Variational principles in physics (how to use them and how not to use)

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I dedicate this work to the memory of Włodzimierz Tulczyjew, an outstanding mathematical physicist, a student of Leopold Infeld and one of my wonderful professors of physics at the University of Warsaw.

Abstract. We show that the standard method of deriving field equations (i.e. Euler-Lagrange equations) from a variational principle, based on imposing *a priori* the field's boundary conditions, cannot be applied to hyperbolic field theory because it leads to a contradiction. This contradiction has been overlooked by theoretical physicists. A way to avoid the problem is proposed. This method consists in describing Lagrangian field theory as a symplectic control system. Implications for various branches of physics are discussed.

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1. Introduction

Variational principles in physics have a long history. The first "least something" principle was likely formulated by Willobord Snell in 1621. He analyzed the phenomenon of light refraction and observed that a light ray, when passing from air to water, behaves like an intelligent horse rider who must move from a favorable grassy terrain to a less favorable, sandy terrain, where the horse's speed is significantly slower. The straight line between the departure point A and the arrival point B is obviously not the quickest one: It is namely worthwhile to lengthen the quick part of the journey over the grass if the slow part, over the sand, can be shortened. It is a matter of a simple, undergraduate mathematics to show, that the quickest trajectory is the one which obeys what we call now the "Snell's law":

$$\frac{\sin\alpha}{\sin\beta} = \frac{V_1}{V_2}\,. \tag{1}$$

Here, α and β are the "angle of incidence" and, respectively, "angle of refraction", i.e. the angles at which the first and, respectively, the second interval of the trajectory intersect the line perpendicular to the boundary of the two regions at the transition point. Moreover, $V_1 > V_2$ denote the speed of the horse over the first and the second terrain. Snell had absolutely no possibility to measure the speed of light (neither in the air, nor in water) but he correctly conjectured, that the refraction of light is due to the same phenomenon: both the horseman and the light ray choose the quickest trajectory.

A simple mathematical proof of the formula (1) consists in minimizing the following function of a single variable:

$$T(X) := \frac{d(A,X)}{V_1} + \frac{d(X,B)}{V_2}\,, \tag{2}$$

where $d(.,.)$ denotes the distance between two points, whereas X is the transition point at the boundary of the two regions (media).

The Snell's "least time" principle was generalized in 1658 by Pierre de Fermat. What we know now as the "Fermat principle" is the following statement: when light travels between two points, it follows the path that requires the least amount of time, which is a stationary time, not necessarily the shortest distance. This means, in modern language, that value of the following integral:

$$T := \int_A^B \frac{\mathrm{d}s}{V} \tag{3}$$

must be minimized, where the integrand $\mathrm{d}T = \frac{\mathrm{d}s}{V}$ denotes the infinitesimal time, i.e. the time it takes for the light to cover the infinitesimal distance $\mathrm{d}s$ with speed V. Fermat formula is a consequence of the Snell's principle, as is easily seen if we approximate the material with the continuous change of the light speed $V(x)$ by the collection of N narrow strips of a homogeneous material with the constant light speed V_i, $i = 0, 1, \ldots, N$. The continuous light ray is then replaced by the polygon line and formula (2) assumes now the following form:

$$T(X_1, X_2, \ldots) := \sum_{i=1}^{N} \frac{d(X_{i-1}, X_i)}{V_i}\,, \tag{4}$$

where X_i denotes the transition point between the strip number n and the strip number $(n+1)$. Because the above function must assume its minimum with respect to each variable X_i separately, it assumes its minimum with respect to the collection of all variables. But as the approximation gets finer and finer, i.e. for $N \to \infty$, we have:

$$\sum_{i=1}^{N} \frac{d(X_{i-1}, X_i)}{V_i} \longrightarrow \int_A^B \frac{\mathrm{d}s}{V}\,. \tag{5}$$

This proves the Fermat principle as the limiting case of the Snell's principle.

But the very advent of calculus of variations is due to Johann Bernoulli. He formulated the "brachistochrone problem" as a challenge to the best mathematicians of his time (Newton, Leibnitz, de l'Hospital and others). The problem is to find the shape of the ramp along which a ball released freely will reach the end point most quickly (without friction, only the force of gravity!!!). Again, the problem consists in minimizing the value of a certain integral T. If we describe the shape of the ramp as a graph of the function

$$[0,L] \ni x \longrightarrow f(x) = y \in \mathbb{R}^1 \,,$$

where L is the horizontal length of the ramp, the infinitesimal distance is given by the formula

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (f')^2}\,\mathrm{d}x\,,$$

and the velocity $V(x)$ at the height $y = f(x)$ is derived from the energy conservation law, where H is the height of the ramp:

$$mgH = mgy + \frac{mV^2}{2} \quad \Longleftrightarrow \quad V = \sqrt{2g\,(H - f(x))}\,.$$

Finally, the value of the following integral has to be minimized:

$$T := \int_0^L \frac{\mathrm{d}s}{V} = \int_0^L \frac{\sqrt{1 + \left(f'(x)\right)^2}}{\sqrt{2g(H - f(x))}}\,\mathrm{d}x\,, \tag{6}$$

where only functions f fulfilling boundary conditions are taken into account:

$$f(0) = H \text{ and } f(L) = 0\,. \tag{7}$$

Classical textbooks on the calculus of variations classify this problem as a "single integral variational problem". Multiple integrals soon entered the theory, and today the calculus of variations is commonly used in economics, logistics, and engineering sciences, wherever people try to find the optimal configuration of a complex system and are able to express "optimization" in terms of the "least something" principle.

However, a turning point in theoretical physics came when Pierre Louis Maupertuis observed that the motion of a classical particle in a potential force field also obeys a certain principle of "least something". Maupertuis Principle was a special case of the principle of "least action", which was later formulated in its full generality by W. R. Hamilton and which is now considered a fundamental law of physics[1].

Maupertuis's motivation was entirely ideological: he was strongly influenced by the optimistic philosophy of Leibniz, so much ridiculed by Voltaire in his *Candide*. Maupertuis saw his "principle of least action" as proof that the world we live in is the best possible one, and that the pursuit of optimization (minimization of action) is a fundamental law of nature. Soon, differential equations were discovered that constitute a necessary condition for the minimum (the so-called Euler-Lagrange equations), and the search for various

[1] A true theoretical physicist, when trying to describe the behavior of a new physical system observed in the laboratory by his experimental colleague, does not even try to guess its equations of motion, but rather tries to guess the corresponding variational principle.

sufficient conditions became the subject of a large branch of mathematics called now “calculus of variations”.

But the **action**, defined as the integral of a **Lagrangian density**, becomes “least” only in problems of a static nature, i.e. those leading to elliptic equations. In truly dynamic problems, the action never reaches a minimum, but assumes its saddle point. Some physicists are aware of this: for example, an important semiclassical quantization method is called the “stationary phase method” and not the “minimum phase method”. Thus, Maupertuis’s idea of “universal optimization” is false, but the calculus of variations remains an important tool in theoretical physics. However, the way the Euler-Lagrange equations are derived and used in field theory is false and leads to a flagrant contradiction with the hyperbolic nature of field equations. This paradox is never discussed in physics textbooks. In this article, we show how to avoid this paradox and correctly use the variational principle in field theory. Finally, we apply those results to the theory of gravity and show what General Relativity Theory would have looked like, had Einstein not abandoned his original idea to relate the phenomenon of gravity with the properties of inertial frames.

2. Euler-Lagrange equations: the “brachistochrone approach”

Consider a physical system whose configuration is described by N functions (dependent variables) $\varphi^K(x^\mu)$ of n independent variables x^μ. Geometrically, these variables are often organized into a fiber bundle

$$\pi \ : \ E \to M \, ,$$

where (x^μ) are (local) coordinates in the basis manifold M, (y^K) are coordinates in its fibres and the physical configuration is understood as the graph of a section of this bundle:

$$y^K = \varphi^K(x^\mu) \, .$$

In physics, the basis is often identified with the spacetime (or space – in statics), whereas the bundle E is usually a bundle of geometric objects (e.g. a tensor bundle) over spacetime. Specific, additional structures of the bundle can in some cases enable us to simplify the theory. But in what follows we consider the most general case where no additional structure is used and the coordinate description is sufficient. Let

$$\mathcal{L} = \mathcal{L}\left(x^\mu; \varphi^K, \varphi^K_{\ \mu}\right) \, , \tag{8}$$

be a “scalar-density-on-M-valued” function on the first jet extension of E, where the “jet-oriented” notation

$$\varphi^K_{\ \mu} := \partial_\mu \varphi^K$$

is used. We call $\mathcal{L}$ the “Lagrangian density”, whereas its integral over a fixed region $\mathcal{O} \subset M$ is called “action”:

$$\mathcal{A} := \int_{\mathcal{O}} \mathcal{L} \, . \tag{9}$$

If the base manifold M is equipped with a (privileged) volume structure $\mathrm{d}\rho$ (e.g. a metric structure) then any scalar density is proportional to the volume and, whence,

$$\mathcal{L} = L \cdot \mathrm{d}\rho \,,$$

where the proportionality coefficient L is a scalar function called the "Lagrangian function". Having in mind application to the Gravity Theory, where no *a priori* volume structure of spacetime is expected, we shall always use the density $\mathcal{L}$. In calculus of variations we consider one-parameter families $\varphi^K(x^\mu, \epsilon)$ of the field configurations and the derivative with respect to the parameter ϵ is called a *variation*:

$$\delta := \frac{\partial}{\partial \epsilon} \,. \tag{10}$$

The obvious identity:

$$\delta \partial_\mu = \partial_\mu \delta \text{ where } \partial_\mu = \frac{\partial}{\partial x^\mu} \tag{11}$$

is traditionally called "the fundamental Lemma of calculus of variations". The theory of calculus of variations begins with the following identity:

$$\begin{aligned}\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi^K}\delta\varphi^K + \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\delta\varphi^K_\mu = \frac{\partial \mathcal{L}}{\partial \varphi^K}\delta\varphi^K + \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\partial_\mu\delta\varphi^K \\ = \left(\frac{\partial \mathcal{L}}{\partial \varphi^K} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\right)\delta\varphi^K + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\delta\varphi^K\right) \,.\end{aligned} \tag{12}$$

Its first term on the right hand side will be called the volume term and the second one – the surface term because, after integration over a volume $\mathcal{O}$, we have:

$$\delta\mathcal{A} = \int_{\mathcal{O}} \left(\frac{\partial \mathcal{L}}{\partial \varphi^K} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\right)\delta\varphi^K + \int_{\partial\mathcal{O}} \left(\frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\delta\varphi^K\right) \,, \tag{13}$$

where $\partial\mathcal{O}$ denotes tho boundary of $\mathcal{O}$. Next, the surface integral is killed by imposing the boundary conditions (cf. formula (7)) on the value of the "dependent variables":

$$\varphi^K\big|_{\partial\mathcal{O}} = f^K \quad \Longrightarrow \quad \delta\varphi^K\big|_{\partial\mathcal{O}} = 0 \,, \tag{14}$$

where f^K are fixed functions on the boundary. Under these conditions, optimization of the action

$$\delta\mathcal{A} = \int_{\mathcal{O}} \left(\frac{\partial \mathcal{L}}{\partial \varphi^K} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu}\right)\delta\varphi^K = 0 \,, \tag{15}$$

implies vanishing of the integrand for every one-parameter family of configurations, i.e. for every value of $\delta\varphi^K$ and, consequently, implies the so called Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \varphi^K} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} = 0 \,. \tag{16}$$

Above method of deriving field equations in physics, based on imposing spacetime boundary conditions, will be called the “brachistochrone approach”. It works perfectly for true optimization problems, i.e. when the resulting Euler-Lagrange equations are elliptic and the boundary-value problem for them is well posed. But in case of a dynamical problem, i.e. when equations are hyperbolic, boundary value of the field cannot be fixed *a priori* and the procedure leads to a paradox (see also [1]). We are going to propose in the sequel a remedy for this paradox.

3. Boundary conditions in hiperbolic field theories

To illustrate the paradox with imposing spacetime boundary conditions in a hyperbolic field theories, consider the “mother of all dynamical field equations”, namely the string equation, i.e. wave equation in two-dimensional spacetime, parameterized by two coordinates (independent variables): time $x^0 = t$, space $x^1 = x$, and equipped with the Lorenzian metric g:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 \, .$$

For a scalar field (a single dependent variable) φ consider the Lagrangian density:

$$\mathcal{L} := -\frac{1}{2}\sqrt{|\det g|}\; g^{\mu\nu}\varphi_\mu\varphi_\nu = \frac{1}{2}\left((\dot\varphi)^2 - (\varphi')^2\right) \, , \tag{17}$$

where “dot” denotes the time derivative and “prime” denotes the space derivative:

$$\dot\varphi := \partial_t\varphi \text{ and } \varphi' := \partial_x\varphi \, .$$

Euler-Lagrange equations (16) read now

$$0 = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\varphi_\mu} = -\partial_t\dot\varphi + \partial_x\varphi' = \varphi'' - \ddot\varphi = \Box\varphi \, , \tag{18}$$

i.e. reduce to the 2D-wave equation, where

$$\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \, ,$$

denotes the two-dimensional D'Alembert operator. It is useful to represent it in “light coordinates”:

$$\begin{array}{l} u := x + t \\ v := x - t \end{array} \quad \Longrightarrow \quad \begin{array}{l} \partial_t = \partial_u - \partial_v \\ \partial_x = \partial_u + \partial_v \end{array} \quad . \tag{19}$$

Consequently, we have:

$$\Box = (\partial_x)^2 - (\partial_t)^2 = (\partial_u + \partial_v)^2 - (\partial_u - \partial_v)^2 = 4\,\frac{\partial^2}{\partial u\partial v} \, . \tag{20}$$

Consider any rectangle $\mathcal{R}$ in coordinates (u, v):

$$\mathcal{R} = \{(u,v) | a \le u \le b \; ; \; c \le v \le d\} \, , \tag{21}$$

and denote by: $A = (u = a, v = d)$, $B = (u = b, v = d)$, $C = (u = b, v = c)$, and $D = (u = a, v = c)$ its corners. For any function φ satisfying the wave equation (18) we have:

$$0 = \frac{\partial^2 \varphi}{\partial u \partial v}$$

and, consequently:

$$\begin{aligned} 0 &= \int_a^b \mathrm{d}u \int_c^d \mathrm{d}v \frac{\partial^2 \varphi}{\partial v \partial u}(u,v) = \int_a^b \left(\frac{\partial \varphi}{\partial u}(u,d) - \frac{\partial \varphi}{\partial u}(u,c) \right) \mathrm{d}u \\ &= \varphi(b,d) - \varphi(a,d) - (\varphi(b,c) - \varphi(a,c)) = -\varphi(A) + \varphi(B) - \varphi(C) + \varphi(D) \\ &\qquad = -\varphi(\text{down}) + \varphi(\text{right}) - \varphi(\text{up}) + \varphi(\text{left}) = 0\,. \end{aligned} \tag{22}$$

It is easy to prove that above equation, satisfied for every rectangle (21) in coordinates (u, v), is not only a *necessary condition* for (18), but actually is *equivalent* to wave equation. Hence, for any square $\mathcal{O}$ in coordinates (t, x), spanned by the corners A, B, C, D, e.g.:

$$\begin{gathered} \mathcal{O} = \{(t,x) | 0 \le t \le T\ ;\ 0 \le x \le T\}; \\ \text{we put: } a = x;\ b = 2T - x;\ c = -x;\ d = x\,. \end{gathered} \tag{23}$$

Identity (22) enables us to calculate the value of φ on each single wall of $\mathcal{O}$, if we only know its value on remaining three walls. Hence, choosing at random boundary conditions at $\partial\mathcal{O}$ there is exactly **zero** probability to capture any solution of the field equation (18). But knowing how to impose correctly boundary conditions, such that admit *any* solution (i.e. knowing equation (22)) is equivalent to know the field equation. This leads to a sad conclusion that in case of a hyperbolic field theory we are able to derive field equations from the variational principle using the "brachistochrone approach" *if and only if* we already know them.

4. "On shell" approach to calculus of variations

Brachistochrone approach is based on killing *a priori* the boundary (i.e. second) term in formula (12):

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \varphi^K} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} \right) \delta\varphi^K + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} \delta\varphi^K \right)\,. \tag{24}$$

This changes the status of (24) from "identity" to "equation" which has to be solved. This way of thinking is so deeply rooted in theoretical physics that Wheeler, Misner and Thorn – authors of the best textbook on the theory of gravity (see [2, p. 520]) – when deriving Einstein's equations from the variational principle do not even bother to calculate the boundary term, so convinced are they that it is completely irrelevant.

Without questioning the validity of this approach to optimization problems (i.e. elliptic), we propose a different, unifying approach that is also valid for dynamic problems (i.e. hyperbolic). We call this approach "on shell" because it is based on killing *a priori* the volume (i.e. first) term in formula (24).

What remain is no longer an identity but, again, an equation to be solved, namely:

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} \delta\varphi^K \right) . \tag{25}$$

It is useful to introduce a new, auxiliary quantity in the theory:

$$p^\mu_K := \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} , \tag{26}$$

which will be called "the canonical momentum". The geometric nature of this quantity is explained in detail in literature (see [3, 4]), but we emphasize that – due to its transformation laws – the formulas written below are completely invariant with respect to the choice of coordinates. Using momenta (26), field equations (25) can be rewritten as:

$$\delta\mathcal{L}\left(\varphi^K, \varphi^K_\mu\right) = \partial_\mu \left(p^\mu_K \delta\varphi^K \right) = \left(\partial_\mu p^\mu_K\right) \delta\varphi^K + p^\mu_K \delta\varphi^K_\mu \tag{27}$$

which is equivalent to the system of first order differential equations:

$$\partial_\mu p^\mu_K = \frac{\partial \mathcal{L}}{\partial \varphi^K} \quad ; \quad p^\mu_K = \frac{\partial \mathcal{L}}{\partial \varphi^K_\mu} . \tag{28}$$

We see that the last equation reproduces definition (26) of momentum, making this way the first equation equivalent to the Euler-Lagrange equation (16). We conclude that the second order equations for configurations φ^K obtained in the "brachistochrone approach", are here replaced by (fully equivalent) first order equations for both configurations and momenta p^μ_K. But, contrary to the previous approach, we avoid here any integration procedure. Therefore, functional analysis is not necessary to determine the class of functions taken into account and to choose their topology. The main advantage of this approach, however, is that we avoid the paradox of imposing space-time boundary conditions, which is strictly forbidden in hyperbolic field theory. Moreover, formula (27) allows us to describe the field dynamics as a symplectic control system. Such a description greatly simplifies the Hamiltonian approach to field theory, clarifies its possible quantization procedures (cf. [5]), and demythologizes the notion of field energy, which is extremely complicated in theory of gravity (cf. [7] and [8]).

5. Symplectic Control Systems

A control system (see also [10]) is a physical system which possesses the "control parameters", say (q^k), which describe the *configuration* of the system and can be freely controlled in the laboratory, and the "response parameters", say (r^α), which can be measured whenever the system assumes its configuration. Relation between them, i.e. "control-response relations" describe entirely the physical properties of the system.

By a *symplectic control system* we mean a control system equipped with an additional structure, namely the *infinitesimal work*, i.e. a differential one-form $w = f_k \mathrm{d}q^k$, which is defined on the space of control parameters

$Q = \{(q^k)\}$ and which measures work W, i.e. the costs necessary to change the configuration along any path $\gamma \subset Q$:

$$W = \int_\gamma w \,. \tag{29}$$

Components f_k of w can be called "forces" – a specific kind of response parameters. This way the *phase space* P describing both control and response parameters: $P = \{(f_k, q^k)\}$, acquires the canonical symplectic structure ω

$$\omega = \mathrm{d}f_k \wedge \mathrm{d}q^k \,, \tag{30}$$

isomorphic[2] to the cotangent bundle T^*Q.

Even more specific case is mainly considered in the literature, in which the work does not depend on the entire path γ but only on its endpoints. Denote by $D \subset P$ the collection of all points (f_k, q^l) which are physically admissible. The submanifold D will be called the dynamics of the system. Independence of the work upon the path means that the pull-back $\theta := \pi^* w$ of the one-form w, from Q to P, is complete when restricted to D:

$$\theta|_D = \mathrm{d}\widetilde{\lambda} \,. \tag{31}$$

Consequently, the submanifold D is a Lagrangian submanifold:

$$\omega|_D = \mathrm{d}\theta|_D = \mathrm{d}\mathrm{d}\widetilde{\lambda} = 0 \,. \tag{32}$$

We are going to restrict our discussion to such systems. To summarize: by a symplectic control system we mean a system: 1) whose phase space P (i.e. space of all parameters describing the state of the system) carries a canonical ("God given") symplectic structure, 2) its physically admissible states constitute a Lagrangian (degenerate with respect to the symplectic form and maximal) submanifold $D \subset P$ called the dynamics, 3) each representation of P as a cotangent bundle, i.e. a symplectomorphism $P \cong T^*Q$ for any manifold Q, is called a "control mode". Once a control mode has been chosen, coordinates (q^k) on Q are called control parameters, parameters (f_k) in the fibers of T^*Q are called response parameters and the function $\widetilde{\lambda}$ (more often: its projection λ onto Q) is called "a generating function of the dynamics".

Example 1. The configuration of a spring balance is measured by its position $x := h_0 - h$, where h_0 is the height (e.g. above the floor) of the pan in the "zero weight" position and h is the actual height. Any infinitesimal change of the state of the balance costs infinitesimal work:

$$w = f\mathrm{d}x \,. \tag{33}$$

Here f denotes the force required to maintain the current position of the pan. It can be measured using a weight placed on the pan. The phase space $P = \{(f, x)\}$ of the system carries a canonical symplectic form:

$$\omega = \mathrm{d}f \wedge \mathrm{d}x \,.$$

[2] Observe that "$\mathrm{d}q^k$" in the definition of w is a covector on the configuration space Q, whereas the same symbol in formula (30) denotes its pull-back to the phase space P.

The Hook Law: $f = Hx$, where H is the elasticity coefficient (Yang modulus), defines the submanifold $D \subset P$ of physically admissible states:

$$D = \{(f, x)|f = Hx\}\,.$$

Choosing x as control and f as response parameters, the generating function of the dynamics λ fulfills on D:

$$\mathrm{d}\lambda(x) = f\mathrm{d}x = \mathrm{d}\left(\tfrac{1}{2}Hx^2\right)\,, \tag{34}$$

and, whence, $\lambda(x) = \frac{1}{2}Hx^2$. However, instead of controlling x "by hand", we can control f by putting different weights on the pan of the balance and treating x as the response parameter. Then

$$\omega = \mathrm{d}f \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}f = \mathrm{d}\left(-x\mathrm{d}p\right)\,, \tag{35}$$

and, instead of (34), we obtain the following generating formula for the new generating function χ:

$$\mathrm{d}\chi(p) = -x\mathrm{d}p\,. \tag{36}$$

But

$$\mathrm{d}\lambda = f\mathrm{d}x = \mathrm{d}(fx) - x\mathrm{d}f \quad \Longleftrightarrow \quad \mathrm{d}\left(\lambda - fx\right) = -x\mathrm{d}f = \mathrm{d}\chi\,, \tag{37}$$

and, whence,

$$\chi = \lambda - fx \quad \Longrightarrow \quad \chi(f) = -\tfrac{1}{2H}f^2\,. \tag{38}$$

Possible additive constants (in both λ and χ) do not play any role since generating function is always defined "up to an additive constant" as it measures the work from a reference state s_0 to the current state s and the choice of the reference is arbitrary. The minus sign in (36) (*mathematically* implied by the sign of the symplectic form (35)) is *physically* implied by the fact, that putting weights on the pan of the balance in order to change its position we actually *get* some work from the system, i.e. we *put* into the system a negative work.

Example 2. The state of a simple thermodynamical body is fully described by four parameters (V, p, S, T), namely: volume, pressure, entropy and the temperature. The first law of thermodynamics:

$$\mathrm{d}U(V, S) = -p\mathrm{d}V + T\mathrm{d}S\,, \tag{39}$$

where U is the internal energy of the body, implies the following interpretation: parameters (V, S) have been chosen as control parameters and the remaining (p, T) play role of the response parameter with respect to the canonical ("God given") symplectic form:

$$\omega = -\mathrm{d}p \wedge \mathrm{d}V + \mathrm{d}T \wedge \mathrm{d}S = \mathrm{d}V \wedge \mathrm{d}p + \mathrm{d}T \wedge \mathrm{d}S\,. \tag{40}$$

Exchanging the role of control and response between V and p we obtain the generating function in a new control mode:

$$\mathrm{d}U = \mathrm{d}(-pV) + V\mathrm{d}p + T\mathrm{d}S \quad \Longleftrightarrow \quad \mathrm{d}\left(U + pV\right) = V\mathrm{d}p + T\mathrm{d}S\,. \tag{41}$$

The generating function $H := U + pV$ in the (p, S)-control mode is called the *enthalpy* of the system. In a similar way we reconstruct the Helmholz *free energy* as the generating function in (V, T)-control mode etc.

Example 3. The "on shell" approach to Lagrangian field theory fits perfectly into this framework. It has been proved (see [3, 4]) that, at every spacetime point (x^μ), the space of first jets of both field configurations φ^K and momenta p_K^μ is equipped with the canonical symplectic structure:

$$\omega = \partial_\mu \left(\mathrm{d}p_K^\mu \wedge \mathrm{d}\varphi^K\right) = \mathrm{d}j_K \wedge \mathrm{d}\varphi^K + \mathrm{d}p_K^\mu \wedge \mathrm{d}\varphi^K_\mu \tag{42}$$

where the "current" j_K is defined as the divergence of the momentum:

$$j_K := \partial_\mu \mathrm{d}p_K^\mu \, . \tag{43}$$

Choosing the first jet of configuration $\left(\varphi^K, \varphi^K_\mu\right)$ as control and jets of momenta as response, we see that, according to (27), the Lagrangian density $\mathcal{L}$ plays role of the generating function in this mode:

$$\mathrm{d}\mathcal{L}\left(\varphi^K, \varphi^K_\mu\right) = \partial_\mu \left(p_K^\mu \mathrm{d}\varphi^K\right) = j_K \mathrm{d}\varphi^K + p_K^\mu \mathrm{d}\varphi^K_\mu \tag{44}$$

Example 4. A particular case of the above construction is obtained if the base manifold is one-dimensional, i.e. when there is only one independent variable, time: $x^0 = t$. Consequently, there is only one momentum $p_K = p_K^0$ for each degree of freedom φ^K, which we rather denote by q^k, and the symplectic form (42) reduces to

$$\omega = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{d}p_K \wedge \mathrm{d}q^K\right) = \mathrm{d}\dot{p}_K \wedge \mathrm{d}q^K + \mathrm{d}p_K \wedge \mathrm{d}\dot{q}^K \, . \tag{45}$$

The symplectic form $\mathrm{d}p_K \wedge \mathrm{d}q^K$ was known long time ago as a necessary ingredient of the Hamiltonian formulation of mechanics. But its time derivative above was invented by W.M. Tulczyjew as a canonic structure arising in calculus of variations (see [9] and [11]):

$$\mathrm{d}\mathcal{L}\left(q^K, \dot{q}^K\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(p_K \mathrm{d}q^K\right) = \dot{p}_K \mathrm{d}q^K + p_K \mathrm{d}\dot{q}^K \, . \tag{46}$$

We stress that the symplectic form $\mathrm{d}p_K \wedge \mathrm{d}q^K$ has no analog in field theory whereas, as shown in [3], the canonical form (45) generalizes in a natural way to (42), i.e. to an arbitrary number of independent variables.

6. Legendre Transformation, constraints, field energy

Whenever a theoretical physicists says "Legendre transformation" he means transition from Lagrangian to Hamiltonian formulation of classical mechanics. But, both formulations describe phenomena which take place in the same phase space $P = \{(p, \dot{p}, q, \dot{q})$ equipped with the symplectic form (45). Having realized this, we see that they both describe dynamics by the same object: a Lagrangian (maximal, singular) submanifold $D \subset P$, but in two different ways:

$$D = \left\{(p, \dot{p}, q, \dot{q}) \middle| p = \tfrac{\partial \mathcal{L}}{\partial \dot{q}}; \dot{p} = \tfrac{\partial \mathcal{L}}{\partial q}\right\} = \left\{(p, \dot{p}, q, \dot{q}) \middle| \dot{p} = -\tfrac{\partial \mathcal{H}}{\partial q}; \dot{q} = \tfrac{\partial \mathcal{H}}{\partial \dot{q}}\right\} \tag{47}$$

The Lagrangian formulation is equivalent to the generating formula (46), when we choose the first jet of configuration: $(q, \dot{q})$, as control parameters and the jet of momenta $(p, \dot{p})$ as response. The transition to the Hamiltonian formulation consists in exchanging the role of p and $\dot{q}$. The corresponding generating formula is obtained from (46) by a procedure analogous to the Legendre transformation (41) between volume V and pressure p in thermodynamics, or the Legendre transformation (37) between position and force in statics:

$$\mathrm{d}\mathcal{L} = \dot{p}\mathrm{d}q + p\mathrm{d}\dot{q} = \mathrm{d}\left(p\dot{q}\right) + \dot{p}\mathrm{d}q - \dot{q}\mathrm{d}p \\ \Rightarrow \quad \mathrm{d}\left(\mathcal{L} - p\dot{q}\right) = \mathrm{d}\left(-\mathcal{H}\right) = \dot{p}\mathrm{d}q - \dot{q}\mathrm{d}p\,, \tag{48}$$

where the function $\mathcal{H} = p\dot{q} - \mathcal{L}$ is traditionally called the *Hamiltonian* function of the system. Physically, the function $\mathcal{H}$ (and *not* the generating function "$-\mathcal{H}$") describes the energy of the system and this is the origin of the above notation. This is how we obtain the standard, textbook version of the Hamiltonian generating formula:

$$\mathrm{d}\mathcal{H} = -\dot{p}\mathrm{d}q + \dot{q}\mathrm{d}p\,. \tag{49}$$

In field theory the concept of "Hamiltonian function" is sometimes used in a misleading way. In classical variational problems (i.e. optimization problems which lead to elliptic equations) the "total Legendre transformation", consisting in exchanging *all* momenta $p_K^{\ \mu}$ with *all* velocities $\varphi^K_{\ \mu}$ is often used (cf. [12, 13, 14]) which leads to the following generating formula as a consequence of (44):

$$\mathrm{d}\left(\mathcal{L} - p_K^{\ \mu}\varphi^K_{\ \mu}\right) = j_K \mathrm{d}\varphi^K + p_K^{\ \mu}\mathrm{d}\varphi^K_{\ \mu} - \mathrm{d}\left(p_K^{\ \mu}\varphi^K_{\ \mu}\right) \\ = j_K \mathrm{d}\varphi^K - \varphi^K_{\ \mu}\mathrm{d}p_K^{\ \mu}\,. \tag{50}$$

The resulting generating function:

$$-\,\mathcal{E} := \mathcal{L} - p_K^{\ \mu}\varphi^K_{\ \mu} \tag{51}$$

(or, rather, its inverse $\mathcal{E}$) is then called the (density of) field Hamiltonian. We emphasize, however, that this quantity, useful in optimization problems, has nothing to do with the physical energy carried by the field, which appears in hyperbolic field theories. Indeed, energy *is not* a *scalar* but a *component of the four-momentum vector* and, whence, depends upon the choice of the reference frame, i.e. upon splitting spacetime into the time $t = x^0$ and the space coordinates (x^k). Given such a splitting, we denote time derivative by a "dot" and select – among all components of momenta – their "time component":

$$\pi_K := p_K^{\ 0}\,. \tag{52}$$

(In classical books on field theory (cf. [15] or [16]), only this component of $p_K^{\ \mu}$ is called by physicists "the canonical momentum of the field".)

Once given this splitting (the reference frame chosen), we perform a "partial" Legendre transformation consisting in exchange between the time-like components π_K of momenta and the time derivatives $\dot{\varphi}^K$ of the fields in formula (44):

$$\begin{aligned} \mathrm{d}\mathcal{L} = \partial_\mu \left(p^\mu_K \mathrm{d}\varphi^K \right) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\pi_K \mathrm{d}\varphi^K \right) + \partial_k \left(p^k_K \mathrm{d}\varphi^K \right) \\ &= \dot{\pi}_K \mathrm{d}\varphi^K + \pi_K \mathrm{d}\dot{\varphi}^K + \partial_k \left(p^k_K \mathrm{d}\varphi^K \right) \\ &= \mathrm{d} \left(\pi_K \dot{\varphi}^K \right) + \dot{\pi}_K \mathrm{d}\varphi^K - \dot{\varphi}^K \mathrm{d}\pi_K + \partial_k \left(p^k_K \mathrm{d}\varphi^K \right) . \end{aligned} \tag{53}$$

Defining the field energy (density):

$$\mathcal{H} := \pi_K \dot{\varphi}^K - \mathcal{L} , \tag{54}$$

we finally obtain

$$\mathrm{d} \left(-\mathcal{H} \right) = -\dot{\pi}_K \mathrm{d}\varphi^K + \dot{\varphi}^K \mathrm{d}\pi_K + \partial_k \left(p^k_K \mathrm{d}\varphi^K \right) . \tag{55}$$

Using the density $\mathcal{H}$ we can describe properly energy carried by the field, analogously to formula (49) in mechanics, with the only difference, namely the last term in (55) being the complete space divergence. Those who want to describe field dynamics as an infinite-dimensional Hamiltonian system, must kill this term imposing specific *space boundary conditions*, which is allowed in hyperbolic theory (and not *spacetime boundary conditions* which is strictly forbidden). However, these are issues related to functional analysis and are beyond the scope of this article.

The above formulas were written as if all the control-response relations considered here were not just any relations, but mappings. If $P \cong T^*Q$ is a control mode, then $S = \mathrm{graph}(\mathrm{d}\lambda)$, where λ is a function of control parameters is, however, a specific case of a Lagrangian submanifold of P. It is characterized by the following properties: 1) the projection of S onto the space of control parameters Q covers the entire Q (is a surjection), 2) for every value $q \in Q$ of control parameters there is a unique response $r \in T^*_q Q$. But, consider a mechanical Lagrangian function $\mathcal{L}\left(q^K, \dot{q}^K\right)$ which contains one irrelevant parameter, say q^n i.e. such that nothing depends upon q^n and also $\dot{q}^n$. Then, formula (46) implies Hamiltonian constraints $p_n = 0$ and, whence, the Hamiltonian function $\mathcal{H}$ resulting from Legendre transformation (48) is defined only on the constraints manifold:

$$C = \left\{ (p_K, q^K) \middle| p_n = 0 \right\} , \tag{56}$$

and the Hamiltonian generating formula (49) seems to have no sense. This situation constitutes the standard "horror" of the authors of textbooks on classical mechanics. They feel obliged to assume constantly that the Legendre transformation is reversible. But, as will be seen in the sequel, nothing wrong happens in this example, even if it is not invertible. For this purpose, we need to better precise our terminology: we declare that the differential $\mathrm{d}f(q)$ of a function defined on a constraints submanifold $C \in Q$ is not a single covector but, possibly, a set of covectors, namely: 1) an empty set if $q \notin C$ and, otherwise, 2) the set of all covectors at $q \in C$ which – when restricted to T_qC – give the same value as the differential of f. With this definition, formula

$$\mathrm{d}\mathcal{H} = -\dot{p}_K \mathrm{d}q^K + \dot{q}^K \mathrm{d}p_K , \tag{57}$$

means the following: 1) The constraint $p_n = 0$ is always valid, 2) for $K \neq n$ we have standard Hamiltonian equations: $-\dot{p}_K = \frac{\partial \mathcal{H}}{\partial q^K}$ and $\dot{q}^K = \frac{\partial \mathcal{H}}{\partial p_K}$, 3) for $K = n$ we have: $\dot{q}^n$ is arbitrary and $\dot{p}_n = 0$ because $\mathcal{H}$ does not depend upon the irrelevant parameter q^n. In classical textbooks, such arbitrary parameters like q^n above, are often called "Lagrange multipliers", in analogy with the *true* Lagrange multipliers that play an important role in real optimization problems. In hyperbolic field theory, they appear as "momenta canonically conjugate to constraints", which simplifies considerably the theory and proves that terminology borrowed from optimization problems and used when *no optimization* is even possible, seriously confuses formulation of field theory[3].

Constraints can appear in the Hamiltonian control mode, as in example above, but can also appear in the Lagrangian control mode, as e.g. in variational problems of higher order, when the Lagrangian function depends upon higher order derivatives of the configurations. Because n-th derivatives can be considered as a submanifold C in the set of "first derivatives of the $(n-1)$-th derivatives", we can simply describe an "n-order Lagrangian" as a "first order Lagrangian", but defined only on constraints C (cf. [4]).

The dynamics S can, however assume a much more general position in the cotangent bundle T^*Q. In particular, it may cover some regions of Q many times. This way the generating function can become "multi valued" and change its character at boundaries of those regions. Such a situation gives rise to the *Theory of Catastrophes* in the sense of René Thom. It often occurs in robotics and similar engineering problems, but not in physically realistic models of the field theory.

7. Example: Theory of Gravity

The authors of [2, p. 520] derive Einstein equations from the Hilbert variational principle (with components $g_{\mu\nu}$ of the spacetime Lorentzian metric taken as field variables). For this purpose, they only calculate the volume part of $\delta\mathcal{L}_{\text{Hilbert}}$ (the first term of formula (24)) and claim that the boundary part is difficult to calculate and does not carry any significant information: its calculation would be a waste of time[4]. But the boundary term can be calculated! The paper [17] contains an easy proof that the variation of the Hilbert Lagrangian is (in the "on-shell approach", i.e. without volume part):

$$\delta\mathcal{L}_{\text{Hilbert}} = \partial_\kappa \left(\pi_\lambda{}^{\mu\nu\kappa} \delta\Gamma^\lambda_{\mu\nu} \right) , \tag{58}$$

where we denote

$$\pi_\lambda{}^{\mu\nu\kappa} := \pi^{\mu\nu}\delta^\kappa_\lambda - \pi^{\kappa(\nu}\delta^{\mu)}_\lambda \quad ; \quad \pi^{\mu\nu} = \frac{1}{2k}\sqrt{\det g}\, g^{\mu\nu} , \tag{59}$$

[3] We stress that the so called "second-type constraints" never appear in our formulation of the field theory.

[4] They are looking for the (non-existing!) **extremum** of the action and declare openly: "Variations of the geometry interior to this surface make no difference in the value of this surface term. Therefore it has no influence on the equations of motion to drop the term (21.88)". Their formula (21.38) is exactly the boundary term of the variation.

and k is the gravitational constant ($k = 8\pi$ in geometric units). Formula (58) proves that, in fact, the symmetric connection $\Gamma^{\lambda}_{\mu\nu}$ and not the metric tensor should play role of configuration in theory of gravity. This agrees with the original idea of Albert Einstein who, after having formulated in 1905 the Special Theory of Relativity which correctly describes electromagnetic field, tried to extend this theory to gravity. As we know from his numerous statements, his starting point was an attempt to understand what an "inertial system" is and how to distinguish gravitational forces from inertial forces. He later abandoned this idea and switched to Riemannian (or pseudo-Riemannian) geometry. However, formula (58) suggests that his original idea was correct.

To follow this idea we remind that Isaac Newton introduced the concept of an inertial frame in order to define "rectilinear motion with constant velocity" i.e. to understand what happens when no force acts on a moving body. The trajectory of such a "free falling body" (imagine a spaceship far away from the Earth) fulfills equation:

$$\ddot{y}^{\alpha} = 0\,. \tag{60}$$

Here (y^{μ}) are spacetime coordinates of the spaceship, and the "dot" denotes the derivative with respect to, for example, the biological clock of its commander. Equation (60) depends, however, upon a choice of coordinates. Indeed, we can "translate" this equation to another system of coordinates, say (x^{λ}):

$$\begin{aligned} \dot{y}^{\alpha} &= \frac{\partial y^{\alpha}}{\partial x^{\nu}}\dot{x}^{\nu} \\ \ddot{y}^{\alpha} &= \frac{\partial y^{\alpha}}{\partial x^{\nu}}\ddot{x}^{\nu} + \frac{\partial^2 y^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}\dot{x}^{\mu}\dot{x}^{\nu} = \frac{\partial y^{\alpha}}{\partial x^{\lambda}}\left(\ddot{x}^{\lambda} + \frac{\partial x^{\lambda}}{\partial y^{\beta}}\frac{\partial^2 y^{\beta}}{\partial x^{\mu}\partial x^{\nu}}\dot{x}^{\mu}\dot{x}^{\nu}\right) \end{aligned}$$

We see that the "rectilinear motion with constant velocity", described by (60) in inertial coordinates (y^{α}), in arbitrary coordinates (x^{λ}) is described by:

$$\ddot{x}^{\lambda} + \frac{\partial x^{\lambda}}{\partial y^{\beta}}\frac{\partial^2 y^{\beta}}{\partial x^{\mu}\partial x^{\nu}}\dot{x}^{\mu}\dot{x}^{\nu} = 0\,. \tag{61}$$

New coordinates (x^{λ}) are also inertial if and only if the second derivatives between them vanish: $\frac{\partial^2 y^{\beta}}{\partial x^{\mu}\partial x^{\nu}} = 0$. When Newton formulated his theory, the Universe was assumed to be nearly empty, with only a small fraction of it filled with matter and coordinates were supposed to be *global*. But, after Einstein, we know, that meaningful statements in physics should be *local*. This suggests the definition of a *local reference frame* at a spacetime point $m \in M$: it is an equivalence class $[(y^{\alpha})]$ of local coordinate systems, defined in a neighborhood of m, with respect to the following equivalence relation:

$$\left\{(y^{\alpha}) \sim_m (x^{\lambda})\right\} \Leftrightarrow \left\{\frac{\partial^2 y^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}(m) = 0\right\}\,. \tag{62}$$

Observe that removing the point m from this definition, we obtain the correct definition of a *global* reference system in the sense of Newton! This observation suggests the following, local version of the Newton's First Law, which we propose to call the Einstein's First Law: at each spacetime point $m \in M$

there is a privileged reference system which we call an inertial frame. Equation (60) describes the motion of the freely falling body at each spacetime point $m \in M$ only in coordinates which are *inertial* at m. Otherwise, it is described by equation (61), where the second term describes the "fictitious forces" (centrifugal, Coriolis etc.) due to the non-inertiality of the coordinates used.

Hence, according to Einstein's First Law, the spacetime is filled with the field of local inertial frames, one at each point m. To find a coordinate description of this field observe that an inertial frame $[(y^\alpha)]$ is uniquely characterized in arbitrary coordinates (x^λ), by the following table of numbers:

$$\Gamma^\lambda_{\mu\nu}(m) := \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m) \, , \tag{63}$$

where (y^α) is a representative of the class $[(y^\alpha)]$. It is easy to prove that the right-hand side does not depend upon a choice of this representative. A physical interpretation of the table $\Gamma^\lambda_{\mu\nu}(m)$ follows from observation that it is a quadratic correction to coordinates (x^λ), necessary and sufficient to "upgrade them" to inertial coordinates (z^λ) at m:

$$z^\lambda := x^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu}(m) x^\mu x^\nu \, . \tag{64}$$

(Without loss of generality, we have assumed that coordinates (x^λ) are centered at m i.e. that $m = (m^\mu) = (0, \ldots, 0)$. Otherwise, x^μ should be replaced by "$(x^\mu - m^\mu)$".)

The numbers $\Gamma^\lambda_{\mu\nu}$ could be interpreted as coefficients of a connections in the tangent bundle TM which (somehow, by chance – see (63)) is symmetric. This would be, however, a very artificial interpretation. We stress that $\Gamma^\lambda_{\mu\nu}(m)$ describe the local *inertial frame* at $m \in M$ – the fundamental concept of contemporary physics (whereas the Newtonian physics was based on its global version!).

The field of local inertial frames Γ describes entirely how the gravitational field influences the motion of heavy objects. Indeed, equation of motion of freely falling bodies is implied by (61):

$$m\ddot{x}^\lambda = -m\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu =: F^\lambda \, , \tag{65}$$

where F^λ is the *gravitational force* and can be eliminated if we use inertial coordinates.

Having identified gravitational field with the field of inertial frames, we would probably like to derive its dynamics from a variational principle. Its Lagrangian density $\mathcal{L}$ should, therefore, depend upon the field configuration $\Gamma^\lambda_{\mu\nu}$ and its derivatives $\Gamma^\lambda_{\mu\nu\kappa} := \partial_\kappa \Gamma^\lambda_{\mu\nu}$. Using transformation laws for Γ, it is easy to prove that $\mathcal{L}$ can depend upon its arguments only *via* the components of the curvature tensor: $\mathcal{L} = \mathcal{L}\left(R^\lambda_{\mu\nu\kappa}\right)$: otherwise the field equations would depend on the choice of a coordinate system.

A considerable simplification of the theory is obtained (see [1]) if, instead of the Riemann tensor $R^{\lambda}_{\mu\nu\kappa}$, we use the following tensor to describe curvature:

$$K^{\lambda}_{\mu\nu\kappa} := -\frac{2}{3} R^{\lambda}_{(\mu\nu)\kappa} \quad ; \quad R^{\lambda}_{(\mu\nu)\kappa} = -2K^{\lambda}_{\mu\nu\kappa}$$

which carry exactly the same information as the Riemann tensor, but is better adapted to canonical structure of the theory. More precisely, the canonical momenta are directly represented by derivatives of the Lagrangian with respect to $K^{\lambda}_{\mu\nu\kappa}$ and not to $R^{\lambda}_{\mu\nu\kappa}$. Finally (cf. [1] or [6]), generating formula for field equations reduces to:

$$\begin{aligned} \delta\mathcal{L}\left(K^{\lambda}_{\mu\nu\kappa}\right) = \partial_{\kappa}\left(P_{\lambda}^{\mu\nu\kappa}\delta\Gamma^{\lambda}_{\mu\nu}\right) &= \left(\partial_{\kappa}P_{\lambda}^{\mu\nu\kappa}\right)\delta\Gamma^{\lambda}_{\mu\nu} + P_{\lambda}^{\mu\nu\kappa}\delta\Gamma^{\lambda}_{\mu\nu\kappa} \\ &= \left(\nabla_{\kappa}P_{\lambda}^{\mu\nu\kappa}\right)\delta\Gamma^{\lambda}_{\mu\nu} + P_{\lambda}^{\mu\nu\kappa}\delta K^{\lambda}_{\mu\nu\kappa} , \end{aligned} \tag{66}$$

and, whence, field equations reduce to:

$$\nabla_{\kappa}P_{\lambda}^{\mu\nu\kappa} = 0 \quad ; \quad P_{\lambda}^{\mu\nu\kappa} = \frac{\partial\mathcal{L}}{\partial K^{\lambda}_{\mu\nu\kappa}} , \tag{67}$$

where nabla represents the covariant derivative with respect to the connection Γ.

But the curvature tensor ($K^{\lambda}_{\mu\nu\kappa}$ or – equivalently – $R^{\lambda}_{\mu\nu\kappa}$) decomposes algebraically into collection of three irreducible objects (see [1] or [6] for details): 1) the symmetric part "$K_{\mu\kappa}$" and 2) the antisymmetric part "$F_{\mu\kappa}$", of the Ricci tensor:

$$R_{\mu\kappa} := R^{\nu}_{\mu\nu\kappa} = K_{\mu\kappa} + F_{\mu\kappa} ,$$

and, finally 3) the remaining trace-free part $W^{\lambda}_{\mu\nu\kappa}$ (the Weyl tensor). It has been proved long ago (see e.g. [18]) that the first sector reproduces exactly the standard equations of General Relativity Theory, with "tiny differences" namely: 1) there is no room for any *ad hoc* modifications of Einstein equations (like e.g. those resulting from the so called "$f(R)$ theories") and, moreover 2) the cosmological constant[5] is necessary from the very beginning of the theory. Indeed, there is only one method to manufacture a scalar density $\mathcal{L}$ from the tensor $K_{\mu\kappa}$, namely:

$$\mathcal{L} = C \cdot \sqrt{\det K} . \tag{68}$$

(The physical constant C is necessary because the curvature tensor and the action have, *a priori*, different physical dimensions). Due to (59), the second field equation (66) implies:

$$P_{\lambda}{}^{\mu\nu\kappa} := \pi^{\mu\nu}\delta^{\kappa}_{\lambda} - \pi^{\kappa(\nu}\delta^{\mu)}_{\lambda} \text{ where } \pi^{\mu\kappa} = \frac{\partial\mathcal{L}}{\partial K_{\mu\kappa}} , \tag{69}$$

and, whence, the first field equation (66) implies:

$$\nabla_{\kappa}\pi^{\mu\nu} = 0 . \tag{70}$$

[5] A. Einstein considered the cosmological constant to be "the greatest mistake of his life", even though he himself had discovered it several years earlier.

Let us use formula (59) as a definition of the metric tensor $g_{\mu\nu}$ in terms of the canonical momentum (69). Then Euler-Lagrange equation (70) of the theory means that the connection $\Gamma^{\lambda}_{\mu\nu}$ is the metric (Levi-Civita) connection of $g_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa} \right) . \tag{71}$$

Moreover, the second equation of (69) implies the following relation between metric (momentum!) and curvature (derivatives of configuration):

$$\pi^{\mu\kappa} = \frac{\partial \mathcal{L}}{\partial K_{\mu\kappa}} = \frac{C}{2} \sqrt{\det K} \left(K^{-1} \right)^{\mu\nu} , \tag{72}$$

which can easily be inverted:

$$\mathcal{G}_{\mu\nu} = \frac{1}{4\pi C} g_{\mu\nu} , \tag{73}$$

where $\mathcal{G}$ is the Einstein tensor:

$$\mathcal{G}_{\mu\nu} := K_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(K_{\alpha\beta} g^{\alpha\beta} \right) . \tag{74}$$

We see that assuming that the Lagrangian density $\mathcal{L}$ feels only the first sector of the theory, we have proved that its unique possible dynamics is the standard Einstein theory with a (non-removable!) cosmological constant $\Lambda = \frac{1}{4\pi C}$. But, what is usually assumed *a priori*, namely the metricity (71) of the connection, is here obtained as the Euler-Lagrange theory, whereas the Einstein equation (74), usually obtained as the Euler-Lagrange equation of the theory, here arises as the definition of the field momenta.

But restriction of the theory to the first sector of curvature (symmetric part of Ricci) does not seem to be realistic. It can be shown that including the second sector and choosing the Lagrangian density depending upon the entire Ricci, for example:

$$\mathcal{L} = C \cdot \sqrt{\det R_{\mu\nu}} = C \cdot \sqrt{\det \left(K_{\mu\nu} + F_{\mu\nu} \right)} , \tag{75}$$

leads to a unified Einstein-Weyl theory of gravity and electromagnetism (see, e.g. [6]), whereas the third sector could possibly describe the "dark matter".

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