Multisymplectic structures as G-structures

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Structure of the talk

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1. Motivation

Motivation I

Let M be a manifold endowed with some geometric structure such as symplectic, Poisson, contact... (we will later restrict to multisymplectic structures). Usually, dynamics on this manifold are described by specifying a Hamiltonian which is usually (but not always) a function $H \in \mathscr{C}^\infty(M)$. There are a lot of results on such dynamics that depend on the existence of symmetries: Both of the geometric structure and the Hamiltonian.

Therefore, it is usually better to understand the symmetries of said geometry and then look for those that are also symmetries of the Hamiltonian.

Motivation II

Here an uncomfortable difference appears between mechanics and field theory:

1. In a symplectic manifold $(M, \omega \in \Omega^2(M))$, (local) symmetries are parametrized by functions $f \in \mathscr{C}^\infty(M)$ by $\iota_{X_f}\omega = \mathrm{d}f$.
2. In a multisymplectic manifold $(M, \omega \in \Omega^{k+1}(M))$, (local) symmetries are parametrized by Hamiltonian forms:

$$\alpha \in \Omega^{k-1}(M) \colon \exists X_\alpha \in \mathfrak{X}(M) \text{ with } \mathrm{d}\alpha = \iota_{X_\alpha}\omega \,.$$

However, depending on the field theory (namely depending on the Lagrangian), the structure of Hamiltonian forms differs drastically.

Motivation III

For instance:

(i) On a real scalar theory, with coordinates (x^μ, ϕ) and momenta (x^μ, ϕ, p^μ), symmetries of the multisymplectic form are

$$X = A^\mu(x, \phi)\frac{\partial}{\partial x^\mu} + B(x, \phi)\frac{\partial}{\partial \phi} + \text{terms with } \frac{\partial}{\partial p_\mu}$$

(ii) However, if the standard fiber is of dimension > 1, we have

$$X = A^\mu(x)\frac{\partial}{\partial x^\mu} + B^i(x, \phi)\frac{\partial}{\partial \phi^i} + \text{terms with } \frac{\partial}{\partial p^i_\mu}\,.$$

This is not exclusive to these examples: The multisymplectic forms appearing in field theories have very different properties.

Motivation IV

We claim that this is an algebraic problem: the existence of symmetries of different nature has to do with the linear type of the multisymplectic form.

In mechanics, we have that every pair of symplectic forms are isomorphic, but multisymplectic forms are not and, in fact, they are not classified (and not expected to be). For these issues, the theory of G-structures is particularly well suited, as we will see.

Furthermore, since existence of symmetries and flatness results (results guaranteeing that a form has constant coefficients) are tightly related, this study also gives results regarding the existence of such charts.

2. G-structures

The bundle of frames

Let M be a smooth manifold of dimension n and V be a real n-dimensional vector space. The bundle of frames is defined as

$$\mathcal{F}(M) := \{\text{linear isomorphisms } \phi\colon V \to \mathrm{T}_x M\}.$$

It is naturally a $\mathrm{GL}(V)$-principal bundle over $\pi\colon \mathcal{F}(M) \to M$.

Definition
Let $G \subseteq \mathrm{GL}(V)$ be a closed Lie group. A G-structure is a reduction of the frame bundle to a bundle with structure group G:

$$B_G \hookrightarrow \mathcal{F}(M).$$

G-structures (II)

Definition
Let $B_G \hookrightarrow \mathcal{F}(M)$ be a G-structure on M. A local frame

$$\phi|_x \colon V \xrightarrow{\sim} \mathrm{T}_x M$$

is said to be adapted if it corresponds to a local section $\phi\colon M \to B_G \subseteq \mathcal{F}(M)$.

Let $\psi\colon V \to M$ be a local diffeomorphism. Then, the pushforward induces a frame $\phi|_x \colon V \xrightarrow{\sim} \mathrm{T}_x M$ by $\phi|_x := (\psi_*)|_{\psi^{-1}(x)}$.

Definition
A G-structure B_G is said to be locally flat if there is an adapted frame induced by a local diffeomorphism $\psi\colon V \to M$.

3. Tableaux and Spencer cohomology

Cartan Tableaux (I)

A Cartan tableau is a subspace $\mathcal{A} \subseteq V^* \otimes W$, where V and W are finite-dimensional vector spaces. Intuitively, these represent homogeneous systems of first-order linear PDEs (with constant coefficients). The system is a PDE on maps $f\colon V \to W$, and reads as

$$\mathrm{d}_v f \in \mathcal{A}\,, \qquad \forall v \in V\,.$$

Namely, if $\{v_i\}$ and $\{w_k\}$ denote bases of V and W, respectively, and

$$\mathcal{A} = \left\{ a_i^k v^i \otimes w_k \text{ such that } \pi^i_k a_i^k = 0 \right\}\,, \qquad \pi^i_k \in \mathbb{R},$$

the PDE it defines is

$$\pi^i_k \frac{\partial f^k}{\partial v^i} = 0\,, \qquad \text{for some} \qquad f \in \mathscr{C}^\infty(V, W)\,.$$

Cartan Tableaux (II)

To find solutions to this PDE, we will try to solve the PDE formally as a sum

$$f = \sum_{j=0}^{\infty} f^{(j)} ,$$

where each $f^{(j)} \in \mathscr{C}^{\infty}(V, W)$ is a polynomial solution of degree j. In terms of multilinear algebra, a polynomial map of degree j $f^{(j)} \colon V \to W$, can be interpreted as an elements of the tensor product of the symmetric product of V^* with W

$$f^{(j)} \in S^j(V^*) \otimes W : \qquad f^{(j)}(v) = a^k_{i_1,\dots,i_j} v^{i_1} \odot \cdots \odot v^{i_j} \otimes w_k$$

The space of elements $f^{(j)} \in S^j(V^*) \otimes W$ that solve the above PDE is called the extension of degree j.

Cartan Tableaux (III)

Definition
The *j-th extension* of $\mathcal{A}$, denoted by $\mathcal{A}^{(j)}$, is the set of polynomial solutions of degree j. Formally,

$$\mathcal{A}^{(j)} = \{f \in S^j(V^*) \otimes W \colon \iota_{v_{j-1}} \cdots \iota_{v_1} f \in \mathcal{A}\,, \forall v_i\}\,.$$

A tableau is called of *finite type* if $\mathcal{A}^{(j)} = 0$ for some finite j. If it is not of finite type, it is called of *infinite type*.

Example
A Lie algebra $\mathfrak{g} \subseteq V^* \otimes V$ can be considered as a Cartan tableau. Its prolongations

$$\mathfrak{g}^{(j)} \subseteq S^j(V^*) \otimes V$$

can be interpreted as the *polynomial symmetries* of degree j of the flat G-structure $V \times G \hookrightarrow \mathcal{F}(V)$.

Spencer Cohomology (I)

Let us define the complex

$$C^{l,j}(\mathcal{A}) := \bigwedge^{l} V^* \otimes \mathcal{A}^{(j)} \subseteq \bigwedge^{l} V^* \otimes S^j(V^*) \otimes W\,,$$

together with the operator
$\partial\colon \bigwedge^l V^* \otimes S^j(V^*) \otimes W \to \bigwedge^{l+1} V^* \otimes S^{j-1}(V^*) \otimes W$ given by the linearization of the de Rham differential:

$$\partial(\alpha \otimes (v^1 \cdots v^j) \otimes w) := \sum_i v^i \wedge \alpha \otimes (v^1 \cdots \widehat{v}^i \cdots v^j) \otimes w\,.$$

It can easily be shown that $\partial C^{l,j}(\mathcal{A}) \subseteq C^{l+1,j-1}(\mathcal{A})$. The spaces

$$H^{l,j}(\mathcal{A}) := \frac{\ker\left(\partial\colon C^{l,j}(\mathcal{A}) \to C^{l+1,j-1}(\mathcal{A})\right)}{\operatorname{Im}\left(\partial\colon C^{l-1,j+1}(\mathcal{A}) \to C^{l,j}(\mathcal{A})\right)}$$

are called the Spencer cohomology groups.

Integrability of G-structures (I)

The Spencer cohomology $H^{2,j}(\mathfrak{g})$ of interpreting a Lie algebra as a tableaux gives obstructions to integrability of order $(j+1)$ for any G-structure. Let us begin with some definitions:

Definition
A G-structure B_G on M is called $(k+1)$-flat at x if there is a coordinate chart

$$\psi \colon V \to M$$

with $\psi(0) = x$ and such that the induced frame $\phi \colon M \to \mathcal{F}(M)$ is tangent to B_G to order k at x.

Intuitively, these are coordinates making the structure flat to order k at a particular point x.

Integrability of G-structures (II)

V. Guillemin. "The Integrability Problem for G-Structures". *Transactions of the American Mathematical Society*, 1965

Let B_G be a G-structure on M. Then, G acts on every space $H^{2,j}(\mathfrak{g})$ so that there are induced vector bundles

$$B_G[H^{2,j}(\mathfrak{g})] \to M\,.$$

Theorem (Guillemin)
A G-structure on M which is globally k-flat has a natural section of $B_G[H^{2,k}(\mathfrak{g})]$, denoted by c_k and called the structure tensor of order k. Then, the G-structure is $(k+1)$-flat at x if and only if $c_k(x) = 0$.

These tensors are constructed via the solder form (and higher order analogues).

Integrability of G-structures (III)

A G-structure is formally flat at x if it is k-flat, for every k. We say that it is formally flat, if it is formally flat at every point. Formal flatness of G-structures can be decided in a finite number of steps by checking whether the previous sections vanish. The existence of true flat coordinates is more involved. In the analytic case, it follows. However, we do not know if there exists a counterexample in the smooth case to the following question:

Is every formally flat G-structure locally flat?

Some positive cases follow from results in the literature:

(i) True if G acts on V irreducibly (or if the action is completely reducible).

(ii) True if G is of finite type, namely $\mathfrak{g}^{(j)} = \{0\}$ for some j.

4. Multisymplectic structures as G-structures

The multisymplectic groups

Let $\omega \in \Omega^{k+1}(M)$ be a differential $(k+1)$-form. When does (M, ω) induce a G-structure?

Proposition
A pair (M, ω) induces a G-structure if and only if it is of constant linear type; namely, there are linear isomorphisms

$$\phi_{x,y} \colon \mathrm{T}_x M \xrightarrow{\sim} \mathrm{T}_y M$$

satisfying $\phi_{x,y}^* \omega|_y = \omega|_x$.

In particular, there exists a linear model (V, ϖ) with $\varpi \in \bigwedge^{k+1} V^*$, and G is the stabilizer group:

$$G = G_\varpi = \{g \in \mathrm{GL}(V) : g^* \varpi = \varpi\}.$$

We call these groups the multisymplectic groups.

The multisymplectic Lie algebras

For most of our considerations, the Lie algebra of G_ϖ, denoted by $\mathfrak{g}_\varpi$, will be the fundamental object. It is given by:

$$\mathfrak{g}_\varpi = \{T \in V^* \otimes V \colon \iota_T \varpi = 0\},$$

where the contraction by a $(1,1)$ tensor is given by

$$(\iota_T \varpi)(v_1, \ldots, v_k) := \sum_{j=1}^{k+1} \varpi(v_1, \ldots, v_{j-1}, T(v_j), v_{j+1}, \ldots, v_k).$$

How can we understand the extensions $\mathfrak{g}_\varpi^{(j)}$?

Let $\ker \varpi = \{v \in V \colon \iota_v \varpi = 0\}$. Then, we have $S^j(V^*) \otimes \ker \varpi \subseteq \mathfrak{g}_\varpi^{(j)}$, and there is a natural isomorphism

$$\mathfrak{g}_\varpi^{(j)} / \left(S^j(V^*) \otimes \ker \varpi \right) \cong$$

$$\{\text{Hamiltonian } (k-1)\text{-forms of homogeneous degree } (j+1)\}.$$

Computing Hamiltonian $(k-1)$-forms (I)

Let (M, ω) be a flat multisymplectic manifold with linear type $\varpi \in \bigwedge^{k+1} V^*$. We compute the space of (local) Hamiltonian forms as follows:

(i) Calculate the Lie algebra $\mathfrak{g}_\varpi$. (In most relevant scenarios, it will be an abelian extension, direct product, quotient, etc., of known Lie algebras, which simplifies the subsequent steps).

(ii) Compute its extensions $\mathfrak{g}_\varpi^{(j)}$. The set of homogeneous polynomial Hamiltonian forms is then given by the previous isomorphism.

The previous discrepancy (scalar field vs. dimension of fibers > 1) can be understood from this point of view:

(i) In the first, $\mathfrak{g}_\varpi$ is an abelian extension of $\mathfrak{gl}(n+1)$.

(ii) In the second, $\mathfrak{g}_\varpi$ is an abelian extension of $\mathfrak{block}(n, m)$.

5. Flatness of multisymplectic forms

The natural representation (I)

Recall that if the natural action of $G \subseteq \mathrm{GL}(V)$ on V is completely irreducible, the vanishing of the structure tensors are a necessary and sufficient condition for the existence of flat coordinates. This can give a (hard to apply in general) strategy to proving this:

(i) Let $W \subsetneq V$ be a G-invariant subspace. Then, we have induced representations

$$\rho_W \colon G \to \mathrm{GL}(W) \qquad \text{and} \qquad \rho_{V/W} \colon G \to \mathrm{GL}(V/W)\,.$$

Denote by G_W and $G_{V/W}$ the image of these morphisms.

(ii) Take an arbitrary formally flat G-structure. The previous decomposition induces an involutive distribution.

(iii) The G-structure induces a G_W-structure on each leaf (leaf geometry) and a $G_{V/W}$-structure on the space of leaves (geometry transversal to leaves).

The natural representation (II)

(iv) If every formally flat G_W and $G_{V/W}$ structures are flat, we can take coordinates on M adapted to the geometry of the leaves and transversal to the leaves.

(iiv) However, these are not coordinates adapted to the original G-structure, as there could be "compatibility conditions". We may try to modify these coordinates to be flat coordinates on the original manifold, but this is hard in full generality.

Nevertheless, for multisymplectic structures, the obstructions for solving this last step can be understood with the Moser trick.

The natural representation (III)

Definition
A subspace $W \subseteq V$ of a multisymplectic vector space (V, ϖ), $\varpi \in \bigwedge^{k+1} V^*$ is called *j-isotropic* if $\iota_{w_1 \wedge \cdots \wedge w_{j+1}} \varpi = 0$, for all $w_i \in W$.

Then, we have the following alternative:

Theorem (de León, Lainz, I-L)
Let $\varpi \in \bigwedge^k V^*$ be multisymplectic. If the natural representation is not irreducible and does not decompose as a direct sum, then (at least) one of the following holds:

(i) *There is a proper invariant j-isotropic subspace.*
(ii) *There is an invariant subspace W for which $W^{\perp,k} = \{0\}$.*

Flatness results (I)

In view of the previous discussion, the alternative given above allows for a systematic study of existence of flat coordinates:

(i) If the representation is irreducible, every formally flat G_ϖ-structure is locally flat.

(ii) If the representation decomposes as a direct sum, it suffices to study each induced representation. In particular, a G_ϖ-structure is locally flat if and only if the induced $G_\varpi|_{W_1}$ and $G_\varpi|_{W_2}$-structures are locally flat.

(iii) The last two scenarios are difficult, in general. However, we may use the Moser trick in some scenarios to understand the precise obstruction to integrability.

Flatness results (II)

The Moser trick implies the following:

Proposition (Moser trick)
Let $\varpi \in \bigwedge^{k+1} V^*$ and let M be endowed with a G_ϖ-structure. Since G_ϖ acts on

$$S = \{\iota_v \varpi \colon v \in V\},$$

we have an induced G_S-structure on M (G_S being the group of linear maps preserving the subspace S). Then, a G_ϖ-structure which is formally flat is locally flat if and only if every formally flat G_S-structure is locally flat.

Namely, ω admits a chart in which it has constant coefficients if and only if there are coordinates in which the bundle $\{\iota_v \omega \colon v \in \mathrm{T}M\}$ is generated by forms of constant coefficients.

Flatness results (III)

In the 1-isotropic case, we have the following:

Theorem (de León, Lainz, I-L)
Let $\varpi \in \bigwedge^{k+1} V^*$ be a multisymplectic form which admits an invariant 1-isotropic subspace and a complementary k-isotropic (non necessarily invariant) subspace. Then, any G_ϖ-structure is flat if and only if the induced $\rho_{V/W}(G_\varpi)$-structure is flat.

What about the case where there is an invariant W with $W^{\perp,k} = \{0\}$?

Proposition
The map induced by restriction

$$\Phi\colon \mathfrak{g}_\varpi \longrightarrow \mathrm{End}(W)$$

is a monomorphism.

Examples (I)

Let us study some examples. The linear models are primarily borrowed from:

L. Ryvkin. "Darboux type theorems in multisymplectic geometry". *Accepted for publication in the Springer INdAM volume 'Poisson Geometry and Mathematical Physics'*, 2025

Disclaimer: Our study only guarantees the existence of tensorial obstructions (namely, the obstruction tensors coming from $H^{2,j}(\mathfrak{g}_{\varpi})$) for the linear types $\varpi \in \bigwedge^k V^*$ to which the method applies. Obtaining the precise obstructions is crucial (requiring a deep understanding of the stabilizer group G_{ϖ}) and heavily depends on ϖ.

Multicotangent bundles

If $V = L \oplus \bigwedge^{k-1} L^*$ and $\varpi \in \bigwedge^k V^*$ is defined by

$$\varpi((l_1, \alpha_1), \ldots, (l_k, \alpha_k)) := \sum_{j=1}^{k} (-1)^{j-1} \alpha_j(l_1, \ldots, \widehat{l_j}, \ldots, l_k)\,,$$

we find that G_ϖ is an abelian extension of $\mathrm{GL}(L)$. In fact, when $k \geq 3$, it leaves invariant the subspace $W = \{0\} \oplus \bigwedge^{k-1} L^*$, which is easily seen to be 1-isotropic. In particular, a G_ϖ-structure is flat if and only if the induced $\rho_W(G_\varpi)$ and $\rho_{V/W}(G_\varpi)$-structures are flat and the PDE above admits a solution. Since the first is of finite type and the second satisfies $\rho_{V/W}(G_\varpi) = \mathrm{GL}(L)$, we conclude that every formally flat G_ϖ-structure is locally flat (the PDE may be solved employing the Poincaré Lemma).

Examples (III)

Product type

Let (V_1, ϖ_1) and (V_2, ϖ_2) be multisymplectic vector spaces with $k \geq 3$. Then, on $V = V_1 \oplus V_2$, take the linear model

$$\varpi = \varpi_1 - \varpi_2 \in \bigwedge^k V^* \,.$$

It is easy to show that $G_\varpi \cong G_{\varpi_1} \times G_{\varpi_2}$, and thus we deduce that a Darboux theorem exists for ϖ if and only if it exists for ϖ_1 and ϖ_2. If the ϖ_i are volume forms, it follows that linear models of the form

$$\varpi = x^1 \wedge \cdots \wedge x^k + y^1 \wedge \cdots \wedge y^k \,,$$

for $k \geq 3$, satisfy a Darboux theorem (and the requirement is simply that the induced distributions are involutive).

Obstructions of arbitrary order (I)

Most known Darboux theorems (at least to the authors) are of first order. It is then a natural question to ask:

Question: Is there a bound j for which a Darboux theorem is necessarily of order $\leq j$?

With the theory of G-structures in mind, this is equivalent to asking:

Question: Is there a bound j for which $H^{2,m}(\mathfrak{g}_{\varpi}) = \{0\}$ for all $m \geq j$ and all ϖ? We answer this question in the negative by proving that for every $j \geq 0$, there is a linear model ϖ_j satisfying

$$H^{2,j}(\mathfrak{g}_{\varpi_j}) \neq \{0\}\,.$$

The construction of this linear model is explicit.

Obstructions of arbitrary order (II)

Let V be a $2j+5$ dimensional vector space and $\{x_0,\dots,x_{j+1},y_0,y_1,p^0,\dots,p^j\}$ denote a basis. Define the 3-form

$$\varpi := \sum_{i=0}^{j} p_i \wedge \alpha^i = \sum_{i=0}^{j} p_i \wedge \left(y^0 \wedge x^i + y^1 \wedge x^{i+1}\right) .$$

Then, we have

Theorem (de León, Lainz, I-L)
The Lie algebra $\mathfrak{g}_\varpi \subseteq V^* \otimes V$ satisfies $H^{2,j}(\mathfrak{g}_\varpi) \neq \{0\}$.

Notice that this is a Theorem of 3-forms. A similar result for k-forms can simply be obtained on a $(2j+k+2)$-dimensional vector space simply by multiplying by a volume element.

Conclusions

(i) There are several relevant questions in the multisymplectic framework for field theories which can benefit from an application of the well developed theory of G-structures.

(ii) In this work we took two main directions:

  (a) The computation of the space of (polynomial) Hamiltonian forms as the set of extensions of $\mathfrak{g}_\varpi$, which allows for a systematic understanding of the possible symmetries, depending on the linear type of $\varpi \in \bigwedge^{k+1} V^*$.

  (b) The existence of Darboux coordinates: We have showed that the study of the natural representation of $\mathfrak{g}_\varpi$ gives a strategy to prove flatness theorems. In fact, the Moser trick allows for a better understanding of the difficult step in proving flatness of a G-structure.

This talk was based on:

M. de León, R. Izquierdo-López, and M. Lainz. ***The Spencer cohomology and integrability of multisymplectic structures*.** 2026. arXiv: 2607.05267 [math.DG]

Bardzo dziękuję za uwagę!

Questions?

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