The multisymplectic structure of the universal covariant phase space

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ongoing work with A. Ibort

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A missing quadrant

	mechanics		
Lagrange	TQ		
Hamilton	T^*Q		

A missing quadrant

	mechanics	first order	
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Lagrange	duals $\downarrow$	TQ	$J^1\pi$	
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A missing quadrant

$\xrightarrow{\text{(co)lim}}$ (first order → all orders)

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A missing quadrant

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$$\text{first order} \xrightarrow{\text{(co)lim}} \text{all orders} \qquad \text{Lagrange} \xrightarrow{\text{duals}} \text{Hamilton}$$

My goal today: Explain what goes in the missing corner

Outline

1. The multisymplectic phase space (1st order)
2. Sheaves and smooth sets
3. The multisymplectic phase space (all orders)

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The affine dual of $J^1\pi$

Fields are sections of $E \xrightarrow{\pi} M$; velocities live in $J^1\pi$, and the Lagrangian is a bundle map $L\colon J^1\pi \to \bigwedge^m T^*M$.

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The affine dual of $J^1\pi$ (cont.)

Theorem

There is an isomorphism

$$\mathrm{Aff}(J^1\pi, \mathbb{R}) \cong \textstyle\bigwedge_1^m T^*E,$$

with the semibasic m-forms on E (at most one vertical leg):

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Also $P^1\pi := M^1\pi / \bigwedge_0^m T^*E$

The first-order picture

$$
\begin{tikzcd}
 & & & \underset{x^\mu, u^a, p, p_a^\mu}{M^1\pi} \arrow[d, "\mu"] \\
\bigwedge^m T^*M & \underset{x^\mu, u^a, u^a_\mu}{J^1\pi} \arrow[l, "L"'] \arrow[dr, "\pi^1_0"] & & \underset{x^\mu, u^a, p_a^\mu}{P^1\pi} \arrow[dl] \arrow[u, bend left, "h"] \\
 & & \underset{x^\mu, u^a}{E} \arrow[d, "\pi"] & \\
 & & \underset{x^\mu}{M} &
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Higher-order theories

First order is often not enough. Many Lagrangians need higher derivatives: GR, EFTs, etc.

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Problem: $J^\infty\pi$ is infinite-dimensional.

Looking beyond manifolds

The fix (after Grothendieck): embed manifolds into *sheaves on manifolds* aka *smooth sets*.

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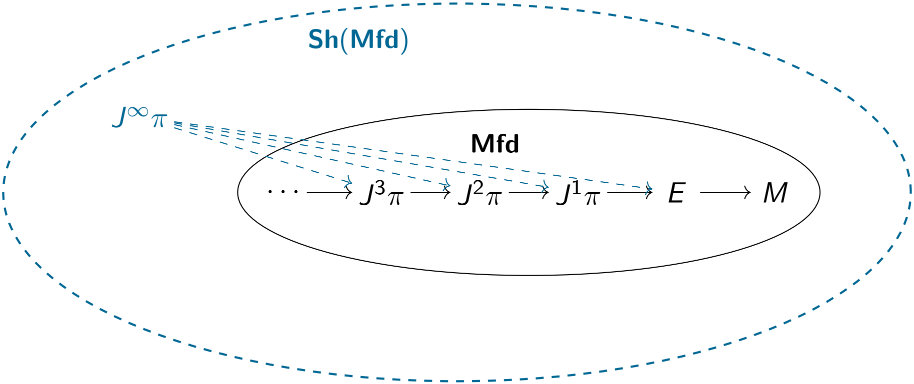

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- For every $M \in \mathbf{Mfd}$, there is a set $X(M) = C^\infty(M, X)$, such that $\phi \colon M \to N$ gives $X(\phi) \colon X(N) = C^\infty(N, X) \to X(M) = C^\infty(M, X)$

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- Elements of $X(U \cup V) = C^\infty(U \cup V, X)$ are the same as pairs in $X(U) \times X(V) = C^\infty(U, X) \times C^\infty(V, X)$ that agree when restricted to $X(U \cap V) = C^\infty(U \cap V, X)$.

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Remark

$\mathbf{\Omega}^n$ has no points!

$$C^\infty(*, \mathbf{\Omega}^n) = \mathbf{\Omega}^n(*) = \Omega^n_{\mathrm{dR}}(*) = 0$$

$J^\infty\pi$ as an honest limit

Unlike **Mfd**, the category of smooth sets is well behaved:

Completeness

Sh(**Mfd**) has all limits and colimits.

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Equivalently: *the points of $J^\infty\pi$ are Taylor series.*

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In coordinates the multisymplectic potential gains a new term at each order:

$$\begin{aligned} p\, d^m x &+ p_a^\mu\, du^a \wedge d^{m-1}x_\mu \\ &+ p_a^{\nu\mu}\, du_\nu^a \wedge d^{m-1}x_\mu \\ &+ p_a^{\nu\mu\lambda}\, du_{\nu\lambda}^a \wedge d^{m-1}x_\mu \\ &+ \cdots \end{aligned}$$

Dualising the tower (cont.)

Taking duals is contravariant: an affine bundle map $J^{k+1}\pi \to J^k\pi$ induces

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$M^\infty\pi = \varinjlim M^k\pi$ is a *co*limit and it also exists as a smooth set!

Bonus track: classifying space of differential forms

Remark

The smooth set $\mathbf{\Omega}^n$ is the classifying space of differential forms, in the sense that maps into it are the same thing as differential forms $\Omega^n_{\mathrm{dR}}(M)$ (by definition!)

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This is how $M^\infty\pi$ gets its multisymplectic structure!

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- G. Giotopoulos, H. Sati, *Field Theory via Higher Geometry I & II*, arXiv:2312.16301 & arXiv:2512.22816.
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Thank you for your attention!