Floer theory in the analysis of Hamiltonian systems

Jagna Wiśniewska,
Universitat Politecnica de Catalunya

February 22nd 2024, Warszawa

From celestial mechanics to Floer theory

Hamiltonian dynamics

T^*M - phase space, cotangent bundle;
H - energy, Hamiltonian function
$H : T^*M \to \mathbb{R}$.

Hamilton's equations:

$$\partial_t q = \partial_p H,$$
$$\partial_t p = -\partial_q H.$$

William R. Hamilton

Hamiltonian dynamics

T^*M - phase space, cotangent bundle;
H - energy, Hamiltonian function
$H : T^*M \to \mathbb{R}$.

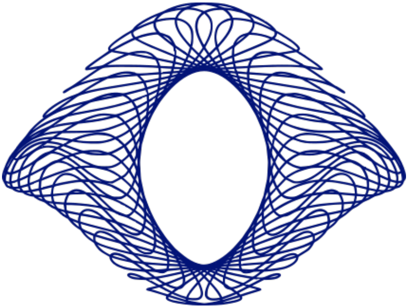
Hamilton's equations:

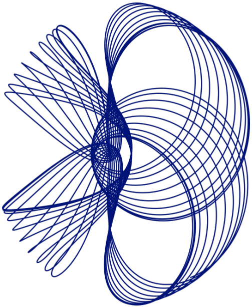
$$\partial_t q = \partial_p H,$$
$$\partial_t p = -\partial_q H.$$

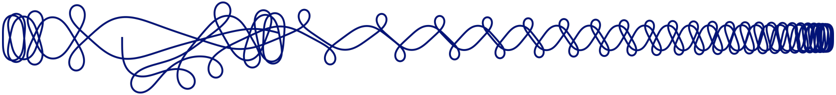
- locally well defined;
- globally hard to compute;
- unstable under small perturbations.

William R. Hamilton

Examples of solutions to the Hamilton's equations:







Symplectic geometry

(M^{2n}, ω) - symplectic manifold, $d\omega = 0$ and $\omega^n \neq 0$,
H - Hamiltonian function, $H : M \to \mathbb{R}$,
X_H - Hamiltonian vector field, $\omega(\cdot, X_H) = dH$.

Symplectic geometry

(M^{2n}, ω) - symplectic manifold, $d\omega = 0$ and $\omega^n \neq 0$,
H - Hamiltonian function, $H : M \to \mathbb{R}$,
X_H - Hamiltonian vector field, $\omega(\cdot, X_H) = dH$.

The cotangent bundle $(T^*M, \omega_0 = dq \wedge dp)$ is an example of a symplectic manifold.

Symplectic geometry

(M^{2n}, ω) - symplectic manifold, $d\omega = 0$ and $\omega^n \neq 0$,
H - Hamiltonian function, $H : M \to \mathbb{R}$,
X_H - Hamiltonian vector field, $\omega(\cdot, X_H) = dH$.

The cotangent bundle $(T^*M, \omega_0 = dq \wedge dp)$ is an example of a symplectic manifold.

Observe: Energy is preserved by the Hamiltonian flow.

Symplectic geometry

(M^{2n}, ω) - symplectic manifold, $d\omega = 0$ and $\omega^n \neq 0$,
H - Hamiltonian function, $H : M \to \mathbb{R}$,
X_H - Hamiltonian vector field, $\omega(\cdot, X_H) = dH$.

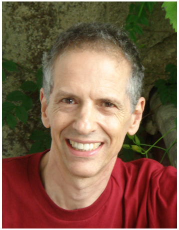
The cotangent bundle $(T^*M, \omega_0 = dq \wedge dp)$ is an example of a symplectic manifold.

Observe: Energy is preserved by the Hamiltonian flow.
Goal: Analyze Hamiltonian flow on a fixed energy level.

Weinstein conjecture

Y - a Liouville vector field, $d\iota_Y\omega = \omega$,

$\Sigma \subseteq M$ - a hypersurface of contact type,

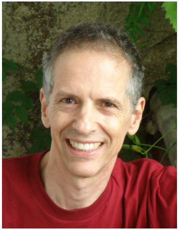


Alan Weinstein

Weinstein conjecture

Y - a Liouville vector field, $d\iota_Y\omega = \omega$,

$\Sigma \subseteq M$ - a hypersurface of contact type, in neighbourhood of Σ exists a Liouville vector field transverse to Σ.



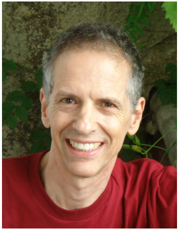
Alan Weinstein

Weinstein conjecture

Y - a Liouville vector field, $d\iota_Y\omega = \omega$,
$\Sigma \subseteq M$ - a hypersurface of contact type, in neighbourhood of Σ exists a Liouville vector field transverse to Σ.

Weinstein conjucture, 1978

Every compact contact type hypersurface admits periodic orbits.



Alan Weinstein

Rabinowizt action functional - variational approach

$(M, \omega = d\lambda)$ - an exact symplectic manifold,

$\Sigma \subseteq M$ - a hypersurface of exact contact type,

H - a Hamiltonian, $H^{-1}(0) = \Sigma$ and $dH\big|_{\Sigma} \neq 0$.

Paul Rabinowitz

Rabinowizt action functional - variational approach

$(M, \omega = d\lambda)$ - an exact symplectic manifold,

$\Sigma \subseteq M$ - a hypersurface of exact contact type,

H - a Hamiltonian, $H^{-1}(0) = \Sigma$ and $dH\big|_\Sigma \neq 0$.

Rabinowitz action functional

$$\mathcal{A}^H(v, \eta) := \int_{S^1} \lambda(\partial_t v)dt - \eta \int_{S^1} H(v)dt \qquad \text{for} \quad v : S^1 \to M, \ \eta \in \mathbb{R}.$$

Rabinowitz action functional - variational approach

$(M, \omega = d\lambda)$ - an exact symplectic manifold,

$\Sigma \subseteq M$ - a hypersurface of exact contact type,

H - a Hamiltonian, $H^{-1}(0) = \Sigma$ and $dH\big|_{\Sigma} \neq 0$.

Rabinowitz action functional

$$\mathcal{A}^H(v, \eta) := \int_{S^1} \lambda(\partial_t v)dt - \eta \int_{S^1} H(v)dt \qquad \text{for} \quad v : S^1 \to M, \;\; \eta \in \mathbb{R}.$$

$$(v, \eta) \in \operatorname{Crit} \mathcal{A}^H \quad \Longleftrightarrow \quad \partial_t v = \eta X_H(v) \quad \text{and} \quad v(t) \in \Sigma \quad \forall\; t.$$

Floer theory

(M, ω) - a compact symplectic manifold

Floer theory

(M, ω) - an aspherical compact symplectic manifold,
$\int_S \omega = 0$ for all $[S] \in \pi_2(M)$,

Andreas Floer

Floer theory

(M, ω) - an aspherical compact symplectic manifold,
$\int_S \omega = 0$ for all $[S] \in \pi_2(M)$,
$\{H_t\}_{t \in S^1}$ - a periodic Hamiltonian, $H_t : M \to \mathbb{R}$.

Andreas Floer

Floer theory

(M, ω) - an aspherical compact symplectic manifold, $\int_S \omega = 0$ for all $[S] \in \pi_2(M)$,

$\{H_t\}_{t \in S^1}$ - a periodic Hamiltonian, $H_t : M \to \mathbb{R}$.

Symplectic action functional

For $v : S^1 \to M, x : D^2 \to M, \partial x = v$:

$$\mathcal{A}(v) := \int_{D^2} x^* \omega - \int_{S^1} H_t(v) dt.$$

$$v \in \operatorname{Crit} \mathcal{A} \iff \partial_t v = X_{H_t}(v).$$

Andreas Floer

Floer theory

Idea:

Constructing Morse-type homology for the symplectic action functional.

Symplectic action functional

For $v : S^1 \to M, x : D^2 \to M, \partial x = v$:

$$\mathcal{A}(v) := \int_{D^2} x^*\omega - \int_{S^1} H_t(v)dt.$$

$$v \in \operatorname{Crit} \mathcal{A} \iff \partial_t v = X_{H_t}(v).$$

Andreas Floer

Floer theory

Idea:

Constructing Morse-type homology for the symplectic action functional.

Floer trajectories

A Floer trajectory is a solution $u : \mathbb{R} \times S^1 \to M$ to

$$\partial_s u(s,t) = -\nabla \mathcal{A}(u)(s,t).$$

Andreas Floer

Floer theory

Idea:

Constructing Morse-type homology for the symplectic action functional.

Floer homology

complex - $\mathrm{Crit}(\mathcal{A}) \otimes \mathbb{Z}$,

boundary operator - counting Floer trajectories.

Andreas Floer

Floer theory

Theorem, Floer 1989:

The Floer homology is well defined and isomorphic to the singular homology.

Floer homology

complex - $\text{Crit}(\mathcal{A}) \otimes \mathbb{Z}$,

boundary operator - counting Floer trajectories.

Andreas Floer

Floer theory

Theorem, Floer 1989:

The Floer homology is well defined and isomorphic to the singular homology.

Estimate the number of 1-periodic orbits

$$\#\,\mathrm{Crit}(\mathcal{A}) \geq \sum_{k=0}^{2n} \dim H_k(M).$$

Andreas Floer

Floer theory

Equations of motion:

$$\partial_t q = \partial_p H_t(p, q),$$
$$\partial_t p = -\partial_q H_t(p.q).$$

Floer homology

$\Longleftrightarrow$

Topology

Singular homology.

$$H_*(M)$$

Critical set of the Rabinowitz action functional

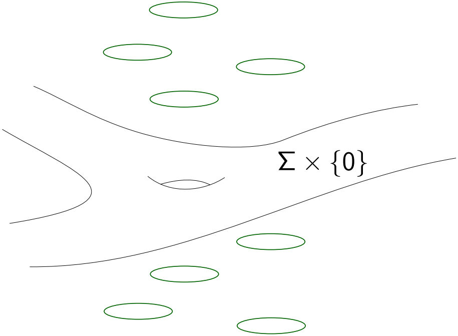
Idea:

Building Floer-type homology for the Rabinowitz action functional.

Critical set of the Rabinowitz action functional

Idea:

Building Floer-type homology for the Rabinowitz action functional.



$(v, \eta) \in \operatorname{Crit} \mathcal{A}^H$ if and only if

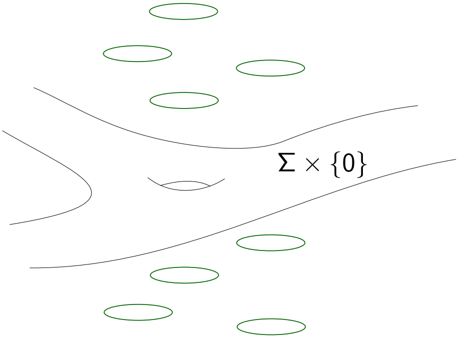
$$\partial_t v = \eta X_H(v) \quad \text{and} \quad v(t) \in \Sigma,$$

for all $t \in S^1$.

Rabinowitz Floer homology

Idea:

Building Floer-type homology for the Rabinowitz action functional.



$f : \mathrm{Crit}(\mathcal{A}^H) \to \mathbb{R}$ - a Morse function

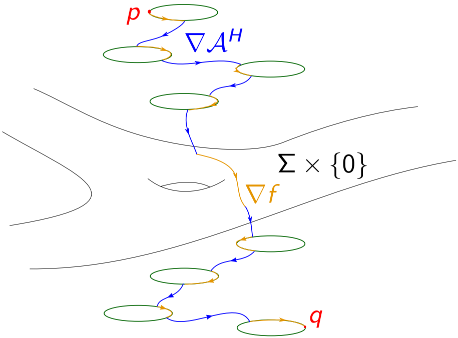
Building RFH:

complex - $\mathrm{Crit}(f) \otimes \mathbb{Z}_2$,

Rabinowitz Floer homology

Idea:

Building Floer-type homology for the Rabinowitz action functional.



$f : \mathrm{Crit}(\mathcal{A}^H) \to \mathbb{R}$ - a Morse function

Building RFH:

complex - $\mathrm{Crit}(f) \otimes \mathbb{Z}_2$,

boundary operator - counting *cascades*.

Rabinowitz Floer homology

Theorem: Cieliebak and Frauenfelder, 2009

Rabinowitz Floer homology is well defined for compact contact type hypersurfaces in exact symplectic manifolds convex at ∞.

Kai Cieliebak and Urs Frauenfelder

Rabinowitz Floer homology

Properties of RFH:

1. If Σ has no periodic orbits then

$$RFH_*(\Sigma) = H_{*+n-1}(\Sigma).$$

Rabinowitz Floer homology

Properties of RFH:

1. If Σ has no periodic orbits then

$$RFH_*(\Sigma) = H_{*+n-1}(\Sigma).$$

Observe:

1st property makes RFH a good tool for answering Weinstein conjecture.

Rabinowitz Floer homology

Properties of RFH:

1. If Σ has no periodic orbits then
$$RFH_*(\Sigma) = H_{*+n-1}(\Sigma).$$
2. RFH is invariant under compact perturbations:
if Σ_s is a smooth family of contact type perturbations of Σ_0, then
$$RFH(\Sigma_s) = RFH(\Sigma_0).$$

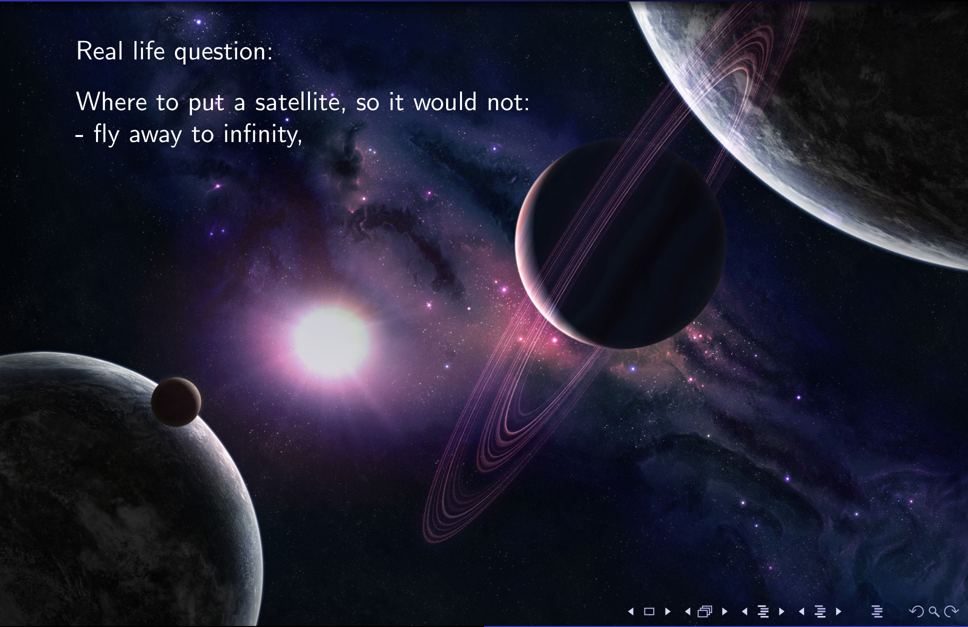
Observe:

1st property makes RFH a good tool for answering Weinstein conjecture.

Back to reality and physical applications

Real life question:

Where to put a satellite, so it would not:
- fly away to infinity,



Back to reality and physical applications

Real life question:

Where to put a satellite, so it would not:
- fly away to infinity,
- crash into one of the planets?

Back to reality and physical applications

Real life question:

Where to put a satellite, so it would not:
- fly away to infinity,
- crash into one of the planets?

Physical systems often have non-compact energy hypersurfaces.

Back to reality and physical applications

Real life question:

Where to put a satellite, so it would not:
- fly away to infinity,
- crash into one of the planets?

Physical systems often have non-compact energy hypersurfaces.

Mathematical question:

Can we define and compute Rabinowitz Floer homology for non-compact hypersurfaces?

Rabinowitz Floer homology for Tentacular hyperboloids

Tentacular hyperboloids

Tentacular hyperboloids are the zero level sets of Hamiltonians:

$$H(x,y) := \frac{1}{2}\left(x^T A_0 x + y^T A_1 y - 1\right) \qquad x \in \mathbb{R}^{2m}, y \in \mathbb{R}^{2k}.$$

where

A_0 - symmetric, positive definite;

A_1 - symmetric, J-hyperbolic,

Federica Pasquotto

Rabinowitz Floer homology for Tentacular hyperboloids

Tentacular hyperboloids

Tentacular hyperboloids are the zero level sets of Hamiltonians:

$$H(x,y) := \frac{1}{2}\left(x^T A_0 x + y^T A_1 y - 1\right) \qquad x \in \mathbb{R}^{2m}, y \in \mathbb{R}^{2k}.$$

where

A_0 - symmetric, positive definite;

A_1 - symmetric, J-hyperbolic, i.e. JA_1 has no imaginary eigenvalues.

$$J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

Federica Pasquotto

Rabinowitz Floer homology for Tentacular hyperboloids

Tentacular hyperboloids

Tentacular hyperboloids are the zero level sets of Hamiltonians:

$$H(x,y) := \frac{1}{2}\left(x^T A_0 x + y^T A_1 y - 1\right) \qquad x \in \mathbb{R}^{2m}, y \in \mathbb{R}^{2k}.$$

Topology of tentacular hyperboloids

$$\Sigma := H^{-1}(0), \quad \Sigma \simeq \mathbb{S}^{2m+k-1} \times \mathbb{R}^k$$

Federica Pasquotto

Rabinowitz Floer homology for Tentacular hyperboloids

Theorem, Pasquotto, W. 2017:

The Rabinowitz Floer homology for tentacular hyperboloids is well defined.

Federica Pasquotto

Rabinowitz Floer homology for Tentacular hyperboloids

Theorem, Pasquotto, W. 2017:

The Rabinowitz Floer homology for tentacular hyperboloids is well defined.

Moreover this Rabinowitz Floer homology is:

invariant under compact perturbations,

Federica Pasquotto

Rabinowitz Floer homology for Tentacular hyperboloids

Theorem, Pasquotto, W. 2017:

The Rabinowitz Floer homology for tentacular hyperboloids is well defined.

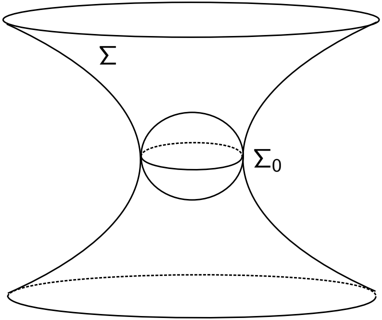
Moreover this Rabinowitz Floer homology is:

- invariant under compact perturbations,
- isomorphic to $H_{*+n-1}(\Sigma)$, whenever Σ has no periodic orbits.

Federica Pasquotto

Computing RFH for tentacular hyperboloids

All the periodic orbits of $\Sigma := H^{-1}(0)$ lie on $\Sigma_0 \times \{0\} \subseteq \Sigma$,

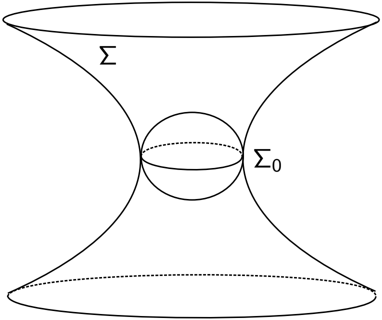


Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

All the periodic orbits of $\Sigma := H^{-1}(0)$ lie on $\Sigma_0 \times \{0\} \subseteq \Sigma$, where

$$\Sigma_0 := H_0^{-1}(0), \qquad H_0(x) := \frac{1}{2}\left(x^T A_0 x - 1\right).$$



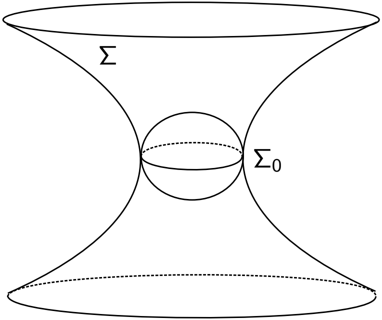
Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1 - k - m, -m, \\ 0 & \text{otherwise.} \end{cases}$$



Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1 - k - m, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Sigma \simeq S^{2m+k-1} \times \mathbb{R}^k$, so

$$H_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 0, 2m + k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1 - k - m, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Sigma \simeq S^{2m+k-1} \times \mathbb{R}^k$, so

$$H_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 0, 2m + k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently

$$RFH_*(\Sigma) \neq H_{*+m+k-1}(\Sigma).$$

Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1-k-m, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary: Weinstein conjecture

If Σ_s is a family of a compact, contact type perturbations

$$RFH_*(\Sigma) \neq H_{*+m+k-1}(\Sigma).$$

Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1-k-m, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary: Weinstein conjecture

If Σ_s is a family of a compact, contact type perturbations of a tentacular hyperboloid Σ,

$$RFH_*(\Sigma) \neq H_{*+m+k-1}(\Sigma).$$


Alex Fauck and Will Merry

Computing RFH for tentacular hyperboloids

Theorem: Fauck, Merry, W. 2020

If Σ is a tentacular hyperboloid, then its Rabinowitz Floer homology is equal to:

$$RFH_*(\Sigma) = \begin{cases} \mathbb{Z}_2 & * = 1-k-m, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary: Weinstein conjecture

If Σ_s is a family of a compact, contact type perturbations of a tentacular hyperboloid Σ, then each Σ_s carries a periodic orbit.

$$RFH_*(\Sigma) \neq H_{*+m+k-1}(\Sigma).$$


Alex Fauck and Will Merry

Thank you for your attention :)