UNIVERSITAT ROVIRA i VIRGILI

Multicontact brackets and multisymplectization

Xavier Rivas

Department of Computer Engineering and Mathematics, Universitat Rovira i Virgili, Tarragona

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Outline

Part 1

Introduction

Motivation and goals

Multicontact geometry has been developed to study **non-conservative field theories** in an intrinsic way.

In this talk, we will deepen in the study of the geometry of multicontact manifolds. In particular, we will

- define and study **brackets** induced by an arbitrary n-form and relate them to the **Poisson brackets** from multisymplectic geometry and the **Jacobi brackets** in contact geometry,
- develop a **multisymplectization** procedure for the geometry induced by an n-form,
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Part 2

Brackets induced by an n-form

Infinitesimal conformal transformations

Let $\Theta \in \Omega^n(M)$ be a smooth n-form on M.

Definition

For $p \leq n$, a multivector field $X \in \mathfrak{X}^p(M)$ is an **infinitesimal conformal transformation** of Θ if there exists a multivector field $V \in \mathfrak{X}^{p-1}(M)$ such that

$$\mathscr{L}_X\Theta = \iota_V\Theta\,,$$

where $\mathscr{L}_X\Theta = \mathrm{d}\iota_X\Theta - (-1)^p\iota_X\,\mathrm{d}\Theta$.

For a vector field $X \in \mathfrak{X}(M)$ the condition reads

$$\mathscr{L}_X\Theta = g\cdot\Theta\,,$$

for $g \in \mathscr{C}^\infty(M)$. We recover the notion of an **infinitesimal conformal symmetry**.

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Conformal Hamiltonian forms

Definition

A **conformal Hamiltonian form** is an $(n-p)$-form $\alpha \in \Omega^{n-p}(M)$ such that

$$\iota_{X_\alpha}\Theta = -\alpha\,,$$

for some infinitesimal conformal transformation $X_\alpha \in \mathfrak{X}^p(M)$. We denote by $\Omega^a_H(M,\Theta)$ the space of conformal Hamiltonian a-forms. If the form Θ is clear from the context we will use $\Omega^a_H(M)$.

Note that an $(n-p)$-form α is conformal Hamiltonian if and only if there exist multivector fields $X_\alpha \in \mathfrak{X}^p(M)$ and $V_\alpha \in \mathfrak{X}^{p-1}(M)$ such that

$$\iota_{X_\alpha}\Theta = -\alpha \qquad \text{and} \qquad \iota_{X_\alpha}\mathrm{d}\Theta = (-1)^{p+1}(\mathrm{d}\alpha + \iota_{V_\alpha}\Theta)\,.$$

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Looking for a bracket

Theorem

Let $\alpha \in \Omega^{n-p}(M)$ and $\beta \in \Omega^{n-q}(M)$ be conformal Hamiltonian forms and let $X_\alpha, V_\alpha, X_\beta$ and V_β be multivector fields satisfying

$$\iota_{X_\alpha}\Theta = -\alpha\,, \qquad \iota_{X_\alpha}\mathrm{d}\Theta = (-1)^{p+1}(\mathrm{d}\alpha + \iota_{V_\alpha}\Theta)\,,$$

and

$$\iota_{X_\beta}\Theta = -\beta\,, \qquad \iota_{X_\beta}\mathrm{d}\Theta = (-1)^{q+1}(\mathrm{d}\beta + \iota_{V_\beta}\Theta)\,.$$

Then, the expression

$$-\iota_{[X_\alpha, X_\beta]}\Theta$$

only depends on α and β and is also a conformal Hamiltonian form.

The bracket

Definition

The **graded Jacobi bracket** (or Jacobi bracket) of two conformal Hamiltonian forms $\alpha \in \Omega^a_H(M)$, $\beta \in \Omega^b_H(M)$ is

$$\{\alpha, \beta\} := -\iota_{[X_\alpha, X_\beta]}\Theta = (-1)^{(p-1)q}\left(\mathscr{L}_{X_\alpha} - \iota_{V_\alpha}\right)\beta\,,$$

where X_α, X_β are conformal symmetries for each of these forms and $a = n - p\,,\ b = n - q$. Its well-definedness follows from the previous theorem.

The bracket defined by a differential n-form $\Theta \in \Omega^n(M)$ is a bilinear operation

$$\{\cdot,\cdot\} : \Omega^a_H(M) \otimes \Omega^b_H(M) \longrightarrow \Omega^{a+b-(n-1)}_H(M)\,.$$

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Recovering the contact Jacobi bracket

Remark. *Suppose $\Theta = \eta$ a* ***contact form*** *on M. Then, the previous bracket generalizes the usual* ***Jacobi bracket*** *of contact geometry. Indeed, we have*

$$\iota_{X_f}\eta = -f \qquad \text{and} \qquad \iota_{X_f}\,\mathrm{d}\eta = \mathrm{d}f - R(f)\eta\,,$$

where R denotes the Reeb vector field, the unique vector field satisfying $\iota_{\mathcal{R}}\eta = 1$, and $\iota_{\mathcal{R}}\,\mathrm{d}\eta = 0$. The graded Jacobi bracket reads

$$\{f, g\} = -\iota_{[X_f, X_g]}\eta\,,$$

which recovers the usual contact Jacobi bracket.

Properties of the bracket

Corollary

Let $\alpha \in \Omega^{n-p}(M)$, $\beta \in \Omega^{n-q}(M)$ and $\gamma \in \Omega^{n-r}(M)$ be conformal Hamiltonian forms. Then, the bracket of conformal Hamiltonian forms is **graded skew-symmetric**, satisfies the **graded Jacobi identity**, and the **weak Leibniz rule**, namely

(i) $\{\alpha, \beta\} = -(-1)^{(p-1)(q-1)}\{\beta, \alpha\}$,

(ii) $(-1)^{(p-1)(r-1)}\{\alpha, \{\beta, \gamma\}\} + (-1)^{(r-1)(q-1)}\{\gamma, \{\alpha, \beta\}\} + (-1)^{(q-1)(p-1)}\{\beta, \{\gamma, \alpha\}\} = 0$,

(iii) $\operatorname{supp}\{\alpha, \beta\} \subseteq \operatorname{supp}\alpha \cap \operatorname{supp}\beta$.

Cup product of conformal Hamiltonian forms

Infinitesimal conformal transformations are closed under exterior products, since

$$\begin{aligned}\mathscr{L}_{X\wedge Y}\Theta &= \mathscr{L}_Y \iota_X \Theta + (-1)^q \iota_Y \mathscr{L}_X \Theta \\ &= (-1)^{p(q-1)} \left(\iota_{[Y,X]}\Theta + \iota_X \mathscr{L}_Y \Theta\right) + (-1)^q \iota_Y \mathscr{L}_X \Theta\,,\end{aligned}$$

where $X \in \mathfrak{X}^p(M)$ and $Y \in \mathfrak{X}^q(M)$. This motivates the following definition.

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$$\alpha \vee \beta := -\iota_{X_\alpha \wedge X_\beta}\Theta\,.$$

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Graded Leibniz rule

Note that the cup product of conformal Hamiltonian forms is well-defined since

$$\alpha \vee \beta = \iota_{X_\beta}\alpha = (-1)^{pq}\iota_{X_\alpha}\beta\,.$$

Remark. *The cup product of two conformal Hamiltonian forms is again conformal Hamiltonian, since $X_\alpha \wedge X_\beta$ is an infinitesimal conformal transformation.*

The cup product defines a bilinear operation

$$\vee\colon \Omega^a_H(M) \otimes \Omega^b_H(M) \longrightarrow \Omega^{a+b-n}_H(M)\,.$$

Corollary (Graded Leibniz rule)

Given $\alpha \in \Omega^{n-p}_H(M)$, $\beta \in \Omega^{n-q}_H(M)$ and $\gamma \in \Omega^{n-r}_H(M)$, then

$$\{\alpha, \beta \vee \gamma\} = \{\alpha, \beta\} \vee \gamma + (-1)^{(r-1)q}\beta \vee \{\alpha, \gamma\}\,.$$

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Some comments

Observe that this endows the set of conformal Hamiltonian forms with a structure of a **Gerstenhaber algebra**, and the map

$$X \mapsto -\iota_X \Theta$$

is a Gerstenhaber algebra homomorphism between infinitesimal conformal transformations and conformal Hamiltonian forms.

Remark. *It is worth noting that, in the case where $\Theta = \eta$ defines a contact form on M, the cup product $f \vee g$ of two functions vanishes. Thus, we only have the weak Leibniz rule in the contact case.*

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(Pre-)Multisymplectic forms

Definition

A **pre-multisymplectic form** on M is a closed $(n+1)$-form $\Omega \in \Omega^{n+1}(M)$. It is said to be **multisymplectic** or **non-degenerate** if the vector bundle map $\flat\colon \mathrm{T}M \to \bigwedge^n M$ defined by $\flat(v) := \iota_v\Omega$ is a monomorphism.

Definition

Let (M, Ω) with $\Omega \in \Omega^{n+1}(M)$ be a (pre-)multisymplectic manifold. Then, a form $\alpha \in \Omega^a(M)$ is **(pre-)multisymplectic Hamiltonian** if there exists a multivector field $X_\alpha \in \mathfrak{X}^{n-a}(M)$ such that $\iota_{X_\alpha}\Omega = \mathrm{d}\alpha$. We denote the space of multisymplectic Hamiltonian a-forms by $\Omega^a_H(M, \Omega)$. If the form to be used is clear from the context we simply write $\Omega^a_H(M)$.

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Poisson bracket of pre-multisymplectic Hamiltonian forms

Definition

Let (M, Ω), where $\Omega \in \Omega^{n+1}(M)$, be a pre-multisymplectic manifold and let $\alpha \in \Omega^{n-p}_H(M)$, $\beta \in \Omega^{n-q}_H(M)$ be pre-multisymplectic Hamiltonian forms. Their **Poisson bracket** is defined as

$$\{\alpha, \beta\}_P = (-1)^{q-1} \iota_{X_\alpha \wedge X_\beta} \Omega \, .$$

Relating both brackets

Let $\widetilde{\alpha} \in \Omega^{n-p}_H(M,\Omega)$ and $\widetilde{\beta} \in \Omega^{n-q}_H(M,\Omega)$ be (pre-)multisymplectic Hamiltonian forms. Their Poisson bracket is

$$\{\widetilde{\alpha}, \widetilde{\beta}\}_P = (-1)^{q-1} \iota_{X_{\widetilde{\alpha}} \wedge X_{\widetilde{\beta}}} \Omega\,.$$

$\mathrm{d}\widetilde{\alpha}$ and $\mathrm{d}\widetilde{\beta}$ are conformal Hamiltonian forms with $V_{\mathrm{d}\widetilde{\alpha}} = 0$ and $V_{\mathrm{d}\widetilde{\beta}} = 0$.

Then, the two brackets are related by

$$\{\mathrm{d}\widetilde{\alpha}, \mathrm{d}\widetilde{\beta}\} = \mathrm{d}\{\widetilde{\alpha}, \widetilde{\beta}\}_P\,.$$

> **Remark.** *If $n = 1$, we have $\{\mathrm{d}f, \mathrm{d}g\} = \mathrm{d}\{f, g\}_P$, where $\{f, g\}_P$ is the Poisson bracket of two functions from symplectic geometry. Thus, the graded Jacobi bracket generalizes the graded Poisson bracket.*

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Part 3

Multicontact structures and multisymplectization

Homogeneous (pre-)multisymplectic manifolds

Definition

An **homogeneous (pre-)multisymplectic** manifold is a pair $(\widetilde{M}, \Upsilon)$, where $\Upsilon \in \Omega^n(\widetilde{M})$ is such that

- $\Omega = -\mathrm{d}\Upsilon$ is a (pre-)multisymplectic form on M, and
- $\exists \Delta \in \mathfrak{X}(\widetilde{M})$, called **a Liouville vector field** satisfying $\iota_\Delta \Omega = -\Upsilon$.

Remark. *A homogeneous symplectic structure is just an exact symplectic structure $\omega = -\mathrm{d}\upsilon$, since there always exists Δ such that $\iota_\Delta \omega = -\upsilon$.*

However, this does not hold in multisymplectic geometry since we cannot guarantee the existence of a vector field Δ satisfying $\iota_\Delta \Omega = -\Upsilon$.

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(Pre-)multisymplectization

Let Θ be an n-form on a smooth manifold M.

Definition

Let $\tau\colon \widetilde{M} \to M$ be a (locally trivial) fiber bundle, where $\widetilde{M}$ is endowed with a homogeneous (pre-)multisymplectic structure $\Omega = -\mathrm{d}\Upsilon$. Then, it is said to be a **(pre-)multisymplectization** of (M, Θ) if:

- it admits a vertical Liouville vector field Δ with respect to the projection τ,
- $\Upsilon = \phi \cdot \tau^*\Theta$ for certain nowhere vanishing function $\phi \in \mathscr{C}^\infty(\widetilde{M})$, called **the conformal factor**.

The **canonical (pre-)multisymplectization** of (M, Θ) is $M \times \mathbb{R}_\times$, with $\Omega = -\mathrm{d}(z\,\Theta)$. In this case, $\Delta = z\dfrac{\partial}{\partial z}$ and $\phi = z$.

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Lifting multivector fields to the multisymplectization

Theorem

Let Θ be a nowhere vanishing n-form on M and let $\tau\colon \widetilde{M} \to M$, with $(\widetilde{M}, -\mathrm{d}\Upsilon)$ be a (pre-)multisymplectization of (M, Θ) with conformal factor $\phi \in \mathscr{C}^\infty(\widetilde{M})$.

Then, for every infinitesimal conformal transformation $X \in \mathfrak{X}^p(M)$ of Θ there exists a unique-up-to-$\ker_p \mathrm{d}\Upsilon$ multivector field $\widetilde{X} \in \mathfrak{X}^p(\widetilde{M})$ such that

$$\mathscr{L}_{\widetilde{X}}\Upsilon = 0 \qquad \text{and} \qquad \tau_*\widetilde{X} = X\,.$$

Lifting the Jacobi bracket to the Poisson bracket

Theorem

Let Θ be a nowhere vanishing n-form on M and let $\tau\colon \widetilde{M} \to M$, with $(\widetilde{M}, -\mathrm{d}\Upsilon)$, be a (pre-)multisymplectization of (M, Θ) with conformal factor $\phi \in \mathscr{C}^\infty(\widetilde{M})$.

Then, the map

$$\begin{array}{rccc} \Psi\colon & \Omega^{n-p}_H(M,\Theta) & \longrightarrow & \Omega^{n-p}_H(\widetilde{M}, \Omega = -\mathrm{d}\Upsilon) \\ & \alpha & \longmapsto & (-1)^{p+1}\phi\cdot\tau^*\alpha \end{array}$$

is well defined and, for every $\alpha \in \Omega^{n-p}_H(M,\Theta)$ and $\beta \in \Omega^{n-q}_H(M,\Theta)$ it satisfies

$$\{\Psi(\alpha), \Psi(\beta)\}_P = \Psi(\{\alpha,\beta\}) + \underbrace{(-1)^q\,\mathrm{d}\Psi(\alpha\vee\beta)}_{\text{exact term}}\,.$$

Some comments

Remark. *If $\Theta = \eta$ is a contact form, taking the symplectization $M \times \mathbb{R}_\times$,*

$$\Psi\colon \begin{array}{ccc} \mathscr{C}^\infty(M) & \longrightarrow & \mathscr{C}^\infty(M \times \mathbb{R}_\times) \\ f & \longmapsto & \Psi(f) = z \cdot f\,, \end{array} \quad \text{and} \quad \{\Psi(f), \Psi(g)\}_P = \Psi(\{f, g\})\,,$$

since $f \vee g = -\iota_{X_f \wedge X_g}\eta = 0$, and we recover the usual symplectization of a contact manifold.

Remark. *If $\Theta \in \Omega^n(M)$ is nowhere-vanishing and $\alpha \in \Omega^a_H(M, \Theta)$ and $\beta \in \Omega^b_H(M, \Theta)$ with $a + b < n$, we have*

$$\{\Psi(\alpha), \Psi(\beta)\}_P = \Psi(\{\alpha, \beta\}).$$

Some comments

Remark. *If $\Theta = \eta$ is a contact form, taking the symplectization $M \times \mathbb{R}_\times$,*

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Multicontact manifolds

Proposition

If $\ker_1 \mathrm{d}\Theta \neq \{0\}$, then $(M \times \mathbb{R}_\times, \Omega := -\mathrm{d}(z\,\Theta))$ is a non-degenerate homogeneous multisymplectic manifold if and only if

$$\ker_1 \Theta \cap \ker_1 \mathrm{d}\Theta = \{0\}\,.$$

Definition

A **multicontact manifold** is a pair (M, Θ) such that

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Remark. *Let $\eta \in \Omega^1(M)$ such that*

$$\ker_1 \eta \cap \ker_1 \mathrm{d}\eta = \{0\} \qquad \text{and} \qquad \ker_1 \mathrm{d}\eta \neq \{0\}\,.$$

M is odd-dimensional since $\dim \ker_1 \eta = \dim M - 1$ and $\operatorname{codim} \ker_1 \mathrm{d}\eta$ is even dimensional. The conditions above guarantee that

$$\begin{aligned} \flat\colon \quad \mathrm{T}\,M &\longrightarrow \mathrm{T}^*\,M \\ v &\longmapsto \iota_v \mathrm{d}\eta + \eta(v)\cdot\eta \end{aligned}$$

defines a vector bundle isomorphism, and thus η defines a ***contact form****.*

Remark. *Note that a multicontact n-form Θ is necessarily nonvanishing: if $\Theta|_p = 0$, then $\ker_1 \Theta|_p = \mathrm{T}_p\,M$ which would yield $\{0\} = \ker_1 \Theta|_p \cap \ker_1 \mathrm{d}\Theta|_p = \ker_1 \mathrm{d}\Theta|_p \neq \{0\}$, contradiction.*

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Proposition

Let $\tau\colon \widetilde{M} \to M$ be a fiber bundle with at least 1-dim. fibers, where $(\widetilde{M}, \Omega = -\mathrm{d}(\phi\,\tau^*\Theta))$ is a homogeneous multisymplectic manifold for some $\phi \in \mathscr{C}^\infty(\widetilde{M})$ nonvanishing. Then, its **Liouville vector field Δ is vertical**.

Remark. *For a multisymplectization $\tau\colon \widetilde{M} \to M$ of a multicontact manifold (M, Θ) with conformal factor ϕ, we have $\Delta(\phi) = \phi$.*

Proposition

If $\tau\colon \widetilde{M} \to M$ is a multisymplectization of the **multicontact manifold** (M, Θ), then the fibers are necessarily **1-dimensional**.

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Part 4

Multicontact field equations

Setting the problem

In this section we study a possible definition of **multicontact field equations** in a particular, but still quite general, subclass of multicontact manifolds.

We look for equations that not only are **dependent on the geometry**, but also depend on a **choice of Hamiltonian**, as is usually done in symplectic and contact mechanics.

In **contact mechanics**, the evolution equation of the motion determined by a Hamiltonian $H \in \mathscr{C}^\infty(M)$ on a contact manifold (M, η) is given by

$$\dot{g} = X_H(g) = \{H, g\} - R(H)g\,,$$

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Going to multicontact

In our case, a Hamiltonian will be a suitable n-form $h \in \Omega^n(M)$, and the **evolution of conformal Hamiltonian $(n-1)$-forms** will be given by

$$\begin{aligned}\psi^*(\mathrm{d}\alpha) &= -\mathcal{R}(\mathrm{d}\alpha)\cdot h - \iota_{X_\alpha}\mathrm{d}h - (\iota_{\mathcal{R}}\,\mathrm{d}h)\wedge\alpha \\ &= -\left(\mathscr{L}_{X_\alpha} + \mathcal{R}(\alpha)\right)h - (\iota_{\mathcal{R}}\,\mathrm{d}h)\wedge\alpha + \mathrm{d}\left(\iota_{X_\alpha}h\right),\end{aligned}$$

where $\mathcal{R}$ is the **Reeb multivector field**, generalizing the Reeb vector field, and $\psi\colon X \to M$ is a smooth map from an n-dimensional manifold to the multicontact manifold (M, Θ).

Extending the domain of definition of brackets to n-forms, the previous equation may be interpreted as

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The Reeb multivector field

Assume $n > 1$. Let us first define the Reeb multivector field.

First attempt: $\mathcal{R}$ is the only n-multivector field up to $\ker_n \Theta \cap \ker_n \mathrm{d}\Theta$ such that $\iota_{\mathcal{R}}\Theta = 1$ and $\iota_{\mathcal{R}} \mathrm{d}\Theta = 0$.

However, a finer definition can be made. We define

$$\begin{array}{rccc} \flat_\Theta : & \ker_1 \mathrm{d}\Theta & \longrightarrow & \bigwedge^{n-1} M \\ & v & \longmapsto & \iota_v \Theta \,. \end{array}$$

It is a monomorphism. We may see its inverse $\flat_\Theta^{-1} : \operatorname{Im} \flat_\Theta \to \ker_1 \mathrm{d}\Theta$ as a tensor

$$\flat_\Theta^{-1} \in \ker_1 \mathrm{d}\Theta \otimes (\operatorname{Im} \flat_\Theta)^* \cong \ker_1 \mathrm{d}\Theta \otimes \left(\bigvee_{n-1} M \,/\, (\operatorname{Im} \flat_\Theta)^{\circ, n-1} \right), \quad \text{where } \bigvee_p M = \bigwedge^p \mathrm{T} M \,.$$

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$\widetilde{\mathcal{R}} := \frac{1}{n} \flat_\Theta^{-1}$, interpreted as a tensor taking values in the vector bundle above.

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Contracting by $\widetilde{\mathcal{R}}$

We want to define **contraction by $\widetilde{\mathcal{R}}$** on a suitable family of n-forms, which will be later identified as the **Hamiltonians**, objects that will define the dynamics.

Remark. *Let*

$$\widetilde{U} = U \otimes [V + (\operatorname{Im} \flat_\Theta)^{\circ, n+1}] \in \ker_1 \mathrm{d}\Theta \otimes \left(\bigvee_{n-1} M \,/\, (\operatorname{Im} \flat_\Theta)^{\circ, n-1} \right) .$$

For a suitable a-form α, we want to define the contraction $\iota_{\widetilde{U}} \alpha$ as $\iota_V \iota_U \alpha$.

- *We can guarantee well-definedness when α is such that $\iota_K \iota_u \alpha = 0$, for every $K \in (\operatorname{Im} \flat_\Theta)^{\circ, n-1}$ and every $u \in \ker_1 \mathrm{d}\Theta$.*
- *This last condition is equivalent to α satisfying $\iota_V \iota_u \alpha \in \operatorname{Im} \flat_\Theta$, for every $u \in \ker_1 \mathrm{d}\Theta$ and $V \in \bigvee_{a-n} M$.*

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Hamiltonians and the dissipation form

Definition

The **Hamiltonian subbundle** is $\mathcal{H} := \left\{ h \in \bigwedge^n M \;\middle|\; \iota_{\mathrm{T}M} h \subseteq \operatorname{Im} \flat_\Theta \right\} \subseteq \bigwedge^n M$.

A **Hamiltonian** is a section $h \in \Gamma(\mathcal{H})$: an n-form h such that $\iota_{\mathrm{T}M} h \subseteq \operatorname{Im} \flat_\Theta$.

Definition

A Hamiltonian $h \in \Gamma(\mathcal{H})$ is called a **good Hamiltonian** if $\iota_{\ker_1 \mathrm{d}\Theta} \mathrm{d}h \subseteq \mathcal{H}$.

For a good Hamiltonian, a fundamental object is the **dissipation 1-form** which is defined as

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Hamilton–de Donder–Weyl equations

Then, we can define the **Hamilton–de Donder–Weyl** equations for a map $\psi\colon X \to M$, where X is and n-dimensional manifold, as

$$\psi^*(\Theta + h) = 0 \qquad \text{and} \qquad \psi^*\iota_\xi(\mathrm{d} + \sigma_h\wedge)(\Theta + h) = 0\,, \quad \forall\xi \in \mathfrak{X}(M)\,.$$

Remark. *These equations generalize the equations found in the literature of multicontact systems. Indeed, denoting by $\Theta_h = \Theta + h$ the equations now read as*

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Evolution of Hamiltonian forms

In order to discuss dissipation phenomena (dissipated forms), it will be useful for us to have an explicit formula for the evolution of a conformal Hamiltonian $(n-1)$-form.

Theorem

Let $h \in \Gamma(\mathcal{H})$ be a good Hamiltonian, $\alpha \in \Omega^{n-1}_H(M)$ a conformal Hamiltonian form and $\psi \colon X \to M$ a solution of the HdDW equations defined by h. Then

$$\psi^*(\mathrm{d}\alpha) = \psi^*\left(-\mathcal{R}(\mathrm{d}\alpha)h - \iota_{X_\alpha}\mathrm{d}h - \sigma_h \wedge \alpha + \sigma_h \wedge \iota_{X_\alpha} h\right) .$$

If X_α takes values in $(\operatorname{Im} \flat_\Theta)^{\circ,1}$, we have

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Definition

A multicontact manifold (M, Θ) is called **variational** if

$$\iota_{\bigwedge^2(\ker_1 \mathrm{d}\Theta)}\Theta = 0\,.$$

Proposition

Let (M, Θ) be a variational multicontact manifold and $h \in \Gamma(\mathcal{H})$. Then,

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Variational multicontact manifolds

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Proposition

Let (M, Θ) be a variational multicontact manifold and $h \in \Gamma(\mathcal{H})$. Then,

$$\iota_{\ker_1 \mathrm{d}\Theta} h = 0 \,.$$

Distortion of a variational multicontact manifold

Definition

The **distortion** of a variational multicontact manifold (M, Θ) is

$$\begin{array}{rccc} C_\Theta : & \Gamma(\ker_1 \mathrm{d}\Theta) \otimes \Gamma(\ker_1 \mathrm{d}\Theta) & \longrightarrow & \Omega^{n-1}(M) \,/\, \Gamma(\operatorname{Im} \flat_\Theta) \\ & R \otimes R' & \longmapsto & \iota_R \, \mathrm{d}\iota_{R'} \Theta \,. \end{array}$$

For a variational multicontact manifold, C_Θ defines a symmetric vector bundle map

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Let (M, Θ) be a variational multicontact manifold such that $\ker_1 \mathrm{d}\Theta$ has constant rank. Then, every Hamiltonian is a good Hamiltonian if and only if its distortion vanishes.

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Dissipated forms

In contact dynamics, a fundamental concept is that of **dissipated quantity**, which is a function $g \in \mathscr{C}^\infty(M)$ satisfying

$$\dot{g} = X_H(g) = -R(H)g\,.$$

Definition

Let (M, Θ) be a multicontact manifold and let $h \in \Gamma(\mathcal{H})$ be a good Hamiltonian. Then, a form $\alpha \in \Omega^a_H(M)$ is said to be **dissipated**, if

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Properties of dissipated forms

Proposition

Let $\alpha \in \Omega^{n-1}_H(M)$. If $-\left(\mathscr{L}_{X_\alpha} + \mathcal{R}(\mathrm{d}\alpha)\right) h + \left(\mathrm{d} + \sigma_h \wedge\right) \iota_{X_\alpha} h = 0$, then α is a dissipated $(n-1)$-form.

Corollary

Let α and β be dissipated conformal Hamiltonian $(n-1)$-forms such that

$$-\left(\mathscr{L}_{X_\alpha} + \mathcal{R}(\mathrm{d}\alpha)\right) h = 0 \qquad \text{and} \qquad -\left(\mathscr{L}_{X_\beta} + \mathcal{R}(\mathrm{d}\beta)\right) h = 0\,.$$

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Multisymplectization of the dynamics

Let (M, η) be contact and $H \in \mathscr{C}^\infty(M)$. We define a **homogeneous Hamiltonian** on its canonical symplectization $(\widetilde{M} := M \times \mathbb{R}_\times, -\mathrm{d}(z\,\eta))$ as $\widetilde{H}(p, z) := zH(p)$.

The **symplectic** Hamiltonian vector field $X_{\widetilde{H}} \in \mathfrak{X}(\widetilde{M})$ is **τ-related** with the **contact** Hamiltonian vector field $X_H \in \mathfrak{X}(M)$.

Definition

Let $(\widetilde{M}, \Omega)$ be a multisymplectic manifold and $\widetilde{h} \in \Omega^n(\widetilde{M})$ be an arbitrary form (which we refer to as the Hamiltonian). Then, a map $\widetilde{\psi}\colon X \to \widetilde{M}$ from an n-dimensional manifold X to $\widetilde{M}$, is said to satisfy the **multisymplectic Hamilton–De Donder–Weyl equations** if

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Let (M, Θ) be variational multicontact and let $(\widetilde{M}, \Omega)$ denote its canonical multisymplectization. Given a good Hamiltonian $h \in \Omega^n(M)$, define the associated homogeneous Hamiltonian as $\widetilde{h}(p, z) := z\tau^* h(p)$. Under technical conditions:

(i) Solutions to the homogeneous HdDW equations project onto solutions to the multicontact HdDW equations.

(ii) Solutions to the multicontact HdDW equations can lifted to solutions to the homogenous HdDW equations provided that the dissipation form is exact over the solutions.

(iii) There is a correspondence between the evolution of observables: for $\alpha \in \Omega^a_H(M)$ conformal Hamiltonian, defining $\widetilde{\alpha}(p, z) := z\,\tau^*\alpha(p)$, we have

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Part 5

Dissipative field theories

Dissipative field theories

Let's apply all these constructions to the case of **dissipative field theories**.

Let $\pi\colon Y \to X$ be a fiber bundle. The **extended phase space** of action dependent field theories is defined as

$$M := \bigwedge\nolimits_2^n Y \oplus \bigwedge\nolimits^{n-1} X\,,$$

where $\bigwedge_2^n Y$ denotes the bundle of 2-horizontal n-forms on Y and $\bigwedge^{n-1} X$ denotes the bundle of $(n-1)$-forms on X.

We can find a **canonical multicontact form** on M given by

$$\Theta = \mathrm{d}\Theta_X^{n-1} - \Theta_Y^n\,,$$

where Θ_Y^n and Θ_X^{n-1} denote the canonical Liouville forms on $\bigwedge_2^n Y$ and $\bigwedge^{n-1} X$.

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Local expressions

Let (x^μ, y^i) denote fibered coordinates on $\pi\colon Y \to X$, and let $(x^\mu, y^i, p, p_i^\mu, s^\mu)$ denote the coordinates on M. **Any element in $\bigwedge_2^n Y$** (resp. in $\bigwedge^{n-1} X$) reads

$$p\,\mathrm{d}^n x + p_i^\mu\,\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu \qquad (\,\text{resp. } s^\mu\,\mathrm{d}^{n-1}x_\mu\,)\,,$$

where $\mathrm{d}^n x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $\mathrm{d}^{n-1}x_\mu = i_{\frac{\partial}{\partial x^\mu}}\,\mathrm{d}^n x$.

The **canonical multicontact form and its exterior differential** read

$$\Theta = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - p\,\mathrm{d}^n x - p_i^\mu\,\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu\,, \qquad \mathrm{d}\Theta = -\,\mathrm{d}p \wedge \mathrm{d}^n x - \mathrm{d}p_i^\mu \wedge dy^i \wedge \mathrm{d}^{n-1}x_\mu\,.$$

And finally, their **kernels** are

$$\ker_1 \Theta = \left\langle \frac{\partial}{\partial y^i} - p_i^\mu \frac{\partial}{\partial s^\mu}\,, \frac{\partial}{\partial p_i^\mu} \right\rangle \qquad \text{and} \qquad \ker_1 \mathrm{d}\Theta = \left\langle \frac{\partial}{\partial s^\mu} \right\rangle\,,$$

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Vertical infinitesimal conformal transformations

As discussed before, the best behaved conformal Hamiltonian forms are the ones whose **infinitesimal conformal transformation** lies in $(\operatorname{Im} \flat_\Theta)^{\circ,1}$.

Proposition

For $n > 1$, **vertical infinitesimal conformal transformations** read

$$X = \left(Fs^\mu + G^\mu - p^\mu_j \frac{\partial G^\nu}{\partial p^\nu_j}\right) \frac{\partial}{\partial s^\mu} - \frac{\partial G^\mu}{\partial p^\mu_i} \frac{\partial}{\partial y^i} + \left(\frac{\partial F}{\partial y^i} s^\mu + \frac{\partial G^\mu}{\partial y^i} + F p^\mu_i\right) \frac{\partial}{\partial p^\mu_i} + \left(\frac{\partial F}{\partial x^\mu} s^\mu + \frac{\partial G^\mu}{\partial x^\mu} + Fp\right) \frac{\partial}{\partial p},$$

where F is independent of s^μ, p and p^μ_i, and G^μ satisfies $\dfrac{\partial G^\mu}{\partial p^\nu_i} = \delta^\mu_\nu \dfrac{\partial G^\alpha}{\partial p^\alpha_i}$.

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Vertical infinitesimal conformal transformations

Corollary

A Hamiltonian form $\alpha \in \Omega^{n-1}_H(M)$ with vertical infinitesimal conformal transformation has the following local expression:

$$\alpha = (-Fs^\mu - G^\mu)\,\mathrm{d}^{n-1}x_\mu\,,$$

where $F = F(x, y)$, and G^μ satisfies $\dfrac{\partial G^\mu}{\partial p^\nu_i} = \delta^\mu_\nu \dfrac{\partial G^\alpha}{\partial p^\alpha_i}$.

Elementary examples of conformal Hamiltonian forms

Form	Infinitesimal conformal transformation	Conformal factor
$s^\mu \, \mathrm{d}^{n-1}x_\mu$	$-s^\mu \dfrac{\partial}{\partial s^\mu} - p_i^\mu \dfrac{\partial}{\partial p_i^\mu} - p \dfrac{\partial}{\partial p}$	-1
$y^i \, \mathrm{d}^{n-1}x_\mu$	$-y^i \dfrac{\partial}{\partial s^\mu} - \dfrac{\partial}{\partial p_i^\mu}$	0
$p_i^\mu \, \mathrm{d}^{n-1}x_\mu$	$\dfrac{\partial}{\partial y^i}$	0
$\mathrm{d}^{n-1}x_\mu$	$-\dfrac{\partial}{\partial s^\mu}$	0

Jacobi brackets of the elementary examples

$\{\cdot,\cdot\}$	$s^\nu\, \mathrm{d}^{n-1}x_\nu$	$y^j\, \mathrm{d}^{n-1}x_\nu$	$p^\nu_j\, \mathrm{d}^{n-1}x_\nu$	$\mathrm{d}^{n-1}x_\nu$
$s^\mu\, \mathrm{d}^{n-1}x_\mu$	0	$y^j\, \mathrm{d}^{n-1}x_\nu$	0	$\mathrm{d}^{n-1}x_\mu$
$y^i\, \mathrm{d}^{n-1}x_\mu$	$-y^i\, \mathrm{d}^{n-1}x_\mu$	0	$\delta^i_j\, \mathrm{d}^{n-1}x_\nu$	0
$p^\mu_i\, \mathrm{d}^{n-1}x_\mu$	0	$-\delta^j_i\, \mathrm{d}^{n-1}x_\mu$	0	0
$\mathrm{d}^{n-1}x_\mu$	$-\mathrm{d}^{n-1}x_\mu$	0	0	0

Hamiltonians

Now, our goal is to write the evolution of the previous Hamiltonian forms with a fixed Hamiltonian (and hence fixed dynamics) using the studied geometry.

Intrinsically, a Hamiltonian is given by a **Hamiltonian section**

$$h\colon \bigwedge_2^n Y / \bigwedge_1^n Y \oplus \bigwedge^{n-1} X \longrightarrow M = \bigwedge_2^n Y \oplus \bigwedge^{n-1} X \,,$$

which then induces a multicontact structure on $\bigwedge_2^n Y / \bigwedge_1^n Y \oplus \bigwedge^{n-1} X$ using $\Theta_h = h^*\Theta$.

As stated before, we prefer to work with the canonical multicontact structure on M and think the Hamiltonian as $\widetilde{h} = \tau^*\Theta_h - \Theta$, where $\tau\colon M \to \bigwedge_2^n Y / \bigwedge_1^n Y \oplus \bigwedge^{n-1} X$ denotes the projection, hence fixing the geometry.

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Local expressions

The canonical coordinates on $\bigwedge_2^n Y / \bigwedge_1^n Y \oplus \bigwedge^{n-1} X$ are $(x^\mu, y^i, p_i^\mu, s^\mu)$ and, hence, the local expression of the Hamiltonian section will be $p = -H(x^\mu, y^i, p_i^\mu, s^\mu)$, so that

$$\Theta_h = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu + H\,\mathrm{d}^n x - p_i^\mu\,\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu\,,$$

and

$$\widetilde{h} = (p + H)\,\mathrm{d}^n x\,.$$

This implies that $\widetilde{h}$ is a good Hamiltonian, so that it could define the following dynamics for some $\psi\colon X \to M$:

$$\psi^*(\Theta + h) = 0 \qquad \text{and} \qquad \psi^* \iota_\xi(\mathrm{d} + \sigma_h \wedge)(\Theta + h) = 0\,, \quad \forall \xi \in \mathfrak{X}(M)\,,$$

where $\sigma_h = \dfrac{\partial H}{\partial s^\mu}\,\mathrm{d}x^\mu$ is the associated dissipation 1-form.

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Multicontact Hamilton–de Donder–Weyl equations

The **local expression of the multicontact HdDW equations** is

$$
\begin{aligned}
\psi^*(\mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu) &= \psi^*\left(\left(p_i^\mu \frac{\partial H}{\partial p_i^\mu} - H\right)\mathrm{d}^n x\right)\,,\\
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\psi^*(\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu) &= \psi^*\left(-\left(\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial s^\mu}p_i^\mu\right)\mathrm{d}^n x\right)\,.
\end{aligned}
$$

In particular, if we look for **sections** $\psi\colon X \to M$, we obtain the so-called multicontact Hamilton–de Donder–Weyl equations for dissipated field theories:

$$
\frac{\partial s^\mu}{\partial x^\mu} = p_i^\mu \frac{\partial H}{\partial p_i^\mu} - H\,, \qquad \frac{\partial y^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu}\,, \qquad \frac{\partial p_i^\mu}{\partial x^\mu} = -\left(\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial s^\mu}p_i^\mu\right)\,.
$$

Multicontact Hamilton–de Donder–Weyl equations

The **local expression of the multicontact HdDW equations** is

$$
\begin{aligned}
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\psi^*(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) &= \psi^*\left(\frac{\partial H}{\partial p_i^\mu}\,\mathrm{d}^n x\right) \,,\\
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Part 6

Conclusions

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In this talk, we have:

- Defined and study **brackets** induced by an arbitrary n-form and related them to the **Poisson brackets** from multisymplectic geometry and the **Jacobi brackets** in contact geometry.
- Developed a **multisymplectization** procedure for the geometry induced by an n-form.
- Introduced an abstract **framework to study multicontact field theories**.

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Dziękuję za uwagę!

Pytania?

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xavier.rivas@urv.cat