Poisson bracket of two functions on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{H, q_i\}, \qquad \dot{p}_i = \{H, p_i\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{H, f\}$$

In particular: f is conserved if and only if $\{H,f\} = 0$.

Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow $\phi^t_{H_i}$: N dynamical systems
- Each H_i is a conserved quantity for all flows.
- The dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- Liouville-Arnold theorem: if this foliation is compact, its leaves are tori.

The proof I gave of the Liouville-Arnold theorem loosely follows the book: [VI Arnold. Mathematical Methods of Classical Mechanics. Springer 1989.]

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Variational principle in multi-time

A simultaneous solution is a function

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

such that $\frac{\partial q}{\partial t_1}$ generates the dynamical system and $\frac{\partial q}{\partial t_i}$ its symmetries.

Pluri-Lagrangian principle

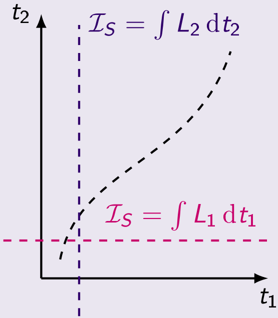
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve S in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

q_I: mixed partial derivative of q defined by a string $I = t_{i_1} \dots t_{i_k}$.

If I is empty then $q_I = q$.

Denote by $\dfrac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\frac{\delta_i f}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial f}{\partial q_{It_i^\alpha}} \qquad = \frac{\partial f}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial f}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial f}{\partial q_{It_i^2}} - \dots$$

Consider $\mathcal{L}[q] = \sum_i L_i[q]\,\mathrm{d}t_i$ with $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$.

Multi-time Euler-Lagrange equations / Multi-time EL eqns

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

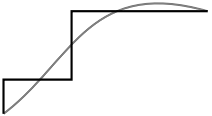
Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{It_i}} = \dfrac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I,$

Derivation of the multi-time Euler-Lagrange equations

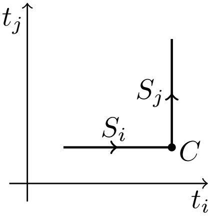
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$.

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.

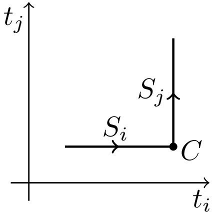


Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i ($i \neq 1$), we get

$$\delta \int_{S_i} L_i \, \mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial q_I} \delta q_I \, \mathrm{d}t_i$$

$$= \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial q_{I t_i^\alpha}} \delta q_{I t_i^\alpha} \, \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \, \mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta q_I} \delta q_I \mathrm{d}t_i + \sum_I \frac{\partial L_i}{\partial q_{I t_i}} \delta q_I \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Multi-time Euler-Lagrange equations

$$\frac{\delta_i L_i}{\delta q_I} = 0 \quad \forall I \not\ni t_i, \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta q_{I t_i}} = \frac{\delta_j L_j}{\delta q_{I t_j}}$$

Newtonian system with symmetries

Consider $\mathcal{L} = \sum_i L_i \, \mathrm{d}t_i$ with

$$\begin{aligned} L_1 &= \tfrac{1}{2}|q_1|^2 - V_i(q) \\ L_i &= q_1 \cdot q_i - H_i(q, q_1) \end{aligned}$$

Multi-time Euler-Lagrange equations

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad \frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_1} = 0 \qquad \Rightarrow \quad q_{11} = -V'(q)$$

$$\frac{\delta_i L_i}{\delta q} = 0 \quad \Rightarrow \quad \frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_i} = 0 \qquad \Rightarrow \quad q_{1i} = -\frac{\partial H_i}{\partial q}$$

$$\frac{\delta_i L_i}{\delta q_1} = 0 \quad \Rightarrow \quad \frac{\partial L_i}{\partial q_1} = 0, \qquad \Rightarrow \quad q_i = \frac{\partial H_i}{\partial q_1}$$

$$\frac{\delta_i L_i}{\delta q_i} = \frac{\delta_j L_j}{\delta q_j} \quad \Rightarrow \quad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}} \qquad \text{trivially satisfied}$$

Exterior derivative of $\mathcal{L}$

As before, consider $L_i = q_1 \cdot q_i - H_i(q, q_1)$

Multi-time Euler-Lagrange equations:

$$q_i = \frac{\partial H_i}{\partial q_1} \qquad \text{and} \qquad q_{1i} = -\frac{\partial H_i}{\partial q}$$

Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_j}{\mathrm{d}t_i} - \frac{\mathrm{d}L_i}{\mathrm{d}t_j} = \left(q_{1i} + \frac{\partial H_i}{\partial q}\right)\left(q_j - \frac{\partial H_j}{\partial q_i}\right) - \left(q_{1j} + \frac{\partial H_j}{\partial q}\right)\left(q_i - \frac{\partial H_i}{\partial q_i}\right)$$
$$- \{H_i, H_j\}$$

Observation: $\mathrm{d}\mathcal{L}$ has a "double zero" on solutions.

$\mathrm{d}\mathcal{L} = 0^2$ is key to the Lagrangian multiform approach.

Relation between $\mathrm{d}\mathcal{L}$, double zeroes, and Poisson brackets is emphasised in:

Caudrelier, Dell'Atti, Singh. Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems arXiv:2307.07339

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Hamiltonian PDEs

∞-dimensional phase space, so geometric aspects are much more subtle.

We can still consider Poisson brackets. For example, for PDEs in two variables (x and t) of KdV-type:

$$\left\{ \int F(q, q_x, q_{xx}, \ldots) \,\mathrm{d}x, \int G(q, q_x, q_{xx}, \ldots) \,\mathrm{d}x \right\} = \int \frac{\delta F}{\delta q} \frac{\partial}{\partial x} \frac{\delta G}{\delta q} \,\mathrm{d}x$$

Then a Hamiltonian H induces dynamics by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int F \,\mathrm{d}x = \left\{ \int H \,\mathrm{d}x, \int F \,\mathrm{d}x \right\}.$$

Integrable if we have an infinite hierarchy $H_2, H_3, \ldots$ such that

$$\left\{ \int H_i \,\mathrm{d}x, \int H_j \,\mathrm{d}x \right\} = 0.$$

This implies that the flows commute: q is a function of $t_1 = x$, t_2, t_3, ...

Side note: often we have a bi-Hamiltonian structure: two “compatible” Poisson brackets, giving rise to a recursion that generates the hierarchy.

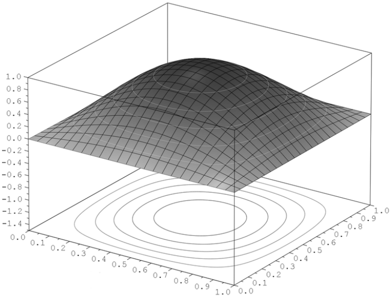
Lagrangian 2-forms

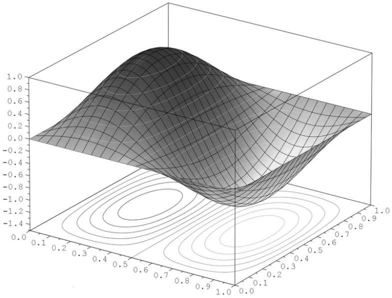
Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_S \mathcal{L}[q]$ is critical on all smooth surfaces S in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





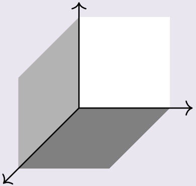
Multi-time Euler-Lagrange equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{I t_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{I t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{I t_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

where we identify $t_1 = x$.

The differentiated equations $q_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxx} + 3q_x^2\right)$ and $q_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$ are Lagrangian with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$
$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!).

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta q} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta q} = 0$ yield

$$q_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxx} + 3q_x^2\right),$$
$$q_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxxxx} + 10 q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3\right).$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta q_x} = \dfrac{\delta_{32} L_{32}}{\delta q_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta q_x} = \dfrac{\delta_{23} L_{23}}{\delta q_{t_2}}$ yield

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10 q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

  the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

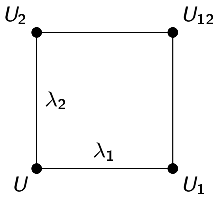
Discretisation of Hamiltonian systems

Hamiltonian ODE	$\rightarrow$	symplectic map
Liouville integrable system	$\rightarrow$	commuting symplectic maps? or symplectic map with conserved quantities?
Hamiltonian PDE	$\rightarrow$	partial difference equation: multisymplectic lattice equation?
Integrable Hamiltonian PDE	$\rightarrow$	?

Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

- Subscripts of U denote lattice shifts.
- λ_1, λ_2 are parameters.
- Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



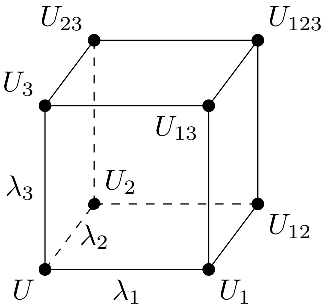
Discrete analogue of commuting flows:

Consistency around the cube

The three ways of calculating U_{123}, using

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

and its shifts, give the same result.



Example: lattice potential KdV:

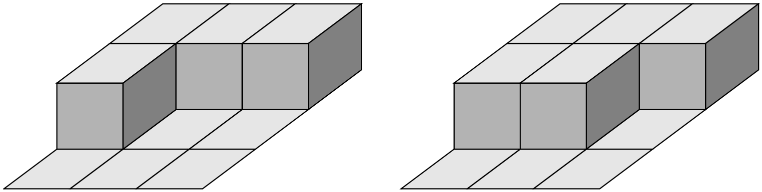
$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$$

Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

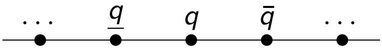
the action $\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.



The discrete and continuous variational principles are the same.

Semi-discrete systems

Consider particles on a line: 1 discrete dimension, many continuous times



Denote $\quad q_1 = q_{t_1} = \frac{\mathrm{d}q}{\mathrm{d}t_1}, \quad q_{11} = q_{t_1 t_1} = \frac{\mathrm{d}^2 q}{\mathrm{d}t_1^2}, \quad$ etc.

Toda lattice: exponential nearest-neighbour interaction

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}).$$

Part of a hierarchy. First symmetry:

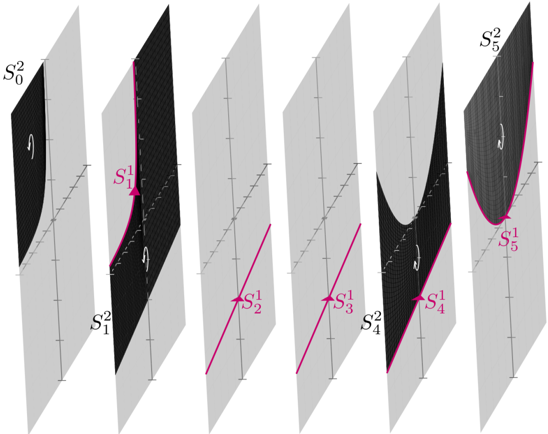
$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

Semi-discrete geometry

Consider the case with only 1 discrete direction: $\mathbb{Z} \times \mathbb{R}^N$

A semi-discrete surface is a collection of surfaces and curves in $\mathbb{R}^N$, each at a specified point in $\mathbb{Z}$

Intuition: curves where the surface jumps to a different value of $\mathbb{Z}$



Semi-discrete geometry

- Consider (scalar) functions q of $\mathbb{Z} \times \mathbb{R}^N$.
  Superscript to emphasise lattice position: $q^{[k]} = q(k, t_1, \ldots, t_N)$
- Semi-discrete 2-form $\mathcal{L}[q]$ is part 1-form and part 2-form:
  components L_{0j} are integrated over curves,
  components L_{ij} integrated over surfaces.
- We have semi-discrete versions of the exterior derivative, the boundary, and Stokes theorem

Variational principle

Look for $q(k, t_1, \ldots, t_N)$ such that the action

$$\int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every semi-discrete surface S.

Toda lattice

Lagrangians ("0" for discrete direction)

$$L_{01} = \frac{1}{2} q_1^2 - \exp(\bar{q} - q)$$

$$L_{02} = q_1 q_2 - \frac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

$$L_{12} = -\frac{1}{4} \left(q_2 - q_{11} - q_1^2 \right)^2$$

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \qquad \rightarrow \qquad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q})$$

$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \qquad \rightarrow \qquad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \qquad \rightarrow \qquad \frac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \frac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0$$

Lagrangian formalism produces a non-trivial PDE at a single lattice site.

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Summary

The Hamiltonian theory of (Liouville-)integrable systems is powerful for ODEs, with some generalisations to PDEs.

Lagrangian multiform (or pluri-Lagrangian) theory applies to ODEs and PDEs, discrete, semi-discrete and continuous.

Much work to do:

- Multiforms as a tool for constructing solutions?
- Full development of semi-discrete case?
- Semi-discrete multiforms in geometric numerical integration?
- Applications to gauge theory?
- Application to quantum integrable systems, path integrals, ...?

Selected references

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