Reconstruction in Lie-Poisson reduction for field theories.

2nd International Workshop on Geometric Field Theory

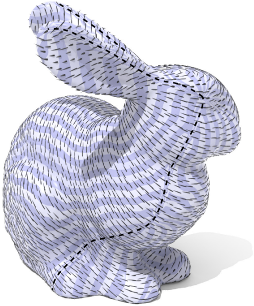
Miguel A. Berbel (Comillas Pontifical University)

July 6, 2026

Table of Contents

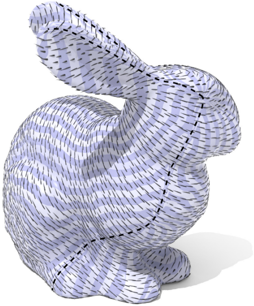
Table of Contents

Sections of bundles



$$
\begin{array}{c} TM \\ \downarrow \\ M \end{array}
$$

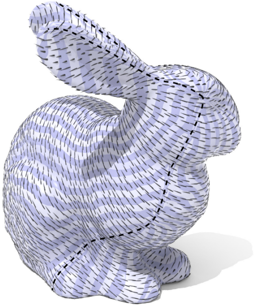
Sections of bundles



$$\begin{array}{c} TM \\ \big\downarrow \\ M \end{array} \qquad\qquad \begin{array}{c} T^*M_4 \\ \big\downarrow \\ M_4 \end{array}$$

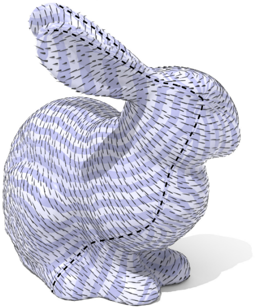
Sections of bundles

1 Euler–Poincaré Reduction



$$\begin{array}{ccc} TM & T^*M_4 & P \\ \downarrow & \downarrow & \downarrow \\ M & M_4 & M \end{array}$$

Sections of bundles



$$
\begin{array}{cccc}
TM & T^*M_4 & P & \mathbb{R}\times Q \\
\downarrow & \downarrow & \downarrow & \downarrow \\
M & M_4 & M & \mathbb{R}
\end{array}
$$

$$
\begin{tikzcd}
J^1P \arrow[d, "\pi_{P,J^1P}"'] \\
P \arrow[d, "\pi_{M,P}"] \\
M
\end{tikzcd}
$$

Jet Bundle

J^1P affine bundle over P, for any $y \in P$

$$J^1_yP = \{\gamma \in T^*_xM \otimes T_yP, T\pi_{M,P} \circ \gamma = \mathrm{id}_xM, x = \pi_{M,P}(y)\}.$$

1st Order Jet Bundles

$$
\begin{tikzcd}
J^1P \arrow[d, "\pi_{P,J^1P}"'] \\
P \arrow[d, "\pi_{M,P}"] \\
M
\end{tikzcd}
$$

Jet Bundle

J^1P affine bundle over P, for any $y \in P$

$$J^1_yP = \{\gamma \in T^*_xM \otimes T_yP, T\pi_{M,P} \circ \gamma = \mathrm{id}_xM, x = \pi_{M,P}(y)\}.$$

Vector bundle: $T^*M \otimes VP$ where $V_yP = \ker(T_y\pi_{M,P})$.

$$
\begin{tikzcd}
J^1P \arrow[d, "\pi_{P,J^1P}"'] \\
P \arrow[d, "\pi_{M,P}"] \\
M
\end{tikzcd}
$$

Hamilton's Variational Principle

A section $s : M \to P$ is a **solution** of the Lagrangian system $(J^1P, \mathcal{L})$ if

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_U \mathcal{L}(j^1 s_\varepsilon) = 0$$

for all compactly supported vertical variations, s_ε, of s.

$$
\begin{tikzcd}
J^1P \arrow[d, "\pi_{P,J^1P}"'] \\
P \arrow[d, "\pi_{M,P}"] \\
M
\end{tikzcd}
$$

Hamilton's Variational Principle

A section $s : M \to P$ is a **solution** of the Lagrangian system $(J^1P, \mathcal{L})$ if

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_U \mathcal{L}(j^1 s_\varepsilon) = 0$$

for all compactly supported vertical variations, s_ε, of s.

Euler–Lagrange equations:

$$\frac{\delta L}{\delta s} - \operatorname{div}^P \frac{\delta L}{\delta j^1 s} = 0,$$

Bundle of Connections

$$\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \left(J^1P\right)/G \arrow[d, "\bar{\pi}"] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M
\end{tikzcd}$$

$L : J^1P \to \mathbb{R}$, G-invariant.

$$\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & (J^1P)/G \arrow[d, "\bar{\pi}"] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M
\end{tikzcd}$$

$L : J^1P \to \mathbb{R}$, G-invariant.

Reduced Lagrangian

$l : (J^1P)/G \to \mathbb{R}, \qquad l\big(q(\gamma)\big) = L(\gamma).$

Bundle of Connections

$$
\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d, "\bar{\pi}"] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M
\end{tikzcd}
$$

$L : J^1P \to \mathbb{R},$ G-invariant.

Reduced Lagrangian

$l : \mathcal{C}(P) \to \mathbb{R}, \qquad l\big(q(\gamma)\big) = L(\gamma).$

$j^1s : U \subset M \to J^1P.$

Bundle of Connections

$$\begin{tikzcd} J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d, "\bar{\pi}"] \\ P \arrow[r, "G\text{-ppal bundle}"'] & M \end{tikzcd}$$

$L : J^1P \to \mathbb{R}$, $\qquad$ G-invariant.

Reduced Lagrangian

$l : \mathcal{C}(P) \to \mathbb{R}, \qquad l\big(q(\gamma)\big) = L(\gamma).$

$j^1s : U \subset M \to J^1P.$

Reduced section

$\sigma : U \to \mathcal{C}(P), \qquad \sigma(x) = q\left(j^1_x s\right).$

Theorem (Euler–Poincaré reduction)

Given principal connection $\mathcal{A}$ on P, the following are equivalent:

1. *The variational principle $\delta \int_U L\big(j^1_x s\big)\, \chi_{\mathrm{vol}} = 0$ holds, for vertical free variations δs.*
2. *$s : U \to P$ satisfies the Euler–Lagrange equations for L.*
3. *The variational principle $\delta \int_U l\big(\sigma(x)\big)\, \chi_{\mathrm{vol}} = 0$ holds, using variations of the form*

$$\delta\sigma = \nabla^{\mathcal{A}}\eta - [\sigma^{\mathcal{A}}, \eta],$$

   where $\eta : U \to \mathrm{ad}\, P$.

4. *$\sigma : U \to \mathcal{P}$ satisfies The Euler–Poincaré equations*

$$\mathcal{EP}(l)(\sigma) := \mathrm{div}^{\mathcal{A}} \frac{\delta l}{\delta\sigma} + \mathrm{ad}^*_{\sigma^{\mathcal{A}}} \frac{\delta l}{\delta\sigma} = 0.$$

Theorem (Euler–Poincaré reduction)

The following are equivalent:

1. *The variational principle* $\delta \int_U L\big(j^1_x s\big)\, \chi_{\mathrm{vol}} = 0$ *holds, for vertical free variations* δs.
2. $s : U \to P$ *satisfies the Euler–Lagrange equations for* L.
3. *The variational principle* $\delta \int_U l\big(\sigma(x)\big)\, \chi_{\mathrm{vol}} = 0$ *holds, using variations of the form*
$$\delta\sigma = \nabla^\sigma \eta,$$
where $\eta : U \to \operatorname{ad} P$.
4. $\sigma : U \to \mathcal{P}$ *satisfies The Euler–Poincaré equations*
$$\mathcal{EP}(l)(\sigma) := \operatorname{div}^\sigma \frac{\delta l}{\delta \sigma} = 0.$$

Reconstruction

$$
\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "j^1 s"'] \arrow[u, bend left, "\sigma"'] \arrow[l, bend right, "s"']
\end{tikzcd}
$$

Connection: $\sigma = q \circ j^1 s$

$\sigma : U \to \mathcal{C}(P)$ comes from a section $s : U \to P$ if and only if σ is a *flat connection* with *trivial holonomy*.

$$
\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "j^1 s"] \arrow[u, bend left, "\sigma"] \arrow[l, bend right, "s"']
\end{tikzcd}
$$

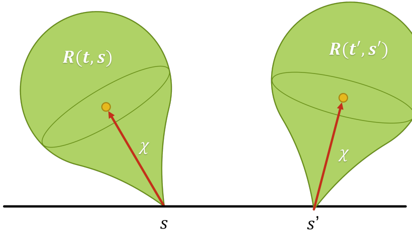
Connection: $\sigma = q \circ j^1 s$

$\sigma : U \to \mathcal{C}(P)$ comes from a section $s : U \to P$ if and only if σ is a *flat connection* with *trivial holonomy*.

$$
\mathcal{EL}(L)(s) = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathcal{EP}(l)(\sigma) = 0, \\ \mathrm{Curv}(\sigma) = 0. \end{cases}
$$

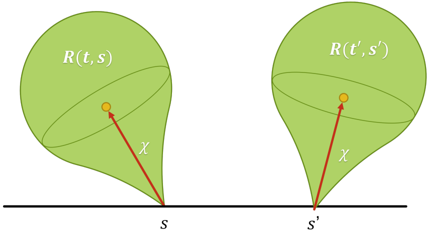
Example: $SO(3)$-Strand

1 Euler–Poincaré Reduction



Example: $SO(3)$-Strand

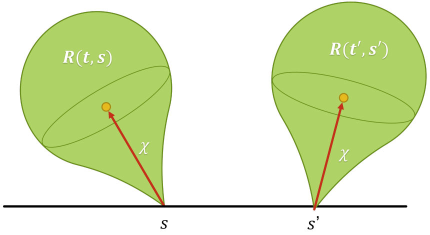
1 Euler–Poincaré Reduction



Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle$$

Example: $SO(3)$-Strand



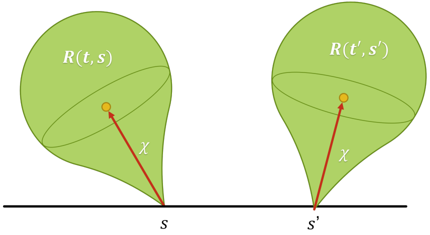
$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Omega = R^{-1}R_t, \quad \Gamma = R^{-1}R_s$$

Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle$$

Example: $SO(3)$-Strand

1 Euler–Poincaré Reduction



$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Omega = R^{-1}R_t, \quad \Gamma = R^{-1}R_s$$

Lagrangian:

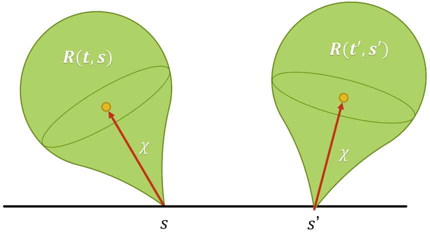
$$l = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega \rangle - \frac{1}{2}\langle \Gamma, \mathbb{J}\Gamma \rangle$$

Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle$$

Example: $SO(3)$-Strand

1 Euler–Poincaré Reduction



Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t\right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s\right\rangle$$

$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Omega = R^{-1}R_t, \quad \Gamma = R^{-1}R_s$$

Lagrangian:

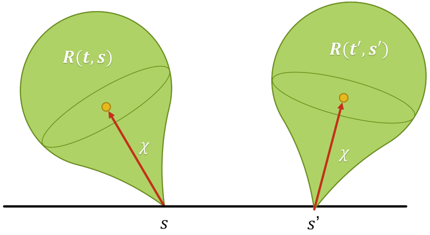
$$l = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega\rangle - \frac{1}{2}\langle \Gamma, \mathbb{J}\Gamma\rangle$$

Equations:

$$\partial_t(\mathbb{I}\Omega) - \mathbb{I}\Omega \times \Omega - \partial_s(\mathbb{J}\Omega) + \mathbb{J}\Gamma \times \Gamma = 0.$$

Example: $SO(3)$-Strand

1 Euler–Poincaré Reduction



Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle$$

$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Omega = R^{-1}R_t, \quad \Gamma = R^{-1}R_s$$

Lagrangian:

$$l = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega \rangle - \frac{1}{2}\langle \Gamma, \mathbb{J}\Gamma \rangle$$

Equations:

$$\partial_t(\mathbb{I}\Omega) - \mathbb{I}\Omega \times \Omega - \partial_s(\mathbb{J}\Omega) + \mathbb{J}\Gamma \times \Gamma = 0.$$

Reconstruction:

$$\partial_t\Gamma - \partial_s\Omega + \Omega \times \Gamma = 0.$$

References

- M. Castrillón López, P. L. García Pérez, and T. S. Ratiu. "Euler–Poincaré Reduction on Principal Bundles". In: *Letters in Mathematical Physics* 58.2 (Nov. 2001), pp. 167–180
- M. Castrillón López and T. S. Ratiu. "Reduction in Principal Bundles: Covariant Lagrange-Poincaré Equations". en. In: *Communications in Mathematical Physics* 236.2 (May 2003), pp. 223–250

Table of Contents

$$
\begin{tikzcd}
J^1P \arrow[r] \arrow[d, "\pi_{P,J^1P}"'] & J^1P^* \arrow[dl, "\pi_{P,J^1P^*}"] \\
P \arrow[d, "\pi_{M,P}"] & \\
M &
\end{tikzcd}
$$

Dual Jet Bundle

J^1P^* affine bundle over P,

$$J^1_yP^* = \mathrm{Aff}(J^1_yP, \bigwedge^n T^*_xM)$$

for any $y \in P$ and $x = \pi_{M,P}(y)$.

$$\dim J^1P^* = \dim J^1P + 1$$

$$\begin{tikzcd} J^1P \arrow[r] \arrow[d, "\pi_{P,J^1P}"'] & J^1P^* \arrow[dl, "\pi_{P,J^1P^*}"] \\ P \arrow[d, "\pi_{M,P}"] & \\ M & \end{tikzcd}$$

Dual Jet Bundle

J^1P^* affine bundle over P,

$$J^1_yP^* = \mathrm{Aff}(J^1_yP, \bigwedge^n T^*_xM)$$

for any $y \in P$ and $x = \pi_{M,P}(y)$.

$\dim J^1P^* = \dim J^1P + 1$
Canonical forms: $J^1P^* \cong Z \subset \bigwedge^n T^*P$
Canonical n-form Θ, $n+1$-form $\Omega = -d\Theta \rightsquigarrow$ *dynamics*

$$
\begin{tikzcd}
J^1P \arrow[r] \arrow[dr] \arrow[dd] & J^1P^* \arrow[d] \\
& \Pi_P \arrow[u, bend right, "\delta"'] \arrow[dl] \\
P \arrow[d] & \\
M &
\end{tikzcd}
$$

Polysymplectic Space

$$\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$$

$$\dim \Pi_P = \dim J^1P$$

$$\begin{tikzcd}
J^1P \arrow[r] \arrow[dr] \arrow[dd] & J^1P^* \arrow[d] \\
& \Pi_P \arrow[u, bend right, "\delta"'] \arrow[dl] \\
P \arrow[d] & \\
M &
\end{tikzcd}$$

Polysymplectic Space

$\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$

$\dim \Pi_P = \dim J^1P$

NO Canonical forms: Given $\delta \in \Gamma(J^1P^* \to \Pi_p)$, n-form $\delta^*\Theta$, $n+1$-form $\delta^*\Omega \rightsquigarrow$ *dynamics*

$$\begin{tikzcd}
J^1P \arrow[r] \arrow[dr] \arrow[dd] & J^1P^* \arrow[d] \\
 & \Pi_P \arrow[u, bend right, "\delta"'] \arrow[dl] \\
P \arrow[d] & \\
M &
\end{tikzcd}$$

Polysymplectic Space

$\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$

$\dim \Pi_P = \dim J^1P$

NO Canonical forms: Given $\delta \in \Gamma(J^1P^* \to \Pi_p)$, n-form $\delta^*\Theta$, $n+1$-form $\delta^*\Omega \rightsquigarrow$ *dynamics*

Hamiltonian system: (Π_P, δ)

$$
\begin{tikzcd}
J^1P \arrow[r] \arrow[dr] \arrow[dd] & J^1P^* \arrow[d, shift left] \\
 & \Pi_P \arrow[u, bend right, "\delta"'] \arrow[dl] \\
P \arrow[d] & \\
M &
\end{tikzcd}
$$

Polysymplectic Space

$\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$

$\dim \Pi_P = \dim J^1P$

NO Canonical forms: Given $\delta \in \Gamma(J^1P^* \to \Pi_p)$, n-form $\delta^*\Theta$, $n+1$-form $\delta^*\Omega \rightsquigarrow$ *dynamics*

Hamiltonian system: (Π_P, δ)

Ehresmann connection Λ on $P \to M$
+ Hamiltonian $\mathcal{H} : \Pi_P \to \bigwedge^n T^*M$.

$$
\begin{tikzcd}
\Pi_P \arrow[d] \\
P \arrow[d, "\Lambda"] \\
M
\end{tikzcd}
$$

Hamiltonian system: $(\Pi_P, \Lambda, \mathcal{H} = H\mathrm{v})$
Hamilton-De Donder equations:

$$\frac{\partial H}{\partial p_a^i} = \frac{\partial y^a}{\partial x^i} - \Lambda_i^a; \qquad -\frac{\partial H}{\partial y^a} = \frac{\partial p_a^i}{\partial x^i} + \frac{\partial \Lambda_i^b}{\partial x^a} p_a^i$$

Polysymplectic Field Theory

$$\begin{tikzcd} \Pi_P \arrow[d] \\ P \arrow[d, "\Lambda"] \\ M \end{tikzcd}$$

Hamiltonian system: $(\Pi_P, \Lambda, \mathcal{H} = H\mathrm{v})$
Hamilton-De Donder equations:

$$\frac{\partial H}{\partial p_a^i} = \frac{\partial y^a}{\partial x^i} - \Lambda_i^a; \qquad -\frac{\partial H}{\partial y^a} = \frac{\partial p_a^i}{\partial x^i} + \frac{\partial \Lambda_i^b}{\partial x^a} p_a^i$$

Covariant Bracket Formulation

$$
\begin{tikzcd}
\Pi_P \arrow[d] \\
P \arrow[d, "\Lambda"] \\
M
\end{tikzcd}
$$

Hamiltonian system: $(\Pi_P, \Lambda, \mathcal{H} = H\mathrm{v})$

Covariant Bracket Formulation

A section p of $\Pi_P \to M$ is a **solution** of a given Hamiltonian system $(\Pi_P, \Lambda, \mathcal{H})$, iff for any horizontal Poisson $(n-1)$-form F:

$$\{F, H\}\mathrm{v} \circ p = d(p^*F) - (d^hF) \circ p,$$

where d^hF is the horizontal differential of F with respect to a connection on Π_P.

Poisson Forms on Π_P

A horizontal r-form F is *Poisson* if there is a vertical $(n-r)$-multivector field χ_F on J^1P^* such that

$$i_{\chi_F}\Omega = dF,$$

Poisson Forms on Π_P

A horizontal r-form F is *Poisson* if there is a vertical $(n-r)$-multivector field χ_F on J^1P^* such that

$$i_{\chi_F}\Omega = dF,$$

Given F a Poisson r-form and E a Poisson s-form their *Poisson bracket* is the $(r+s+1-n)$-form on J^1P^*,

$$\{F, E\} = (-1)^{r(s-1)} i_{\chi_E} i_{\chi_F}\Omega.$$

Proposition

A Poisson r-form on J^1P^* is projectable to $\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$.

Proposition

A Poisson r-form on J^1P^* is projectable to $\Pi_P = TM \otimes V^*P \otimes \bigwedge^n T^*M$.

Proposition

Any Poisson $(n-1)$-form on Π_P is of the kind

$$F = \Theta_X + \pi^*_{P,\Pi_P}\omega + Z,$$

where Z is a closed $(n-1)$-form on Π_P, ω is a $(n-1)$-form on P, $X \in \mathfrak{X}^V(P)$ and $(\Theta_X)_p = \langle p, X \rangle$.

Symmetry: *Lie Group G acts on P*

$$\Pi_P = TM \otimes V^*P \otimes \wedge^n T^*M$$

$$\begin{tikzcd}
\Pi_P \arrow[d] \arrow[dr] & \\
 & \Pi_P/G \arrow[dd] \\
P \arrow[dr, "\Lambda"] & \\
 & M
\end{tikzcd}$$

Symmetry: *Lie Group G acts on P*

$$\Pi_P = TM \otimes V^*P \otimes \wedge^n T^*M$$

$$\Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^*M$$

$$\begin{tikzcd}
\Pi_P \arrow[d] \arrow[dr] & \\
 & \Pi_P/G \arrow[dd] \\
P \arrow[dr, "\Lambda"] & \\
 & M
\end{tikzcd}$$

Symmetry: *Lie Group G acts on P*

$$\Pi_P = TM \otimes V^*P \otimes \wedge^n T^*M$$

$$\Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^*M$$

$$\begin{tikzcd}
\Pi_P \arrow[d] \arrow[dr] & \\
 & \Pi_P/G \arrow[dd] \\
P \arrow[dr, "\Lambda"] & \\
 & M
\end{tikzcd}$$

Reduced Poisson $(n-1)$-forms

$F = \Theta_X + \pi^*_{P,\Pi_P}\omega + Z$, G-invariant Poisson $(n-1)$-form on Π_P

$$f = \Theta_{\bar{\xi}} + \pi^*_{M,\Pi_P/G}\bar{\omega} + \bar{Z},$$

where $\bar{\xi} = \Gamma(\tilde{\mathfrak{g}})$, $\bar{\omega} \in \Omega^{n-1}(M)$, and $\bar{Z} \in Z^{n-1}(\Pi_P/G)$.

$$
\begin{tikzcd}
\Pi_P \arrow[dd] \arrow[dr] & \\
& \Pi_P/G \arrow[dd] \\
P \arrow[dr, "\Lambda"] & \\
& M
\end{tikzcd}
$$

Symmetry: *Lie Group G acts on P*

$$\Pi_P = TM \otimes V^* P \otimes \wedge^n T^* M$$
$$\Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^* M$$

Reduced Poisson $(n-1)$-forms

$F = \Theta_X + \pi^*_{P,\Pi_P}\omega + Z$, G-invariant Poisson $(n-1)$-form on Π_P

$$f = \Theta_{\bar{\xi}} + \pi^*_{M,\Pi_P/G}\bar{\omega} + \bar{Z},$$

where $\bar{\xi} = \Gamma(\tilde{\mathfrak{g}})$, $\bar{\omega} \in \Omega^{n-1}(M)$, and $\bar{Z} \in Z^{n-1}(\Pi_P/G)$.

Reduced bracket:

$$\{f, h\} = -\left\langle \mu, \left[\bar{\xi}, \frac{\delta h}{\delta \mu}\right]\right\rangle .$$

Theorem (Lie–Poisson reduction)

For any section p of $\Pi_P \to M$, let $\mu = \kappa \circ p \in \Gamma(\Pi_P/G)$ and let $\mathcal{H} = H\mathrm{v}$ be a G-invariant Hamiltonian reducing to h. Then, the following are equivalent:

1. *for every Poisson $(n-1)$-form F on Π_P,*

$$p^*\{F, H\}\mathrm{v} = d(F \circ p) - d^h F \circ p.$$

2. *for every Poisson $(n-1)$-form f on Π_P/G,*

$$\mu^*\{f, h\}\mathrm{v} = d(f \circ \mu) - d^h f \circ \mu$$

3. *the section μ satisfies the Lie–Poisson equations*

$$\mathrm{div}^\Lambda \mu = \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu$$

References

- I. V. Kanatchikov. “Canonical structure of classical field theory in the polymomentum phase space”. In: *Rep. Math. Phys.* 41.1 (1998), pp. 49–90
- J. E. Marsden et al. “Covariant Poisson brackets for classical fields”. In: *Ann. Physics* 169.1 (1986), pp. 29–47
- M. Castrillón López and J. E. Marsden. “Some remarks on Lagrangian and Poisson reduction for field theories”. In: *J. Geom. Phys.* 48.1 (2003), pp. 52–83

$$
\begin{tikzcd}
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, dashed, "p"] \arrow[u, bend left, "\mu"] \arrow[l, dashed, bend right, "s"']
\end{tikzcd}
$$

$$(\Pi_P)/G \cong TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \cong \Omega^{n-1}(M, \tilde{\mathfrak{g}}^*)$$

When does $\mu : U \to \Pi_P/G$ comes from a section $p : U \to \Pi_P$?

$$
\begin{tikzcd}
\Pi_P \arrow[rr, "G\text{-ppal bundle}"] \arrow[dd, "\pi_{P,J^1P}"'] & & (\Pi_P)/G \arrow[dd] \\
& & \\
P \arrow[rr, "G\text{-ppal bundle}"'] & & M \arrow[uull, dashed, "p"] \arrow[uu, bend left, "\mu"] \arrow[ll, dashed, bend right, "s"']
\end{tikzcd}
$$

$$(\Pi_P)/G \cong TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \cong \Omega^{n-1}(M, \tilde{\mathfrak{g}}^*)$$

When does $\mu : U \to \Pi_P/G$ comes from a section $p : U \to \Pi_P$?

Problems

1. $\Gamma(\Pi_P/G)$ are not connections
2. Positions and mommenta are not related

Vertical derivative

$$\frac{\delta h}{\delta \mu} : \Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM$$

Vertical derivative

$$\frac{\delta h}{\delta \mu} : \Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM$$

$$\widehat{\frac{\delta h}{\delta \mu}} = \left\langle \frac{\delta h}{\delta \mu}, \chi_{\mathrm{vol}} \right\rangle : \Pi_P/G \longrightarrow T^*M \otimes \tilde{\mathfrak{g}}$$

Vertical derivative

$$\frac{\delta h}{\delta \mu} : \Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM$$

$$\widehat{\frac{\delta h}{\delta \mu}} = \left\langle \frac{\delta h}{\delta \mu}, \chi_{\mathrm{vol}} \right\rangle : \Pi_P/G \longrightarrow T^*M \otimes \tilde{\mathfrak{g}}$$

$$\mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right) : M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}}$$

Vertical derivative

$$\frac{\delta h}{\delta \mu} : \Pi_P/G = TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM$$

$$\widehat{\frac{\delta h}{\delta \mu}} = \left\langle \frac{\delta h}{\delta \mu}, \chi_{\mathrm{vol}} \right\rangle : \Pi_P/G \longrightarrow T^*M \otimes \tilde{\mathfrak{g}}$$

$$\mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right) : M \longrightarrow T^*M \otimes \tilde{\mathfrak{g}}$$

Connection: $\sigma = \Lambda + \mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

Reconstruction

$$
\begin{tikzcd}
\Pi_P \arrow[rr, "G\text{-ppal bundle}"] \arrow[dd, "\pi_{P,J^1P}"'] & & (\Pi_P)/G \arrow[dd] \\
& & \\
P \arrow[rr, "G\text{-ppal bundle}"'] & & M \arrow[uull, dashed, "p"] \arrow[uu, bend left, "\mu"] \arrow[ll, dashed, bend right, "s"']
\end{tikzcd}
$$

Connection: $\sigma = \Lambda + \mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

Reconstruction

$$
\begin{tikzcd}
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, dashed, "p"] \arrow[l, dashed, bend right, "s"'] \arrow[u, bend left, "\mu"]
\end{tikzcd}
$$

Connection: $\sigma = \Lambda + \mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

$\mu : U \to \Pi_P / G$ comes from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

Reconstruction

$$
\begin{tikzcd}
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "p"] \arrow[u, bend left, "\mu"] \arrow[l, bend right, "s"']
\end{tikzcd}
$$

Connection: $\sigma = \Lambda + \mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

$\mu : U \to \Pi_P/G$ comes from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

Reconstruction

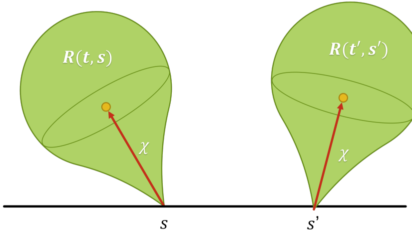
$$\begin{tikzcd}
\Pi_P \arrow[rr, "G\text{-ppal bundle}"] \arrow[dd, "\pi_{P,J^1P}"'] & & (\Pi_P)/G \arrow[dd] \\
& & \\
P \arrow[rr, "G\text{-ppal bundle}"'] & & M \arrow[uull, "p"'] \arrow[uu, bend left, "\mu"] \arrow[ll, bend right, "s"']
\end{tikzcd}$$

Connection: $\sigma = \Lambda + \mu^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

$\mu : U \to \Pi_P/G$ comes from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

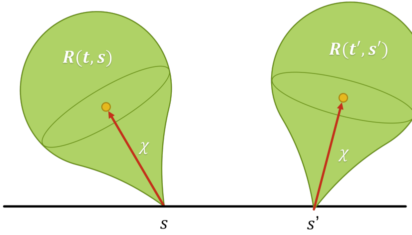
$$\mathcal{HDW}(\mathcal{H})(p) = 0 \iff \begin{cases} \operatorname{div}^\Lambda \mu = \operatorname{ad}^*_{\delta h/\delta \mu}\mu, \\ \operatorname{Curv}(\sigma) = 0. \end{cases}$$

Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

Example: $SO(3)$-Strand

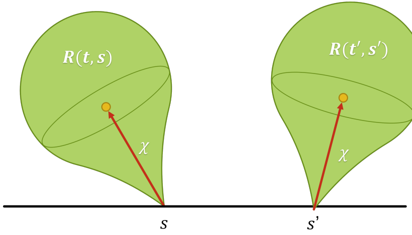


Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle$$

Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

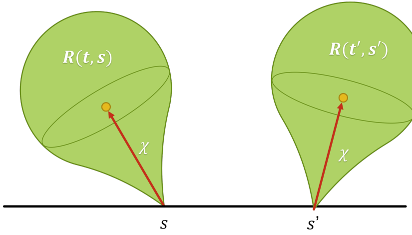
Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s.$$

Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

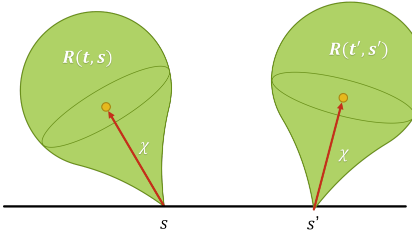
Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s.$$

$$\Lambda = 0$$



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

Hamiltonian:

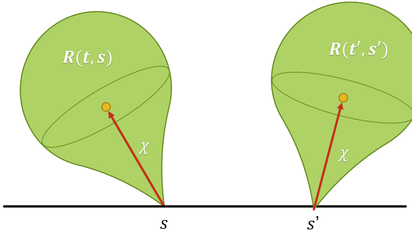
$$h = \frac{1}{2}\langle \mu^t, \mathbb{I}^{-1}\mu^t \rangle - \frac{1}{2}\langle \mu^s, \mathbb{J}^{-1}\mu^s \rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s.$$

$$\Lambda = 0 \quad \sigma = \mathbb{I}^{-1}\mu^t \mathrm{d}t - \mathbb{J}^{-1}\mu^s \mathrm{d}s,$$

Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(3)$

Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s.$$

$$\Lambda = 0 \quad \sigma = \mathbb{I}^{-1}\mu^t \mathrm{d}t - \mathbb{J}^{-1}\mu^s \mathrm{d}s,$$

Reconstruction:

$$\partial_t(\mathbb{I}^{-1}\mu^t) + \partial_s(\mathbb{J}^{-1}\mu^s) - (\mathbb{I}^{-1}\mu^t) \times (\mathbb{J}^{-1}\mu^s) = 0.$$

Table of Contents

$$
\begin{tikzcd}
J^1P \arrow[r, "q_G"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & P/G = M
\end{tikzcd}
$$

$L : J^1P \to \mathbb{R}$, $\quad$ K-invariant, $\quad$ $K < G$

$$L : J^1P \to \mathbb{R}, \qquad K\text{-invariant}, \qquad K < G$$

$$\begin{tikzcd}
J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[d] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "K\text{-ppal}"'] & P/K & P/G = M
\end{tikzcd}$$

Bundle of Connections

$$L : J^1P \to \mathbb{R}, \qquad K\text{-invariant}, \qquad K < G$$

$$\begin{tikzcd}
J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[r] \arrow[d] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "K\text{-ppal}"'] & P/K \arrow[r] & P/G = M
\end{tikzcd}$$

Bundle of Connections

$$\begin{tikzcd}
J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[r] \arrow[d] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "K\text{-ppal}"'] & P/K \arrow[r] & P/G = M
\end{tikzcd}$$

$L : J^1P \to \mathbb{R}$, $\quad K$-invariant, $\quad K < G$

Reduced configuration bundle

$$J^1P/K \simeq \mathcal{C}(P) \times_M P/K.$$

$$
\begin{tikzcd}
J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[r] \arrow[d] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "K\text{-ppal}"'] & P/K \arrow[r] & P/G = M
\end{tikzcd}
$$

$L : J^1P \to \mathbb{R}$, $\quad K$-invariant, $\quad K < G$

Reduced configuration bundle

$$J^1P/K \simeq \mathcal{C}(P) \times_M P/K.$$

$j^1s : U \subset M \to J^1P.$

Bundle of Connections

$$\begin{tikzcd} J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[r] \arrow[d] & \mathcal{C}(P) \arrow[d] \\ P \arrow[r, "K\text{-ppal}"'] & P/K \arrow[r] & P/G = M \end{tikzcd}$$

$L : J^1P \to \mathbb{R}$, K-invariant, $K < G$

Reduced configuration bundle

$$J^1P/K \simeq \mathcal{C}(P) \times_M P/K.$$

$$j^1s : U \subset M \to J^1P.$$

Reduced sections

$$\sigma : U \to \mathcal{C}(P), \qquad \bar{s} : U \to P/K$$

Bundle of Connections

$$
\begin{tikzcd}
J^1P \arrow[r, "q_K"] \arrow[d, "\pi_{P,J^1P}"'] & J^1P/K \arrow[r] \arrow[d] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "K\text{-ppal}"'] \arrow[rr, bend right, "G\text{-ppal}"'] & P/K \arrow[r] & P/G = M
\end{tikzcd}
$$

$$L : J^1P \to \mathbb{R}, \qquad K\text{-invariant}, \qquad K < G$$

Reduced configuration bundle

$$J^1P/K \simeq \mathcal{C}(P) \times_M P/K.$$

$$j^1s : U \subset M \to J^1P.$$

Reduced sections

$$\sigma : U \to \mathcal{C}(P), \qquad \bar{s} : U \to P/K$$

Theorem (Euler–Poincaré reduction)

The following are equivalent:

1. *The variational principle* $\delta \int_U L\left(j^1_x s\right)\, \chi_{\mathrm{vol}} = 0$ *holds, for vertical free variations* δs.
2. $s : U \to P$ *satisfies the Euler–Lagrange equations for* L.
3. *The variational principle* $\delta \int_U l\left(\sigma(x), \bar{s}(x)\right)\, \chi_{\mathrm{vol}} = 0$ *holds, using variations of the form*

$$\delta\sigma = \nabla^\sigma \eta, \qquad \delta\bar{s} = \eta_{P/K}$$

   where $\eta : U \to \operatorname{ad} P$.
4. *The Euler–Poincaré equations hold*

$$\mathcal{EP}(l)(\sigma, \bar{s}) := \operatorname{div}^\sigma \frac{\delta l}{\delta \sigma} - \mathsf{P}^+_{\bar{s}}\left(\frac{\delta l}{\delta \bar{s}}\right) = 0.$$

$$
\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "j^1 s"] \arrow[u, bend right, "\sigma"'] \arrow[l, bend right, "s"']
\end{tikzcd}
$$

$$j^1 s \rightsquigarrow (\sigma, \bar{s}) \rightsquigarrow s.$$

> $\sigma : U \to \mathcal{C}(P)$ comes from a section $s : U \to P$ if and only if σ is a *flat connection* with *trivial holonomy* and $\bar{s}$ is parallel with respect to σ.

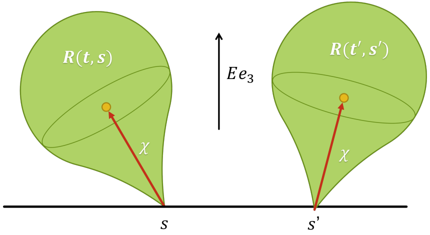
$$
\begin{tikzcd}
J^1P \arrow[r, "q"] \arrow[d, "\pi_{P,J^1P}"'] & \mathcal{C}(P) \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "j^1 s"'] \arrow[u, bend left, "\sigma"'] \arrow[l, bend right, "s"']
\end{tikzcd}
$$

$$j^1 s \rightsquigarrow (\sigma, \bar{s}) \rightsquigarrow s.$$

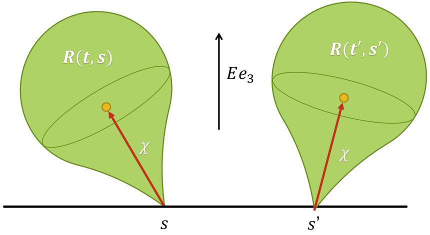
$\sigma : U \to \mathcal{C}(P)$ comes from a section $s : U \to P$ if and only if σ is a *flat connection* with *trivial holonomy* and $\bar{s}$ is parallel with respect to σ.

$$\mathcal{EL}(L)(s) = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathcal{EP}(l)(\sigma, \bar{s}) = 0, \\ \mathrm{Curv}(\sigma) = 0 \\ \nabla^{\sigma} \bar{s} = 0. \end{cases}$$

Example: $SO(3)$-Strand



Example: $SO(3)$-Strand

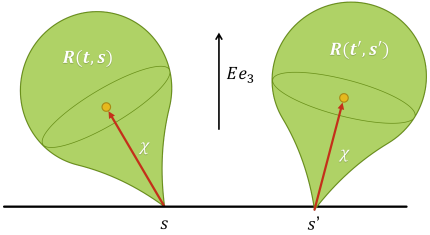


Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle - mg\left\langle Re_3, \chi \right\rangle$$

Example: $SO(3)$-Strand

3 Euler–Poincaré Reduction by a subgroup

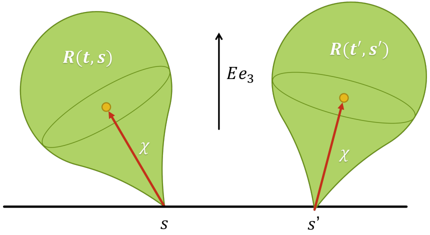


$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Sigma \in S^2 = SO(3)/SO(2)$$

Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle - mg\,\langle Re_3, \chi\rangle$$

Example: $SO(3)$-Strand



$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Sigma \in S^2 = SO(3)/SO(2)$$

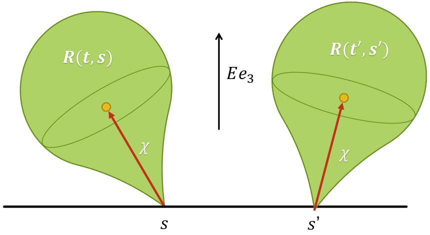
Lagrangian:

$$l = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega \rangle - \frac{1}{2}\langle \Gamma, \mathbb{J}\Gamma \rangle - mg\langle \Sigma, \chi \rangle$$

Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle - mg\langle Re_3, \chi \rangle$$

Example: $SO(3)$-Strand



Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle - mg\left\langle Re_3, \chi \right\rangle$$

$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Sigma \in S^2 = SO(3)/SO(2)$$

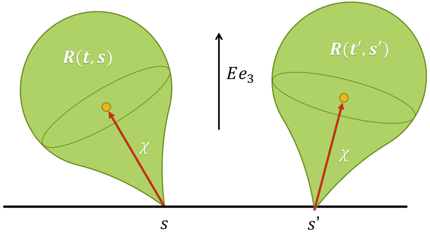
Lagrangian:

$$l = \frac{1}{2}\left\langle \Omega, \mathbb{I}\Omega \right\rangle - \frac{1}{2}\left\langle \Gamma, \mathbb{J}\Gamma \right\rangle - mg\left\langle \Sigma, \chi \right\rangle$$

Equations:

$$\partial_t(\mathbb{I}\Omega) - \mathbb{I}\Omega \times \Omega - \partial_s(\mathbb{J}\Omega) + \mathbb{J}\Gamma \times \Gamma + mg\Sigma \times \chi = 0.$$

Example: $SO(3)$-Strand



Config. bundle: J^1P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

$$L = \frac{1}{2}\left\langle R^{-1}R_t, \mathbb{I}R^{-1}R_t \right\rangle - \frac{1}{2}\left\langle R^{-1}R_s, \mathbb{J}R^{-1}R_s \right\rangle - mg\,\langle Re_3, \chi\rangle$$

$$\sigma = \Omega \mathrm{d}t + \Gamma \mathrm{d}s, \quad \Sigma \in S^2 = SO(3)/SO(2)$$

Lagrangian:

$$l = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega\rangle - \frac{1}{2}\langle \Gamma, \mathbb{J}\Gamma\rangle - mg\,\langle \Sigma, \chi\rangle$$

Equations:

$$\partial_t(\mathbb{I}\Omega) - \mathbb{I}\Omega\times\Omega - \partial_s(\mathbb{J}\Omega) + \mathbb{J}\Gamma\times\Gamma + mg\Sigma\times\chi = 0.$$

Reconstruction:

$$\Sigma_t + \mathbb{I}^{-1}\mu^t \times \Sigma = 0 \quad \Sigma_s - \mathbb{J}^{-1}\mu^s \times \Sigma = 0$$

$$\partial_t\Gamma - \partial_s\Omega + \Omega\times\Gamma = 0.$$

References

- M. Castrillón López, P. M. Chacón, and P. L. García. "Lagrange-Poincaré reduction in affine principal bundles". In: *J. Geom. Mech.* 5.4 (2013), pp. 399–414
- M. Castrillón López, P. García, and C. Rodrigo. "Euler–Poincaré Reduction in Principal Bundles by a Subgroup of the Structure Group". In: *Journal of Geometry and Physics* 74 (Dec. 2013), pp. 352–369. ISSN: 03930440

Table of Contents

$(\Pi_P, \Lambda, \mathcal{H})$, Hamiltonian $\mathcal{H}$ is K-invariant, $K < G$.

$$
\begin{tikzcd}
\Pi_P \arrow[d] \\
P \arrow[d, "\Lambda"] \\
M
\end{tikzcd}
$$

$(\Pi_P, \Lambda, \mathcal{H})$, Hamiltonian $\mathcal{H}$ is K-invariant, $K < G$.

$$
\begin{tikzcd}
\Pi_P \arrow[dd] \arrow[dr] & \\
 & {\color{blue}\Pi_P/K} \arrow[dd] \\
P \arrow[dd, "\Lambda"'] \arrow[dr] & \\
 & P/K \arrow[dl] \\
M &
\end{tikzcd}
$$

$(\Pi_P, \Lambda, \mathcal{H})$, Hamiltonian $\mathcal{H}$ is K-invariant, $K < G$.

$$
\begin{tikzcd}
\Pi_P \arrow[dr] \arrow[dd] & \\
& \Pi_P/K \arrow[dd] \\
P \arrow[dr] \arrow[dd, "\Lambda"] & \\
& P/K \arrow[dl] \\
M &
\end{tikzcd}
$$

$$\Pi_P/K \simeq (TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^*M) \times_M P/K$$

Reduction

$$
\begin{tikzcd}
\Pi_P \arrow[dr] \arrow[dd] & \\
& \Pi_P/K \arrow[dd] \\
P \arrow[dr] \arrow[dd, "\Lambda"] & \\
& P/K \arrow[dl] \\
M &
\end{tikzcd}
$$

$(\Pi_P, \Lambda, \mathcal{H})$, Hamiltonian $\mathcal{H}$ is K-invariant, $K < G$.

$$\Pi_P/K \simeq (TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^*M) \times_M P/K$$

Reduced Poisson $(n-1)$-forms

$F = \Theta_X + \pi^*_{P,\Pi_P}\omega + Z$, K-invariant Poisson $(n-1)$-form on Π_P

$$f = \Theta_{\bar{\xi}} + \pi^*_{P/K,\Pi_P/K}\bar{\omega} + \bar{Z},$$

where $\bar{\xi} = \Gamma(\pi^*_{P/K}\mathrm{Ad}(P))$, $\bar{\omega} \in \Omega^{n-1}(P/K)$, $\bar{Z} \in Z^{n-1}(\Pi_P/K)$.

$$
\begin{tikzcd}
\Pi_P \arrow[dr] \arrow[dd] & \\
& \Pi_P/K \arrow[dd] \\
P \arrow[dr] \arrow[dd, "\Lambda"] & \\
& P/K \arrow[dl] \\
M &
\end{tikzcd}
$$

$(\Pi_P, \Lambda, \mathcal{H})$, Hamiltonian $\mathcal{H}$ is K-invariant, $K < G$.

$$\Pi_P/K \simeq (TM \otimes \tilde{\mathfrak{g}}^* \otimes \wedge^n T^*M) \times_M P/K$$

Reduced bracket

$$\{f, h\} = \{\bar{\xi}, h\}_{LP} + \{f, h\}_{P/K}$$

where,

$$\{f, h\}_{P/K} = \left\langle \frac{\delta f}{\delta \bar{s}}, \mathsf{P}_{\bar{s}} \left(\frac{\delta h}{\delta \mu} \right) \right\rangle - \left\langle \frac{\delta h}{\delta \bar{s}}, \mathsf{P}_{\bar{s}}(\bar{\xi}) \right\rangle .$$

Theorem (Lie–Poisson reduction)

For any section p of $\Pi_P \to M$, let $(\mu, \bar{s}) = \kappa \circ p \in \Gamma(\Pi_P/K)$ and $\mathcal{H} = H\mathrm{v}$ is a K-invariant Hamiltonian reducing to h. Then, the following are equivalent:

1. *for every Poisson $(n-1)$-form F on Π_P, $p^*\{F, H\}\mathrm{v} = d(F \circ p) - d^h F \circ p$.*
2. *the section $p : M \to \Pi_P$ satisfies HDW-equations.*
3. *for every affine Poisson $(n-1)$-form f on Π_P/K,*

$$(\mu, \bar{s})^*\{f, h\}\mathrm{v} = d(f \circ (\mu, \bar{s})) - d^h f \circ (\mu, \bar{s})p$$

4. *sections μ and $\bar{s}$ satisfy*

$$\mathrm{div}^\Lambda \mu - \mathrm{ad}^*_{\delta h/\delta \mu}\mu + P^+_{\bar{s}}\left(\frac{\delta h}{\delta \bar{s}}\right) = 0, \quad \nabla^{\sigma_K}\bar{s} = 0,$$

Reconstruction

$$\begin{tikzcd}
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, dashed, "p"] \arrow[u, bend left, "\mu"] \arrow[l, dashed, bend right, "s"']
\end{tikzcd}$$

Connection: $\sigma = \Lambda + (\mu, \overline{s})^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

Reconstruction

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, dashed, "p"] \arrow[l, dashed, bend right, "s"'] \arrow[u, bend left, "\mu"]
\end{tikzcd}$$

Connection: $\sigma = \Lambda + (\mu, \bar{s})^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

$\mu : U \to \Pi_P / G$ and $\bar{s} : U \to P/K$ come from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

Reconstruction

$$
\begin{tikzcd}
\Pi_P \arrow[rr, "G\text{-ppal bundle}"] \arrow[dd, "\pi_{P,J^1P}"'] & & (\Pi_P)/G \arrow[dd] \\
& & \\
P \arrow[rr, "G\text{-ppal bundle}"'] & & M \arrow[uull, "p"'] \arrow[uu, bend left, "\mu"] \arrow[ll, bend right, "s"']
\end{tikzcd}
$$

Connection: $\sigma = \Lambda + (\mu, \bar{s})^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

$\mu : U \to \Pi_P/G$ and $\bar{s} : U \to P/K$ come from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

Reconstruction

$$\begin{tikzcd}
\Pi_P \arrow[r, "G\text{-ppal bundle}"] \arrow[d, "\pi_{P,J^1P}"'] & (\Pi_P)/G \arrow[d] \\
P \arrow[r, "G\text{-ppal bundle}"'] & M \arrow[ul, "p"'] \arrow[u, bend left, "\mu"'] \arrow[l, bend right, "s"']
\end{tikzcd}$$

Connection: $\sigma = \Lambda + (\mu, \bar{s})^* \left(\widehat{\frac{\delta h}{\delta \mu}} \right)$

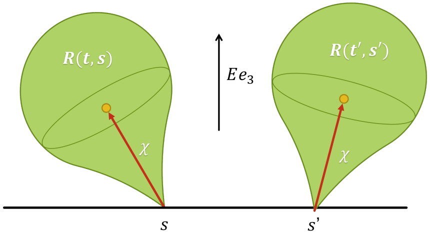
$\mu : U \to \Pi_P/G$ and $\bar{s} : U \to P/K$ come from a section $\pi : U \to \Pi_P$ if and only if σ is a *flat connection* with *trivial holonomy*.

$$\mathcal{HDW}(\mathcal{H})(p) = 0$$

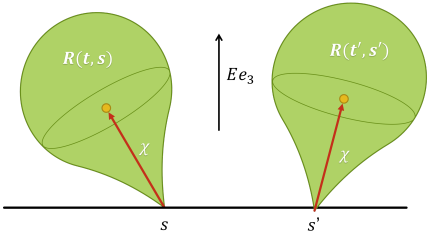
$$\Updownarrow$$

$$\operatorname{div}^{\Lambda}\mu - \operatorname{ad}^*_{\delta h/\delta\mu}\mu + P^+_{\bar{s}}\left(\frac{\delta h}{\delta \bar{s}}\right) = 0, \quad \nabla^{\sigma_K}\bar{s} = 0, \quad \operatorname{Curv}(\sigma) = 0.$$

Example: $SO(3)$-Strand

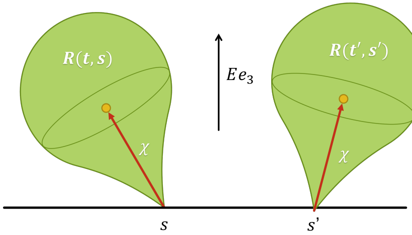


Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

Example: $SO(3)$-Strand

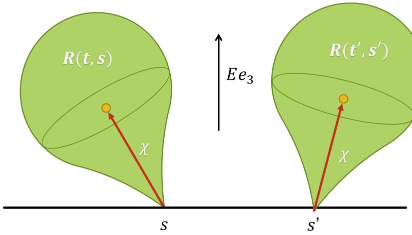


Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle - mg\left\langle \Sigma, \chi \right\rangle$$

Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

Example: $SO(3)$-Strand



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

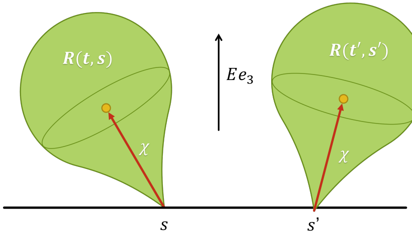
Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle - mg\left\langle \Sigma, \chi \right\rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s - mg\Sigma \times \chi =$$

$$\Sigma_t + \mathbb{I}^{-1}\mu^t \times \Sigma = 0 \quad \Sigma_s - \mathbb{J}^{-1}\mu^s \times \Sigma = 0$$



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle - mg\left\langle \Sigma, \chi \right\rangle$$

Equations:

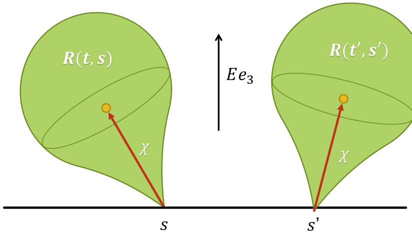
$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s - mg\Sigma \times \chi =$$

$$\Sigma_t + \mathbb{I}^{-1}\mu^t \times \Sigma = 0 \quad \Sigma_s - \mathbb{J}^{-1}\mu^s \times \Sigma = 0$$

$$\Lambda = 0 \quad \sigma = \mathbb{I}^{-1}\mu^t \mathrm{d}t - \mathbb{J}^{-1}\mu^s \mathrm{d}s,$$

Example: $SO(3)$-Strand

4 Lie–Poisson Reduction by a subgroup



Config. bundle: Π_P, $P = \mathbb{R}^2 \to \mathbb{R}^2 \times SO(3)$
Symmetry group: $SO(2)$

Hamiltonian:

$$h = \frac{1}{2}\left\langle \mu^t, \mathbb{I}^{-1}\mu^t \right\rangle - \frac{1}{2}\left\langle \mu^s, \mathbb{J}^{-1}\mu^s \right\rangle - mg\left\langle \Sigma, \chi \right\rangle$$

Equations:

$$\partial_t(\mu^t) + \partial_s(\mu^s) = \mu^t \times \mathbb{I}^{-1}\mu^t + \mu^s \times \mathbb{J}^{-1}\mu^s - mg\Sigma \times \chi =$$

$$\Sigma_t + \mathbb{I}^{-1}\mu^t \times \Sigma = 0 \quad \Sigma_s - \mathbb{J}^{-1}\mu^s \times \Sigma = 0$$

$$\Lambda = 0 \quad \sigma = \mathbb{I}^{-1}\mu^t \mathrm{d}t - \mathbb{J}^{-1}\mu^s \mathrm{d}s,$$

Reconstruction:

$$\partial_t(\mathbb{I}^{-1}\mu^t) + \partial_s(\mathbb{J}^{-1}\mu^s) - (\mathbb{I}^{-1}\mu^t) \times (\mathbb{J}^{-1}\mu^s) = 0.$$

References

- M. Á. Berbel and M. Castrillón López. "Hamiltonian Reduction in Affine Principal Bundles". In: *Geometric Mechanics* 02.04 (Dec. 2025), pp. 519–533
- M. Á. Berbel and M. Castrillón López. *Lie-Poisson reduction in principal bundles by a subgroup of the structure group*. 2026. arXiv: 2604.08288 [math.DG]. URL: https://arxiv.org/abs/2604.08288

Table of Contents

Open Problems

- Lie–Poisson Reduction without a choice of connection Λ on $P \to M$ nor volume form χ_{vol} on M.

  F. Gay-Balmaz, J. C. Marrero, and N. Martínez Alba. "A New Canonical Affine Bracket Formulation of Hamiltonian Classical Field Theories of First Order". In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 118.3 (July 2024), p. 103

- Reduced Tulczyjew triplet in Field Theory.

  K. Grabowska. "A Tulczyjew Triple for Classical Fields". In: *Journal of Physics A: Mathematical and Theoretical* 45.14 (Apr. 2012), p. 145207

Thank you for listening!
Any questions?