Skinner–Rusk formalism of action-dependent field theories

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Outline

Where we are

Intuition from contact mechanics

Contact manifolds are the odd-dimensional counterpart of symplectic manifolds: building their geometry requires adding one extra variable s to the symplectic phase space (q, p).

The extra variable is absorbed by the dynamics:

$$\eta(\mathcal{X}_{\mathrm{H}}) = -\mathrm{H}\,, \quad i_{\mathcal{X}_{\mathrm{H}}}\mathrm{d}\eta = \mathrm{dH} - \mathcal{R}(\mathrm{H})\eta$$

whose flows give:

$$\dot{q} = \frac{\partial \mathrm{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathrm{H}}{\partial q} - \frac{\partial \mathrm{H}}{\partial s}\, p, \qquad \dot{s} = p\frac{\partial \mathrm{H}}{\partial p} - \mathrm{H}.$$

The extra term $\frac{\partial \mathrm{H}}{\partial s}\, p$ represents the dissipation. If the Hamiltonian is independent of the s, then the usual symplectic Hamilton's equations are recovered.

Multicontact geometry aims at achieving the same generalization with a family s^μ instead of a single s.

What does “action-dependent” mean?

In the usual Hamilton's variational principle, the action is given by

$$S[\phi] = \int_M L(x, \phi, \partial\phi)\, \mathrm{d}^m x.$$

The Herglotz variational principle generalizes the classical one, allowing the Lagrangian L to depend explicitly on the action. Namely, it introduces new variables s^μ that describe the flow of the action:

$$L = L\big(x^\mu, y^A, y^A_\mu, s^\mu\big), \qquad \frac{\partial s^\mu}{\partial x^\mu} = L.$$

The new Lagrangian yields the dissipative terms: the dependence on s^μ breaks conservation.

The big picture

Conservative field theory

- Described by *multisymplectic geometry*
- Conservation laws: $\partial_\mu j^\mu = 0$
- Generalization of symplectic formalism for conservative field theory

Action-dependent theory

- Described by *multicontact geometry*
- Non-conservation: nonzero divergences
- Generalization of contact formalism for dissipative field theory

The Lagrangian / Hamiltonian may depend on extra action variables s^μ, which encode the flow of action $\Rightarrow$ dissipation.

Where we are

Multicontact manifolds

Definition. A form $\Theta \in \Omega^m(\mathcal{M})$ with $\dim \mathcal{M} > m$ is

- **premulticontact** if $\quad \ker \mathrm{d}\Theta \neq \{0\}$;
- **multicontact** if, in addition, $\quad \ker \Theta \cap \ker \mathrm{d}\Theta = \{0\}$.

On a premulticontact manifold, the generalized definition of the Reeb distribution is:

$$\mathfrak{R} = \left\{ R \in \Gamma(\ker \omega) \;\middle|\; \iota_R \mathrm{d}\Theta \in \mathcal{A}^m(\ker \omega) \right\}$$

where $\mathcal{A}^m(\ker \omega)$ is the set of m-forms that vanishes on $\ker \omega$.

If ω is closed, $\mathfrak{R}$ is involutive.

$\mathfrak{R}$ is what makes the structure "contact-like" rather than purely multisymplectic.

Special multicontact manifolds

Definition. Given $\mathcal{M}$ with $\omega \in \Omega^m(\mathcal{M})$ a closed m-form and $\Theta \in \Omega^m(\mathcal{M})$ s.t.

(1) $\operatorname{rank} \ker \omega = N$,

(2) $\operatorname{rank} \mathfrak{R} = m + k$,

(3) $\operatorname{rank} (\ker \omega \cap \ker \Theta \cap \ker \mathrm{d}\Theta) = k$,

(4) $\mathcal{A}^{m-1}(\ker \omega) = \{\iota_R \Theta \mid R \in \mathfrak{R}\}$.

then we call $(\mathcal{M},\ \Theta,\ \omega)$ a **special (pre)multicontact manifold**.

If we restrict ourselves to a bundle $\mathcal{M} \xrightarrow{\kappa} M$ with $\omega = \kappa^* \omega_M$ and $\Theta \in \Omega^m(\mathcal{M})$ that form a special (pre)multicontact manifold, then Darboux-type coordinates are guaranteed to exist.

The dissipation form and the modified differential

For a special (pre)multicontact manifold, there exists a unique 1-form σ_Θ, the dissipation form such that

$$\sigma_\Theta \wedge \iota_R \Theta = \iota_R \mathrm{d}\Theta, \qquad \forall\, R \in \mathfrak{R}.$$

It defines a modified (Lichnerowicz) differential that replaces d in every field equation:

$$\bar{\mathrm{d}}\beta = \mathrm{d}\beta + \sigma_\Theta \wedge \beta$$

- $\sigma_\Theta = 0 \Rightarrow$ recover the conservative (multisymplectic) theory.
- $\sigma_\Theta \neq 0 \Rightarrow$ the $\sigma_\Theta \wedge \beta$ term is exactly the dissipation.
- If σ_Θ is closed $\Rightarrow \bar{\mathrm{d}}^2 = 0$ and defines a cohomology.

Where we are

Lagrangian formalism

Multivelocities phase bundle $\mathcal{P} = J^1\pi \times_M \bigwedge^{m-1} \mathrm{T}^* M$, coordinates $(x^\mu, y^A, y^A_\mu, s^\mu)$.

Lagrangian form: $\Theta_{\mathcal{L}} = -\iota_{\mathrm{J}} \mathrm{d}\mathcal{L} - \mathcal{L} + \mathrm{d}S$. In particular:

$$\left(\frac{\partial^2 L}{\partial y^A_\mu \partial y^B_\nu}\right) \text{ invertible } \iff \Theta_{\mathcal{L}} \text{ is a special multicontact form.}$$

The **Lagrangian problem**: finding holonomic sections $\psi_{\mathcal{L}} = j^1\phi : M \to \mathcal{P}$ solution to

$$(j^1\phi)^* \Theta_{\mathcal{L}} = 0 \quad \text{and} \quad (j^1\phi)^* \iota_{\boldsymbol{X}} \bar{\mathrm{d}} \Theta_{\mathcal{L}} = 0 \quad \forall \boldsymbol{X} \in \mathfrak{X}(\mathcal{P})$$

The same problem can be stated equivalently using multivector fields and connections.

Herglotz–Euler–Lagrange equations

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^A_\mu}\right) = \frac{\partial L}{\partial y^A} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^A_\mu}, \qquad \frac{\partial s^\mu}{\partial x^\mu} = L.$$

The dissipative term was absent in the multisymplectic case.

The SOPDE condition

We need fields that satisfy the second-order condition $y^A_\mu = \dfrac{\partial y^A}{\partial x^\mu}$.

- For regular L: holonomy is automatic.
- For singular L in the Lagrangian formalism: it must be imposed.

Keep this in mind: the unified formalism gets this for free, straight from the geometry.

Hamiltonian formalism

Restricted and extended multimomenta phase bundle $\mathcal{P}^* = J^{1*}\pi \times_M \bigwedge^{m-1} \mathrm{T}^* M$ and $\tilde{\mathcal{P}} = \mathcal{M}\pi \times_M \bigwedge^{m-1} \mathrm{T}^* M$, with coordinates $(x^\mu, y^A, p_A^\mu, s^\mu)$ and $(x^\mu, y^A, p_A^\mu, p, s^\mu)$.

From a Hamiltonian section h (function H) one builds the Hamiltonian form:

$$\Theta_{\mathcal{H}} = -p_A^\mu \, \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{H}\,\mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu .$$

The **Hamiltonian problem**: finding sections $\psi_{\mathcal{H}} : M \to \mathcal{P}^*$ solution to

$$(\psi_{\mathcal{H}})^*\Theta_{\mathcal{H}} = 0 \quad \text{and} \quad \psi_{\mathcal{H}}^* \, \iota_{\mathbf{Y}} \, \bar{\mathrm{d}}\Theta_{\mathcal{H}} = 0 \quad \forall \mathbf{Y} \in \mathfrak{X}(\mathcal{P}^*)$$

Herglotz–Hamilton–de Donder–Weyl equations

$$\frac{\partial y^A}{\partial x^\mu} = \frac{\partial \mathrm{H}}{\partial p_A^\mu}, \qquad \frac{\partial p_A^\mu}{\partial x^\mu} = -\left(\frac{\partial \mathrm{H}}{\partial y^A} + p_A^\mu \frac{\partial \mathrm{H}}{\partial s^\mu}\right).$$

- **(Hyper)regular** L: $\mathcal{F}L$ is a (local) diffeomorphism.
- **Singular** L: restrict to $\mathcal{P}^*_\circ = \mathcal{F}L(\mathcal{P})$ (almost-regular case).

Two separate pictures

Lagrangian		Hamiltonian
velocities y^A_μ		momenta p^μ_A
$\Theta_{\mathcal{L}}$ on $\mathcal{P}$	$\xrightarrow{\text{Legendre}}$	$\Theta_{\mathcal{H}}$ on $\mathcal{P}^*$
Herglotz–Euler–Lagrange		Herglotz–HDW

- Linked by the Legendre map $\mathcal{F}L$ that is introduced *by hand* as a fibre derivative.
- Singular case: each picture needs its own care (constraints, holonomy...).

Can we treat both at once, and get $\mathcal{F}L$ and holonomy automatically?

Where we are

The unified idea

Skinner–Rusk: keep velocities y^A_μ and momenta p^μ_A together on one bundle.

1. Treat them as *independent*; impose their link as a *compatibility condition*.
2. The theory is always singular, regardless of whether L is regular.
3. Holonomy and the Legendre map emerge from the equations, not by hand.
4. Lagrangian & Hamiltonian formalisms are recovered as *projections*.

Setup: the Pontryagin bundles

Extended and **restricted** jet–multimomentum (Pontryagin) bundles:

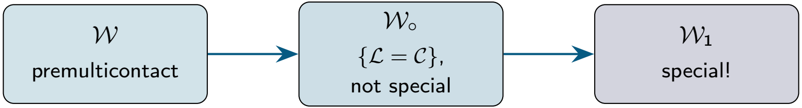
$$\mathcal{W} = (J^1\pi \times_E \mathcal{M}\pi) \times_M \bigwedge^{m-1} \mathrm{T}^* M,$$
$$\mathcal{W}_r = (J^1\pi \times_E J^{1*}\pi) \times_M \bigwedge^{m-1} \mathrm{T}^* M.$$

Canonical structures on $\mathcal{W}$:

$$\underbrace{\mathcal{C}(w) = (p + p_A^\mu y_\mu^A)\,\mathrm{d}^m x}_{\text{coupling form}} \qquad \underbrace{\Theta_{\mathcal{W}} = -\rho_2^*\Theta + \mathrm{d}(\bar{\tau}_2^*\theta)}_{\text{canonical } m\text{-form}}$$

Dynamics lives on the Hamiltonian submanifold $\mathcal{W}_\circ = \{\, w \in \mathcal{W} \mid \mathcal{L}(w) = \mathcal{C}(w) \,\}$
(1-codimensional, diffeomorphic to $\mathcal{W}_r$).

The key step: from $\mathcal{W}_\circ$ to the special submanifold $\mathcal{W}_1$



$\Theta_\circ = \mathrm{j}_\circ^*\Theta_{\mathcal{W}}$ is premulticontact on $\mathcal{W}_\circ$, but not special. It becomes special exactly on

$$\mathcal{W}_1 = \left\{ \frac{\partial L}{\partial y^A_\mu} = p^\mu_A \right\} = \operatorname{graph} \mathcal{F}L$$

On $\mathcal{W}_1$, the rank conditions for a special premulticontact structure hold, so there exists a unique

$$\sigma_\Theta = \frac{\partial L}{\partial s^\mu}\, \mathrm{d}x^\mu \,.$$

Three equivalent ways to state the dynamics

The Premulticontact Lagrange–Hamilton problem stated in terms of Multivector fields:

The unified field equations

$$\iota_{\mathbf{X}_\circ}\Theta_\circ \underset{\mathcal{W}_1}{=} 0, \qquad \iota_{\mathbf{X}_\circ}\bar{\mathrm{d}}\Theta_\circ \underset{\mathcal{W}_1}{=} 0 \quad \text{for } \{X_\circ\} \subset \mathfrak{X}^m(\mathcal{W}_\circ).$$

The same problem can be stated in three equivalently ways using sections, $\psi_\circ : M \to \mathcal{W}_\circ$, multivector fields, $\mathbf{X}_\circ \in \mathfrak{X}^m(\mathcal{W}_\circ)$ or Ehresmann connections: $\nabla_\circ$.

The resultant solutions automatically satisfy:

- Holonomy: $(X_\circ)^A_\mu = y^A_\mu \;\Rightarrow\; y^A_\mu = \partial y^A/\partial x^\mu$
- Legendre map: $\partial L/\partial y^A_\mu - p^\mu_A = 0$ (compatibility constraint $= \mathcal{W}_1$)
- Herglotz–Lagrange–Hamilton equations: $\partial_\mu s^\mu = L, \quad \partial_\mu p^\mu_A = \frac{\partial L}{\partial y^A} + \frac{\partial L}{\partial s^\mu} p^\mu_A$

Singular theories: the constraint algorithm

If the Hessian is singular, the field equations may be incompatible on $\mathcal{W}_1$.

$$\mathcal{W}_1 \xrightarrow{\text{tangency}} \mathcal{W}_2 \longrightarrow \cdots \longrightarrow \mathcal{W}_f$$

- Impose tangency of the solutions $\Rightarrow$ new constraints $\Rightarrow$ smaller submanifold.
- Iterate until a final constraint submanifold $\mathcal{W}_f$ where tangent solutions exist.
- Crucially: semi-holonomy is preserved at every step.

Recovering both classical formalisms

$$\begin{tikzcd}
& \mathcal{W}_\circ \supset \mathcal{W}_1 \arrow[dl, "\mathrm{pr}_\circ"'] \arrow[dr, "\bar{\mathrm{pr}}_\circ"] & \\
\text{Lagrangian } \mathcal{P} \arrow[rr, "\mathcal{F}L"'] & & \text{Hamiltonian } \mathcal{P}^*
\end{tikzcd}$$

- $\psi_{\mathcal{L}} = \mathrm{pr}_\circ \circ \psi_\circ$ is holonomic and solves the Lagrangian equations (canonical lift of $\phi : M \to E$).
- $\psi_{\mathcal{H}} = \mathcal{F}L \circ \psi_{\mathcal{L}}$ solves the Hamilton–de Donder–Weyl equations.
- Converse holds too and it is valid for regular and singular cases.

Why the unified formalism is worth it

- No ad hoc Legendre map: $\mathcal{F}L$ is a compatibility condition of the equations.
- Holonomy for free: no need to impose the SOPDE condition by hand.
- One algorithm: a single constraint procedure handles both Lagrangian and Hamiltonian formalisms for the singular case.
- Both pictures at once: Lagrangian and Hamiltonian are projections of one object.

Where we are

Example: Maxwell

Spacetime M (dim 4); $C \to M$ the bundle of connections of a $U(1)$ principal bundle. Coordinates $(x^\mu, A_\mu, A_{\mu,\nu}, p^{\mu,\nu}, s^\mu)$.

Electromagnetic Lagrangian with a linear dissipation term:

$$L = -\frac{1}{4\mu_\circ} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta} - A_\alpha J^\alpha - \gamma_\alpha s^\alpha, \qquad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

- J^α: four-current; γ_α: dissipation coefficients; g: Lorentzian metric.
- The dissipation form is $\sigma_\Theta = \gamma_\mu \, \mathrm{d}x^\mu$ (closed iff $\partial_\nu \gamma_\mu = \partial_\mu \gamma_\nu$).

Example: Maxwell

The Skinner–Rusk equations give, on $\mathcal{W}_1$,

$$\underbrace{p^{\mu,\alpha} = \frac{1}{\mu_\circ} g^{\mu\nu} g^{\alpha\beta} F_{\beta\nu},}_{\text{Legendre map}} \qquad \underbrace{A_{\mu,\alpha} = \partial_\alpha A_\mu,}_{\text{holonomy}}$$

with no new constraints, and the field equation

$$\partial_\alpha F^{\alpha\mu} = \mu_\circ J^\mu - \gamma_\alpha F^{\alpha\mu} \quad \text{(dissipative Maxwell).}$$

Together with the Bianchi identity, in 3+1 language:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_\circ} - \boldsymbol{\gamma} \cdot \boldsymbol{E}, \quad \nabla \times \boldsymbol{B} = \mu_\circ \boldsymbol{j} + \mu_\circ \epsilon_\circ \partial_t \boldsymbol{E} - \boldsymbol{\gamma} \times \boldsymbol{B} + \frac{\gamma_\circ}{c} \boldsymbol{E}.$$

Example: Maxwell

Taking $\gamma_\mu = (\gamma_\circ, 0)$ and applying the curl:

Damped wave equation

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2 \nabla^2 \boldsymbol{E} + c\,\gamma_\circ \frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{\epsilon_\circ}\frac{\partial \boldsymbol{j}}{\partial t} = 0$$

(and an analogous equation for $\boldsymbol{B}$).

- The dissipative term has a clear physical meaning: it damps the waves.
- $\gamma = 0$ recovers ordinary Maxwell.

Where we are

Conclusions

What we did

- A unified Skinner–Rusk formalism for action-dependent field theories, built on $\mathcal{W}_\circ$ and made *special* on $\mathcal{W}_1$.
- The Legendre map appears as a compatibility constraint.
- Holonomy is automatic.
- A single constraint algorithm handles both theories.
- Both classical formalisms recovered (for the regular *and* singular cases).
- Validated on *dissipative Maxwell* theory.

Open problems & future work

Extensions of the formalism

- Higher-order action-dependent field theories.
- Study the cohomology given by the $\bar{\mathrm{d}}$ operator when the dissipative 1-form is closed.

Physical applications

- Modified, action-dependent theories of gravitation.
- Action-dependent Kaluza–Klein & Yang–Mills.

Thank you!