Multisymplectic Arnold Conjecture

Based on the work of
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Outline

Hamiltonian Mechanics

Let (M, ω) be a symplectic manifold and

$$H : S^1 \times M \to \mathbb{R}$$

a time-dependent Hamiltonian.

Hamilton's equation for a periodic orbit $Z : S^1 \to M$ is

$$\omega(\cdot, \dot{Z}(t)) = dH_t(Z(t)).$$

Equivalently,

$$\dot{Z} = X_{H_t}(Z), \qquad \iota_{X_{H_t}}\omega = dH_t.$$

- Symplectic geometry studies periodic solutions.
- Dynamics is encoded in the symplectic form.
- Periodic orbits become critical points of an action functional.

The Cotangent Bundle Example

Let Q be a closed manifold. Its cotangent bundle

$$M = T^*Q$$

is the fundamental example of a symplectic manifold.
In local coordinates (q, p),

$$\lambda = p\,dq, \qquad \omega = d\lambda.$$

For the kinetic energy Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2,$$

the Hamiltonian flow is the geodesic flow of Q.

Hence periodic Hamiltonian orbits in T^*Q correspond to closed geodesics on Q.

The Arnold Conjecture

For a closed symplectic manifold (M, ω):

Arnold Conjecture: A Hamiltonian diffeomorphism has at least as many fixed points as required by the topology of M.

Typical lower bounds:

$$\#\mathrm{Fix}(\varphi_H) \geq \sum_i b_i(M),$$

or

$$\#\mathrm{Fix}(\varphi_H) \geq \mathrm{cl}(M) + 1.$$

Floer's breakthrough was to construct an infinite-dimensional Morse theory whose critical points are Hamiltonian periodic orbits.

The Symplectic Action Functional

Let $Z : S^1 \to M$ be contractible and assume $\omega = d\lambda$.
The action functional is

$$\mathcal{A}_H(Z) = \int_{S^1} Z^*\lambda - \int_0^1 H_t(Z(t))\,dt.$$

Its differential is

$$d\mathcal{A}_H(Z)\xi = \int_0^1 \omega(\xi, \dot{Z} - X_H(Z))\,dt.$$

Hence

$$d\mathcal{A}_H(Z) = 0 \iff \dot{Z} = X_H(Z).$$

Thus Hamiltonian periodic orbits are critical points of $\mathcal{A}_H$.

The L^2 Gradient

Choose an ω-compatible almost complex structure

$$J : TM \to TM.$$

The metric

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot)$$

induces an L^2 metric on the loop space,

$$\langle \xi, \eta \rangle_{L^2} = \int_0^1 g(\xi, \eta)\, dt.$$

With respect to this metric,

$$\nabla \mathcal{A}_H(Z) = J(Z)\big(\dot{Z} - X_H(Z)\big).$$

Therefore,

$$\nabla \mathcal{A}_H(Z) = 0 \iff \dot{Z} = X_H(Z).$$

Floer Theory

The negative L^2-gradient flow equation is

$$\partial_s \widetilde{Z} + J(\widetilde{Z})\big(\partial_t \widetilde{Z} - X_H(\widetilde{Z})\big) = 0.$$

Equivalently,

$$\partial_s \widetilde{Z} + J(\widetilde{Z})\partial_t \widetilde{Z} - \nabla H_t(\widetilde{Z}) = 0.$$

Floer trajectories satisfy

$$\lim_{s \to \pm\infty} \widetilde{Z}(s, \cdot) = Z_{\pm}.$$

- Critical points: periodic Hamiltonian orbits.
- In T^*Q, these include closed geodesics.
- Gradient lines: Floer trajectories.
- Chain complex generated by periodic orbits.

Why Ellipticity Is Fundamental

The principal symbol of

$$\bar{\partial}_J = \partial_s + J\partial_t$$

is

$$\sigma(\xi_s, \xi_t) = \xi_s + J\xi_t.$$

It is invertible whenever $(\xi_s, \xi_t) \neq 0$.
Hence:

- $\bar{\partial}_J$ is elliptic.
- Linearizations are Fredholm.
- Moduli spaces are finite-dimensional.
- Compactness follows from energy estimates.
- Floer homology can be defined.

$$\boxed{\text{Ellipticity} \implies \text{Arnold conjecture}}$$

From Newton's Equation to nonlinear Laplace equations

Classical mechanics studies trajectories

$$q : \mathbb{R} \to Q.$$

The fundamental equation is

$$-\partial_t^2 q = \nabla V(q).$$

For two-dimensional domains one obtains

$$q : \Sigma \to Q$$

and

$$-\Delta q = \nabla V(q).$$

- Geodesics $\leftrightarrow$ one-dimensional objects.
- Harmonic maps $\leftrightarrow$ two-dimensional objects.

We seek a Hamiltonian formalism suitable for these PDEs.

De Donder–Weyl equations

Introduce momenta p_1, p_2.
The De Donder–Weyl equations are

$$-\partial_1 p_1 - \partial_2 p_2 = \frac{\partial H}{\partial q},$$

$$\partial_1 q = \frac{\partial H}{\partial p_1}, \qquad \partial_2 q = \frac{\partial H}{\partial p_2}.$$

For

$$H(q, p_1, p_2) = \frac{1}{2}|p_1|^2 + \frac{1}{2}|p_2|^2 + V(q),$$

one recovers the nonlinear Laplace equation.

The Main Analytic Obstruction

The De Donder–Weyl operator is not elliptic.
Indeed, for arbitrary ψ,

$$q \equiv \text{const},$$

$$p_1 = \partial_2 \psi, \qquad p_2 = -\partial_1 \psi$$

satisfies the homogeneous system.
Therefore the operator has an infinite-dimensional kernel.

Consequences:

- no Fredholm theory,
- no compact moduli spaces,
- no Floer homology,
- no Arnold-type theorem.

A different formulation is required.

Complex Regularization

Assume Q carries a complex structure i.
Introduce

$$p = 2\partial_t q, \qquad \partial_t = \frac{1}{2}(\partial_1 - i\partial_2).$$

Then

$$-4\partial_{\bar{t}}\partial_t q = \nabla V(q)$$

becomes

$$-2\partial_{\bar{t}} p = \nabla V(q).$$

The variables (q, p) satisfy a first-order Dirac-type system.

The Bridges Equations

Writing

$$Z = (q, p),$$

one obtains

$$\partial Z = \nabla H(Z),$$

where

$$\partial = \begin{pmatrix} 0 & -2\partial_{\bar{t}} \\ 2\partial_t & 0 \end{pmatrix}.$$

Since

$$\partial^2 = -\Delta,$$

∂ is a Dirac operator.

Unlike De Donder–Weyl theory, the resulting system is elliptic.

CRPS Structures

A complex-regularized polysymplectic structure consists of

$$(W, I, \Omega)$$

with

$$\Omega = \omega_1 \otimes e_1 + \omega_2 \otimes e_2,$$

such that

$$\omega_2 = -\omega_1(\cdot, I\cdot).$$

Key fact:

$$\omega_1, \omega_2$$

are both symplectic forms.
The cotangent bundle of a complex manifold provides the fundamental example.

CRPS Morphisms are Holomorphic

Assume

$$\psi : (W, I, \Omega) \to (W', I', \Omega')$$

satisfies

$$\psi^* \Omega' = \Omega.$$

Since

$$\omega_2 = -\omega_1(\cdot, I\cdot), \qquad \omega_2' = -\omega_1'(\cdot, I'\cdot),$$

and $\psi^* \omega_i' = \omega_i$, we obtain

$$\omega_1'(d\psi\cdot, I' d\psi\cdot) = \omega_1'(d\psi\cdot, d\psi I\cdot).$$

Because ω_1' is symplectic and $d\psi$ is invertible,

$$I' d\psi = d\psi I.$$

Example: Harmonic Maps in Kähler manifolds

Let (Q, i, g) be Kähler and consider

$$H(q,p) = \frac{1}{2}\|p\|_g^2$$

on T^*Q with its canonical CRPS structure.

The Hamiltonian equation

$$\omega_1(\cdot, \partial_x Z) + \omega_2(\cdot, \partial_y Z) = dH_0(Z)$$

becomes, in holomorphic coordinates,

$$-\partial_{\bar{t}}\bar{p}^j = \frac{\partial H_0}{\partial \bar{q}^j}, \qquad \partial_t q^j = h^{j\bar{k}}\bar{p}^k.$$

Using the Kähler identity we get

$$-\partial_{\bar{t}}\partial_t q^l = \Gamma^l_{jk}\, \partial_{\bar{t}} q^k\, \partial_t q^j.$$

Relation to Holomorphic Symplectic Geometry

Define

$$\omega_{\mathbb{C}} = \omega_1 + i\omega_2.$$

Then

$$\Omega \longleftrightarrow \omega_{\mathbb{C}}$$

gives a bijection between

- CRPS structures,
- holomorphic symplectic structures.

However:

Holomorphic Floer theory studies holomorphic Hamiltonians, while CRPS studies real-valued Hamiltonians associated to PDEs.

Action Functional on the Torus

Assume

$$\omega_i = d\lambda_i.$$

The action is

$$\mathcal{A}_H(Z) = \int_{T^2} \big(\lambda_1(\partial_1 Z) + \lambda_2(\partial_2 Z) - H(Z)\big)\, dV.$$

Its critical points satisfy

$$dH = \omega_1(\cdot, \partial_1 Z) + \omega_2(\cdot, \partial_2 Z).$$

This is the CRPS Hamiltonian equation.

The Fueter Operator

There exist structures I, J, K with

$$IJ = K, \qquad JK = I, \qquad KI = J.$$

The gradient of the action is

$$\nabla \mathcal{A}_H(Z) = J\partial_1 Z + K\partial_2 Z - \nabla H(Z).$$

and the negative L^2-gradient flow equation is

$$\partial_s \widetilde{Z} + J\partial_1 \widetilde{Z} + K\partial_2 \widetilde{Z} - \nabla H(\widetilde{Z}).$$

When $H = 0$ we get

$$I\partial_s \widetilde{Z} - J\partial_2 \widetilde{Z} + K\partial_1 \widetilde{Z} = 0.$$

This is the Fueter equation, the quaternionic analogue of the Cauchy–Riemann equation.

Why a Multisymplectic Extension?

CRPS geometry is naturally formulated for maps

$$T^2 \to W,$$

using the global frame on the torus.

For a general Riemann surface Σ:

- $T\Sigma$ need not be trivial,
- the polysymplectic description is no longer intrinsic,
- one must use multisymplectic geometry.

Fields are now sections of a bundle

$$\pi : Y \to \Sigma,$$

and the cotangent bundle is replaced by the *extended multimomentum bundle*

$$\Lambda^2_2(Y).$$

Extended Multimomentum Bundles

The multisymplectic analogue of T^*Q is

$$\Lambda^2_2(Y) = \{\lambda \in \Lambda^2(Y) \mid u \lrcorner (v \lrcorner \lambda) = 0 \; \forall \, u, v \in \ker d\pi\}.$$

In local coordinates (t^1, t^2, q^a) on Y:

$$(t^1, t^2, q^a, p^a_1, p^a_2, p),$$

where p^a_i are multimomenta and p is an energy variable.

The canonical forms are

$$\Lambda = p \, dt^1 \wedge dt^2 + \sum_a \left(p^a_1 \, dq^a \wedge dt^2 - p^a_2 \, dq^a \wedge dt^1 \right), \;\; \Omega = d\Lambda.$$

A Hamiltonian section $p = -H$ induces the De Donder–Weyl equations

$$-\partial_i p^a_i = \frac{\partial H}{\partial q^a}, \qquad \partial_i q^a = \frac{\partial H}{\partial p^a_i}.$$

Complex Structures on the Multimomentum Bundle

Let

$$\pi : (Y, i) \to (\Sigma, j)$$

be a holomorphic fibre bundle.
Now consider the complex-invariant subbundle

$$\Lambda^C(Y) = \{\lambda \in \Lambda_2^2(Y) \mid \lambda(i\cdot, i\cdot) = \lambda\}.$$

In local holomorphic coordinates

$$(t^1, t^2, q_1^a, q_2^a),$$

the constraints become

$$p_{1,1}^a = p_{2,2}^a, \qquad p_{1,2}^a = -p_{2,1}^a.$$

Hence only two momentum variables survive:

$$P_1^a := p_{1,1}^a, \qquad P_2^a := -p_{1,2}^a.$$

The Elliptic Hamilton Equations

Using the variables $(q_1^a, q_2^a, P_1^a, P_2^a)$, the induced multisymplectic form is

$$\Omega = \sum_a (dP_1^a \wedge dq_1^a + dP_2^a \wedge dq_2^a) \wedge dt^2$$
$$- \sum_a (dP_1^a \wedge dq_2^a - dP_2^a \wedge dq_1^a) \wedge dt^1.$$

For a Hamiltonian H, the corresponding equations are

$$\frac{\partial H}{\partial q_1^a} = -\partial_1 P_1^a + \partial_2 P_2^a, \qquad \frac{\partial H}{\partial q_2^a} = -\partial_1 P_2^a - \partial_2 P_1^a,$$
$$\frac{\partial H}{\partial P_1^a} = \partial_1 q_1^a + \partial_2 q_2^a, \qquad \frac{\partial H}{\partial P_2^a} = \partial_1 q_2^a - \partial_2 q_1^a.$$

These are precisely the Bridges equations, which form an elliptic Dirac-type system.

Complex-Regularized Multisymplectic Forms

This motivates the definition of a *complex-regularized multisymplectic form*:
A CRMS bundle

$$\sigma : (W, I) \to (\Sigma, j)$$

carries a closed 3-form Ω satisfying:

1. $d\Omega = 0$;
2. Ω is 1-horizontal;
3. fiberwise nondegeneracy;
4. $$\Omega(I\xi, v_1, v_2) = -\Omega(\xi, v_1, Iv_2).$$

These axioms generalize CRPS geometry to arbitrary Riemann surfaces.

A Darboux Theorem for CRMS Bundles

Around every point there exist local coordinates, compatible with the complex structures, such that

$$\Omega = \Omega_{\mathrm{std}} - dH_t \wedge dt^1 \wedge dt^2,$$

where

$$\Omega_{\mathrm{std}} = \omega_1 \wedge dt^2 - \omega_2 \wedge dt^1,$$

and (ω_1, ω_2) are the standard CRPS forms on $\mathbb{C}^{2n}$.

Consequently, for every non-vertical vector ξ,

$$\Omega(\xi, \cdot, \cdot)$$

induces the standard CRPS structure on the fibers.

Compatible Metrics and Quaternionic Structure

For every non-zero tangent vector

$$\rho \in T\Sigma$$

there is an operator

$$J(\rho) : V \to V$$

such that

$$J(\rho)^2 = -|\rho|^2 \,\mathrm{Id},$$

and

$$\omega_\rho = g(\cdot, J(\rho)\cdot).$$

Moreover,

$$J(j\rho) = I\, J(\rho).$$

The CRMS Action Functional

Assume

$$\Omega = d\Lambda.$$

For a section $Z : \Sigma \to W$, define

$$\mathcal{A}(Z) = \int_\Sigma Z^* \Lambda.$$

For a variation $\dot{Z} \in \Gamma(Z^* V)$,

$$d\mathcal{A}(Z)(\dot{Z}) = \int_\Sigma Z^*(\iota_{\dot{Z}} \Omega).$$

Choose a metric h on Σ, a fiber metric g, and an Ehresmann connection.

Then $\mathcal{A}$ admits an L^2-gradient, and its critical points are solutions of the multisymplectic Hamiltonian equations.

Computing the Gradient

Let $\{\rho_1, \rho_2 = j\rho_1\}$ be a local orthonormal frame.
The first variation can be rewritten as

$$\Omega(\dot{Z}, dZ(\rho_1), dZ(\rho_2)) = g\Big(\dot{Z}, J(\rho_2)\nabla_{\rho_1} Z - J(\rho_1)\nabla_{\rho_2} Z - \alpha(Z)\Big).$$

Define

$$J^{\nabla} Z(\rho) = J(\rho)\big(-I\nabla_{\rho} Z - \nabla_{j\rho} Z\big).$$

Hence

$$\nabla\mathcal{A}(Z) = J^{\nabla} Z - \alpha(Z).$$

Floer Sections

Negative L^2-gradient trajectories

$$s \mapsto \widetilde{Z}(s, \cdot)$$

satisfy

$$\nabla_{\partial_s} \widetilde{Z} + J^{\nabla} \widetilde{Z} = \alpha(\widetilde{Z}).$$

Introducing

$$\widetilde{I}(\rho_s, \rho) = \rho_s I + J(\rho),$$

the equation becomes

$$\sum_{l=1}^{3} \widetilde{I}(v_l) \nabla_{v_l} \widetilde{Z} = I\alpha(\widetilde{Z}).$$

Floer equation $\iff$ perturbed Fueter equation.

Ellipticity Recovered

Classical Floer theory:

$$\bar{\partial}_J = \partial_s + J\partial_t.$$

CRMS Floer theory:

$$\mathcal{F} = \sum_{l=1}^{3} \widetilde{I}(v_l)\nabla_{v_l}.$$

- Cauchy–Riemann operator $\to$ Fueter operator.
- Holomorphic curves $\to$ pseudo-Fueter curves.
- Ellipticity survives.
- Fredholm and compactness theories become available.

This is the key conceptual advance of CRMS geometry.

Cuplength Results in CRPS

For

$$H(q,p) = \frac{1}{2}|p|^2 + h(q,p),$$

on

$$T^*T^{2n},$$

Brilleslijper and Fabert prove a lower bound

$$\#\mathrm{Crit}(A_H) \geq \mathrm{cl}(T^{2n}) + 1 = 2n + 1.$$

This is a direct analogue of the Arnold philosophy:

$$\text{Topology} \implies \text{Hamiltonian solutions.}$$

A Multisymplectic Arnold Conjecture

As in symplectic geometry, one expects a Hamiltonian Floer homology independent of the Hamiltonian, hence computable for $H = 0$.

Symplectic case:

$$0 = \nabla \mathcal{A}(Z) = J(Z)\dot{Z} \quad \Longleftrightarrow \quad \dot{Z} = 0,$$

so the Floer generators are points of phase space $\Rightarrow$ topology.

Multisymplectic case:

$$0 = \nabla \mathcal{A}(Z) = J(\rho)\big(-I\nabla_\rho Z - \nabla_{j\rho} Z\big)$$

$$\Longleftrightarrow \qquad -I\nabla_\rho Z - \nabla_{j\rho} Z = 0,$$

so the generators are $(-I)$-holomorphic curves $\Rightarrow$ complex geometry.

Holomorphic Floer Theory (Kontsevich–Soibelman)

Holomorphic Floer Theory (ongoing) associates Floer-type invariants to holomorphic symplectic manifolds with holomorphic Hamiltonians.

- For $H = 0$, the relevant solutions are Fueter-type curves
- It aims to relate Floer theory, wall-crossing phenomena, deformation quantization, and resurgence theory.
- They consider holomorphic Hamiltonians only, which correspond to symmetries of complex-regularised multisymplectic bundles.
- Physical and geometric applications typically involve *real* Hamiltonians (e.g. harmonic maps and field theories).
- These lie *outside* the present framework of Kontsevich-Soibelman.

Combining the holomorphic symplectic viewpoint of Kontsevich-Soibelman with multisymplectic geometry may be fruitful for both theories!

Conclusions

1. The Arnold conjecture relies on elliptic Floer theory.
2. Standard multisymplectic equations are not elliptic.
3. CRPS geometry restores ellipticity via Dirac/Fueter operators.
4. CRMS geometry extends this framework to arbitrary Riemann surfaces.
5. Floer sections satisfy perturbed Fueter equations.
6. This provides a natural route toward a multisymplectic Arnold conjecture.

$$\boxed{\text{Cauchy–Riemann} \Longrightarrow \text{Fueter}}$$

$$\boxed{\text{Arnold} \Longrightarrow \text{Multisymplectic Arnold}}$$