

Lie Theorem for superposition rules for PDEs revisited

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Joint work with J. de Lucas

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- ① PDE Lie systems
- ② The foliation approach
- ③ Action Lie algebroids
- ④ Conclusions and further work
- ⑤ References

Outline

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Let $J^r \pi$ be the r^{th} -jet space of a fibre bundle $\pi: E \rightarrow M$.

Under which conditions the general solution of a PDE

$$\mathcal{E} \subset J^r \pi$$

can be expressed in terms of a finite number of particular solutions?

This is the case of two relevant families:

- Darboux integrable PDEs (Anderson, Fels, Vassiliou, Ivey):

$$u_{xy} = e^u \rightsquigarrow u(x, y) := \log \frac{2\dot{f}(x)\dot{g}(y)}{(f(x) + g(y))^2}.$$

- **Lie systems** (Cariñena, Grabowski, Marmo, Winternitz, Odziejewicz, Grundland, de Lucas, Sardón):

$$\dot{x} = b_0(t) + b_1(t)x + b_2(t)x^2 \rightsquigarrow x(t) := \frac{x_{(1)}(x_{(3)} - x_{(2)}) - kx_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k(x_{(3)} - x_{(1)})}.$$

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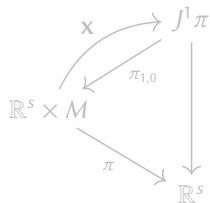
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First-order PDEs

- $\pi: \mathbb{R}^s \times M \rightarrow \mathbb{R}^s$ with adapted coordinates (t^i, u^α) .
- $J^1\pi$ with adapted coordinates $(t^i, u^\alpha, u_i^\alpha)$.

Let now $\mathbf{X}$ be a section of $\pi_{1,0}: J^1\pi \rightarrow \mathbb{R}^s \times M$:



$\mathcal{E} := \mathbf{X}(M) \subset J^1\pi$ is a **PDE on M in normal form**

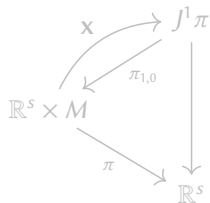
A **(local) solution** of $\mathcal{E}$ is a local section $\phi: \mathbb{R}^s \dashrightarrow \mathbb{R}^s \times M$ of $\pi: \mathbb{R}^s \times M \rightarrow \mathbb{R}^s$ such that $\text{Im}(j^1\phi) \subset \mathcal{E}$.

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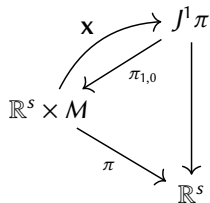
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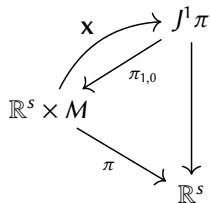
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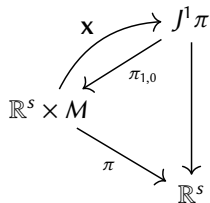
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s-vector fields

- An **s-vector field** is a section of the Whitney sum $\oplus^s TM \rightarrow M$.
- A **t-dependent s-vector field**, with $t = (t^1, \dots, t^s) \in \mathbb{R}^s$, is a section $\mathbf{X}$ of the Whitney sum

$$\oplus^s \text{pr}_M^* TM \rightarrow \mathbb{R}^s \times M, \quad (\text{pr}_M: \mathbb{R}^s \times M \rightarrow M).$$

- Equivalently, $\mathbf{X}: \mathbb{R}^s \times M \rightarrow \oplus^s TM$ such that $\mathbf{X} = (X_1, \dots, X_s)$, with

$$\begin{array}{ccc} \mathbb{R}^s \times M & \xrightarrow{X_i} & TM \\ & \searrow \text{pr}_M & \downarrow \tau_M \\ & & M \end{array}$$

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$$\frac{\partial u}{\partial t^i} = X_i(t, u).$$

A particular solution $\gamma: \mathbb{R}^s \dashrightarrow M$ with $\gamma(0) = u_0$ is called an **integral section** of $\mathbf{X}$ with initial condition u_0 at $t = 0$.

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Superposition rules

Let us consider the **Riccati PDE system** on $\mathbb{R}$:

$$\begin{cases} \frac{\partial u}{\partial t^1} = b_0^1(t) + b_1^1(t)u + b_2^1(t)u^2, \\ \frac{\partial u}{\partial t^2} = b_0^2(t) + b_1^2(t)u + b_2^2(t)u^2. \end{cases}$$

Locally defined map $\Phi: \mathbb{R}^3 \times \mathbb{R} \dashrightarrow \mathbb{R}$

$$\Phi(u_{(1)}, u_{(2)}, u_{(3)}; k) := \frac{u_{(1)}(u_{(3)} - u_{(2)}) - ku_{(2)}(u_{(3)} - u_{(1)})}{(u_{(3)} - u_{(2)}) - k(u_{(3)} - u_{(1)})}.$$

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every particular solution $u(t)$ with $u(0) \in W := \mathbb{R} - \{u_{(2)}(0)\}$ can be cast in the form

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Let $\mathbf{X}$ be an integrable t -dependent s -vector field on M .

Definition (Superposition rule)

A **superposition rule** for $\mathbf{X}$ depending on $\ell \geq 1$ particular solutions is

$$\Phi: M^\ell \times M \dashrightarrow M$$

satisfying

- (i) $\Phi_p := \Phi(p; \cdot): M \dashrightarrow M$ is étale for every $p \in M^\ell$ where it is defined and,
- (ii) there exist open subsets $U \subset M^\ell$ and $W \subset M$ such that, given particular solutions $u_{(1)}(t), \dots, u_{(\ell)}(t)$ with $(u_{(1)}(0), \dots, u_{(\ell)}(0)) \in U$, every particular solution $u(t)$ with $u(0) \in W$ can be cast in the form

$$u(t) = \Phi(u_{(1)}(t), \dots, u_{(\ell)}(t); k),$$

where $k \in M$ is related to the initial conditions.

Let $\mathbf{X}$ be an integrable t -dependent s -vector field on M .

Definition (Superposition rule)

A **superposition rule** for $\mathbf{X}$ depending on $\ell \geq 1$ particular solutions is

$$\Phi: M^\ell \times M \dashrightarrow M$$

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Definition (PDE Lie system)

$\mathbf{X} = (X_1, \dots, X_s)$ is a **PDE Lie system** if there exists a finite-dimensional Lie algebra of vector fields $V \subset \mathfrak{X}(M)$ (a so-called **VG Lie algebra**) such that $(X_i)_t \in V$ for every $t \in \mathbb{R}^s$ and $1 \leq i \leq s$.

Theorem (Lie; Odziejewicz–Grundland; Cariñena–Grabowski–Marmo)

$\mathbf{X}$ admits a superposition rule if and only if it is a PDE Lie system.

Riccati PDE system

$\mathbf{X} = (X_1, X_2)$, where

$$X_i := \left(b_0^i(t) + b_1^i(t)u + b_2^i(t)u^2 \right) \partial_u$$

take values in

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Diagonal prolongations

- The ℓ^{th} -**diagonal prolongation** of a VB $\tau_E: E \rightarrow M$ is the VB over M^ℓ with:
 - total space $E^{[\ell]} := E \times \cdots \times E$;
 - projection $\tau_E^{[\ell]} := \tau_E \times \cdots \times \tau_E$, and
 - the fibre-wise vector space structure on $E_{(u_{(1)}, \dots, u_{(\ell)})}^{[\ell]} = E_{u_{(1)}} \times \cdots \times E_{u_{(\ell)}}$.
- It can be identified with the Whitney sum

$$E^{[\ell]} = \text{pr}_1^* E \oplus \cdots \oplus \text{pr}_\ell^* E \leadsto (\text{TM})^{[\ell]} = \text{TM}^\ell$$

$$\Gamma(E) \ni e \mapsto e^{[\ell]} \in \Gamma(E^{[\ell]})$$

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- Given a VG Lie algebra $V \subset \mathfrak{X}(M)$, there exists $\ell \geq 1$ such that

$$V^{[\ell]} := \langle Y^{[\ell]} : Y \in V \rangle \simeq V$$

possesses a **modular basis**; namely, a basis $\{X_1^{[\ell]}, \dots, X_r^{[\ell]}\}$ such that

$$X_1^{[\ell]} \wedge \dots \wedge X_r^{[\ell]}$$

is nowhere vanishing on an open and dense subset $U \subset M^\ell$.

Let $\mathbf{X} = (X_1, \dots, X_s)$ be a PDE Lie system on M .

Definition (Trivialising submersion)

A **trivialising submersion** for $\mathbf{X}$ is a submersion $\Psi: M \times M^\ell \dashrightarrow M$ such that

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$\mathbf{X}$ admits a trivialising submersion $\Psi: M \times M^\ell \dashrightarrow M$ if and only if there exists a VG Lie algebra $V \subset \mathfrak{X}(M)$ for $\mathbf{X}$ such that $V^{[\ell]}$ possesses a modular basis.

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Given $(p_0, k_0) \in M^\ell \times M$ and $u_0 \in M$, there is a 1-1 correspondence

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$\ell = 3$ and $\Psi: \mathbb{R} \times \mathbb{R}^3 \dashrightarrow \mathbb{R}$ given by

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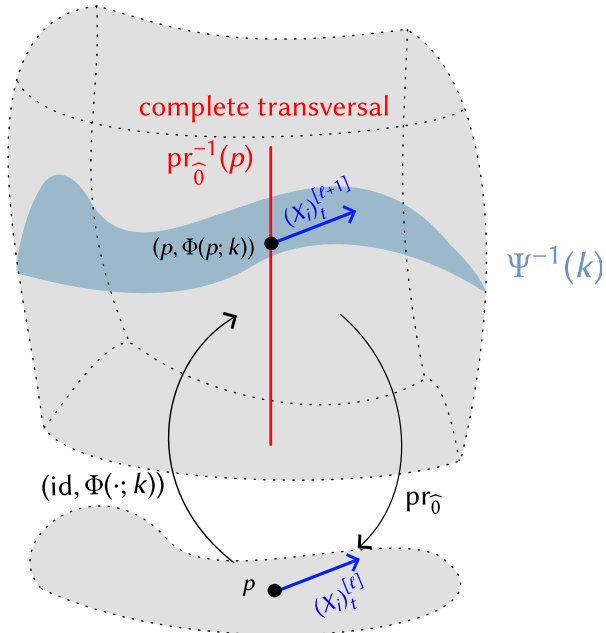
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$D \subset M \times M^\ell$
 generic points of $V^{[\ell+1]}$

$$\text{pr}_{\widehat{0}}(D) \subset M^\ell$$



The global clase: Vessiot's philosophy

- $\Phi: M^\ell \times M \dashrightarrow M$ is **global** if every $\Phi_p: M \dashrightarrow M$ is a global diffeomorphism. In particular,

$$\text{Dom}(\Phi) = U \times M, \quad U \subset M^\ell.$$

- $(U, \Phi) \preceq (U', \Phi')$ if $U \subset U'$ and $\Phi = \Phi'|_{U \times M}$.

Assume that $\Phi: U \times M \rightarrow M$ is maximal with respect to $\preceq$

Then:

- (1) $G_\Phi := \{\Phi_p: p \in U\}$ is a Lie group and $G_\Phi \curvearrowright M$ given by $\Phi_p \cdot k := \Phi(p; k)$ is a Lie group action whose infinitesimal generators span a VG Lie algebra for $\mathbf{X}$.
- (2) The diagonal action $G_\Phi \curvearrowright M^\ell$ is such that $U \subset M^\ell$ is union of principal orbits.

Such an action is referred to as a **pretransitive Lie group action**.

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Let $V \subset \mathfrak{X}(M)$ be a VG Lie algebra.

- Consider $\mathfrak{g}$ an (abstract) Lie algebra isomorphic to V via an infinitesimal action

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$$\rho(v, u) := \mathcal{a}(v)_u,$$

- the Lie bracket on $\Gamma(\mathfrak{g} \times M) = C^\infty(M, \mathfrak{g})$ extending the Lie bracket on constant sections.

It follows that

$$[\alpha, f\beta] = (\mathcal{L}_{\rho(\alpha)}f) \beta + f[\alpha, \beta].$$

Action Lie algebroid $\mathfrak{g} \ltimes M \Rightarrow M$

This Lie algebroid is **integrable**!

- (Palais'57) $\leadsto$ Lie groupoid determined by a local Lie group action integrating V .
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Global Lie Theorem: global superposition rules $\leadsto$ VG Lie algebras formed by **complete** vector fields.

What about **local** superposition rules?

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Submersions $s, t: \mathcal{G} \rightarrow \mathbb{R}$ defined by

$$s(u_{(0)}, p) := u_{(0)}, \quad t(u_{(0)}, p) := \Phi(p; \Psi(u_{(0)}, p_0)).$$

Closed embedding $u: \mathbb{R} \hookrightarrow \mathcal{G}$ given by

$$u(u) := (u, p_0).$$

We have that

$$s \circ u = t \circ u = \text{id}_{\mathbb{R}}.$$

Lie groupoid structure?

The target is a Möbius-type map:

$$t(u_{(0)}, p) := \frac{a(p)u_{(0)} + b(p)}{c(p)u_{(0)} + d(p)} \sim \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix} \begin{pmatrix} u_{(0)} \\ 1 \end{pmatrix}.$$

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These maps endow $\mathcal{G}$ with the structure of a **local Lie groupoid** over $\mathbb{R}$. Its Lie algebroid, $\text{Lie}(\mathcal{G}) \Rightarrow M$, is given by

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Remarkably, $\text{Lie}(\mathcal{G}) = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}$ is the action Lie algebroid associated with the $\mathfrak{sl}(2, \mathbb{R})$ -VG Lie algebra of the Riccati PDE system!

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Outline

- ① PDE Lie systems
- ② The foliation approach
- ③ Action Lie algebroids
- ④ Conclusions and further work
- ⑤ References

Conclusions and further work

- The 4th-diagonal prolongation of $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R} \Rightarrow \mathbb{R}$ is a Hamiltonian Lie algebroid.



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Properties of the 4th-diagonal prolongation of the local Lie groupoid $\mathcal{G}$?

- When is the multiplication globally associative?



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Dziękuję bardzo!
Thank you for your kind attention!

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