INTRODUCTION

Classical geometrical field theory is usually associated with the concept of a multisymplectic structure. It appeared first in the papers by Gawędzki, Kijowski, Szczyrba, Tulczyjew and then there were works by Gotay and Marsden supported by Isenberg and Montgomery and with some input by Śniatycki and Yasskin – these works are now called GIMMSY and have 5 parts. When I was growing up scientifically, I was actively discouraged from reading GIMMSY, probably due to personal animosities between Włodek Tulczyjew and Jerrold Marsden. When I now come back to these papeprs, I find them rather valuable and well written. In the first part of GIMMSY the autors declare that they „develop the geometry of the jet bundle and the Euler-Lagrange equations, which is analogous to the geometry of the tangent bundle and the Lagrange equations of motion in the case of particle mechanics" and then there appears the chapter about multisymplectic manifolds that starts with the opening that there are two major approaches of the classical field theory – the covariant (multisymplectic) and instantaneous – in 3+1 decomposition of the space-time. In this classification I will be talking about the covariant approach, but is it really multisymplectic? Here is the outline of my talk:

① Introduction ← already done

② Mechanics

← short overview to point out the most important structures being used

③ Lagrangian and momenta

↑ Lagrangian, momenta, Hamiltonian + structures

④ Multisymplectic approach

← How to get from Hamiltonian to field equations using multisymplectic form

⑤ Non-multisymplectic approach

↑ How to get from Hamiltonian to equations NOT using multisymplectic form at any moment

MECHANICS

Let us go over the structures of classical mechanics. The system lives on a manifold M that may or may not include the time dimension. Let us consider the autonomous case - no explicit time dependence in the Lagrangian. M comes with the tangent bundle and cotangent bundle. We usually end up with the following identifications: $q \in M \to$ positions

$\delta q \in \underline{TM} \to$ variations $v \in \underline{TM} \to$ velocities Note, that both - variations and velocities are of the same geometric nature $p \in T^*M \to$ momenta

$\langle p, v \rangle$ appears in the Legendre transform and $\langle p, \delta q \rangle$ appears in the variational formula at the beginning and at the end of a motion.

The canonical structure of TM is that of a Lie algebroid i.e. the bracket of vector fields, the canonical structure of T^*M is a linear symplectic form ω_M. It turnes out that from the point of view of differential geometry we can look at the pair of dual bundles $\tau_M : TM \to M$, $\pi_M : T^*M \to M$ together with $[\cdot,\cdot]$ - the bracket on $\mathfrak{X}(M)$, $\omega_M \in \Lambda^2 T^*M$ $\beta_M : TT^*M \longrightarrow T^*T^*M$ $\beta_M = \omega_M^b$, $\alpha_M : TT^*M \longrightarrow T^*TM$ as one system of geometric structures. ~~M~~ We may add deRham differential to the list and observe that each of the structures: ω_M, $[\cdot,\cdot]$, β_M, α_M, d defines the other, so in fact we may treat it as one structure~~s~~ expressed differently. This is the special case of a more general situation which is the pair of dual vector bundles $\tau : E \to M$, $\pi : E^* \to M$

Together with $([\cdot,\cdot],\rho)$, Λ, d_E, $\Lambda^\#$, ε - a simmilar list of structures (bracket, Poisson bivector, algebroid differential, dvb morphism $\Lambda^\#$ and ε) that are more general and that may appear when we consider symmetries or constraints. We may build on it some more, e.g. a Dirac algebroid, but this is still within the same environment.

As a picture representing this environment I am choosing the classical Tulczyjew triple

$$\begin{tikzcd}
T^*T^*M \arrow[dr] & & TT^*M \arrow[ll, "\beta_M"'] \arrow[rr, "\alpha_M"] \arrow[dl] \arrow[dr] & & T^*TM \arrow[dl] \\
& T^*M \arrow[ul, bend left, "\mathrm{d}H"] \arrow[ur, bend right, "X_H"'] & & TM \arrow[ll, "\lambda"'] \arrow[ur, bend right, "\mathrm{d}L"'] &
\end{tikzcd}$$

symplectic
Poisson
Dirac

Lie algebroid
α_M

Can we have such „environment" in field theory?

FIELD THEORY - LAGRANGIAN AND MOMENTA

Let us try to build similar environment for the field theory. We have to arrive quite promptly to the hamiltonian description, therefore we have to go fast through the lagrangian approach:

We live on a fibre bundle $\tau: E \to M$. For a very general version of the theory, we do not need to put any additional structure on E or M. Sections of τ are fields (~~p~~ field potentials) with the infinitesimal parts being jets →

$$J^1(\tau) \longrightarrow E \longrightarrow M.$$

affine bundle modelled on $VE \otimes_E T^*M$

If we approach the theory from the variational point of view, we start from a density valued Lagrangian $\mathcal{L}: J^1(\tau) \longrightarrow \Omega^m = \Lambda^m T^*M$

Our action is of the form (x^α, y^a, y^a_μ)

$i_{\partial_{x^\mu}} \eta$

$\eta_\mu = i_{\partial_{x^\mu}} \eta$

$$S[\varphi] = \int_D \mathcal{L}(j^1\varphi)$$

From $\delta S[\varphi] = \int_D \mathcal{E}\mathcal{L}(j^2\varphi) + \int_{\partial D} \boxed{\frac{\partial \mathcal{L}}{\partial y^a_\mu}\delta y^a} \eta_\mu$ Looking at the boundary term we define momenta

$$P = V^*E \otimes_E \Omega^{m-1}$$

$$\begin{tikzcd}
 & V^*J^1\overset{(\tau)}{\cancel{E}} \otimes \Omega^m \arrow[dl] \arrow[dr] & \\
P & & J^1(\tau) \arrow[ll, "\lambda"'] \arrow[ul, bend right, <->, "\delta\mathcal{L}"']
\end{tikzcd}$$

vertical differential of the Lagrangian

$P^\nu_a \, \delta y^a \otimes \eta_\mu$

$$\delta L = \left(\frac{\partial L}{\partial y^a}\delta y^a + \frac{\partial L}{\partial y^b_\nu}\delta y^b_\nu\right) \otimes \eta$$

As we can see, momenta are dual to variations (vertical) rather then velocities. Phase equations on sections of $P \longrightarrow M$ are first order partial differential equations, geometrically the subset of J^1P. This subset can be obtained from δL by means of the map $\alpha: J^1P \longrightarrow V^*J^1E \otimes \Omega^m$ which is dual to $\kappa: \underbrace{J^1VE} \longrightarrow \underbrace{VJ^1E}$

J^1P is dual to this

$V^*J^1(\tau) \otimes \Omega^m$ is dual to this

$$\langle\langle j^1p, j^1\delta\varphi \rangle\rangle = D_m \underbrace{\langle p, \delta\varphi \rangle}_{\in \Omega^{m-1}}$$

total differential

This looks very much as in the mechanical case

$$\alpha : J^1P \longrightarrow V^*(J^1(\tau)) \otimes \Omega^m$$

$$(x^\mu, y^a, p^\nu_b, y^a_\mu, p^\alpha_{c\beta}) \qquad (x^\mu, y^a, y^b_\nu, \pi_a, \pi^\alpha_c)$$

$$(x^\mu, y^a, y^b_\nu, \pi_a, \pi^\alpha_c)\circ\alpha = (x^\mu, y^a, y^b_\nu, p^\beta_{a\beta}, p^\alpha_c)$$

As we can see, α is not an isomorphism, since only the divergence part of ~~∂~~ j^1p matters. As for the natural structure of $P = V^*E \otimes \Omega^{m-1}$, we have a tautological form $\theta_P(v) = \langle \tau_P(v), T\pi_E(v) \rangle \in \Omega^{m-1}$

$v \in VP$ $\tau_P : VP \longrightarrow P$, $\pi_E : V^*E \otimes \Omega^{m-1} = P \longrightarrow E$

Verte →

$T\pi_E : V$~~$\times E$~~$P \longrightarrow VE$

$$\theta_P = p^\mu_a \partial y^a \otimes \eta_\mu$$

$\partial y^a \in V^*E$ it is a vertical differential that can act only on vertical vectors!

We can consider also a vertical differential $\omega_P = \partial\theta_P = \partial p^\mu_a \wedge \partial y^a \otimes \eta_\mu$. It is a closed, vertical, canonical two-form with values in Ω^{m-1}, ~~therefore~~ it will be needed later on.

MULTISYMPLECTIC STUFF

The form ω_P is not a multisymplectic form. It is rather fiberwise over M symplectic (~~or in general, presymplectic~~) with values in Ω^{m-1}. however not in the traditional sense, since it acts only on vertical tangent vector. Multisymplectic structure that is used in the Hamiltonian approach ~~is~~ lives not on P but somewhere „over P". The Hamiltonian, i.e. dual description of fields theory is based on the affine dual bundle to to $J^1(\tau) \longrightarrow E$.

$$D \subset J^1 P$$

$$\mathcal{D} = \alpha^{-1}\left(\partial L(J^1(I))\right)$$

$$P^{\alpha}_{a\alpha} = \frac{\partial L}{\partial y^a} \qquad P^{\nu}_{b} = \frac{\partial L}{\partial y^b{}_\nu}$$

$$\Downarrow$$

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^a{}_\nu}\right) = \frac{\partial L}{\partial y^a}$$ E-L equations

$$\begin{tikzcd} J^1(\tau) \arrow[d, "affine"] \\ E \end{tikzcd}$$

$y \in E$

$x \in M \quad x = \tau(y)$

$J^+_yE = \mathrm{Aff}(J^1_y(\tau), \Omega^m_x)$

vector bundle

Collecting J^+_yE point by point in E we get $J^+E \longrightarrow E$

Every affine map has a linear part. which is a linear map from $V_yE \otimes T^*_xM \longrightarrow \Omega^m_x$ i.e. an element of $V_y^*E \otimes \underbrace{T_xM \otimes \Omega^m_x}_{\cong\, \Omega^{m-1}_x} = P$

We have then $J^+E \longrightarrow P$. It is a ~~one~~ ~~m~~ one dimensional fibre bundle (affine) modelled on $P \times_M \Omega^{m} \longrightarrow P$. It is well known that J^+E can be identified with a particular bundle of multicovectors, namely m covectors on E with at most one vertical component

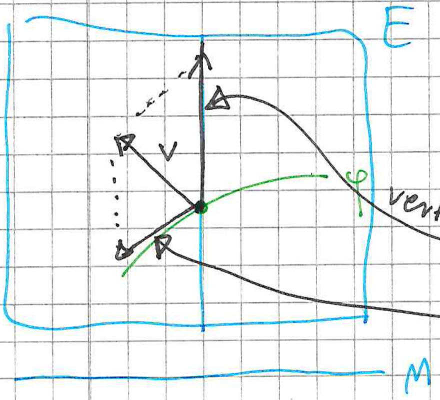
$$\Lambda^m_1 T^*E = \{ \cancel{\varphi}\, A \in \Lambda^n T^*E : i_v i_u \varphi = 0,\ u, v \in VE \}$$

in coordinates such a ~~φ~~ A reads

$$A\, \cancel{\varphi} = p^\mu_a\, dy^a \wedge \eta_\mu + \rho\, \eta \qquad \eta = dx^1 \wedge \cdots \wedge dx^m$$

$$\eta_\mu = i_{\partial_{x^\mu}} \eta$$

To see this identification, we have to be able to evaluate ~~A~~ on $j^1\varphi = (x^\mu, y^a, y^b_\nu)$ and obtain an m-covector.



$$v = \dot{y}^a \partial_a + \dot{x}^\mu \partial_\mu =$$

$$= \underbrace{(\dot{y}^a - y^a_\mu \dot{x}^\mu)\partial_a}_{\text{vertical}} + \underbrace{y^a_\mu \dot{x}^\mu \partial_a + \dot{x}^\nu \partial_\nu}_{\text{horizontal}}$$

$T_yE = V_yE \oplus H_yE$ relative to the jet. We can do the same for the dual bundle T^*_yE

$$T^*_y E = \underset{\underset{\langle dy^a - y^a_\mu d\lambda^\mu \rangle}{\nearrow}}{V^*_y E} \oplus \underset{\nwarrow\, \langle dx^\mu \rangle}{H^*_y E}$$

This propagates to m-covectors

$$A = p^\mu_a dy^a \wedge \eta_\mu + \rho\eta =$$

$$= p^\nu_a (dy^a - y^a_\mu dx^\mu + y^a_\mu dx^\mu) \wedge \eta_\nu + \rho\eta = \dots$$

$$\dots = p^\nu_a (dy^a - y^a_\mu dx^\mu) \wedge \eta_\nu + (\rho + p^\nu_a y^a_\nu)\eta$$

Using the jet we can geometrically recover the horizontal part which is a basic m-covector depending on the jet → a value of A on $j^1\varphi$.

We also have the projection $\Lambda^m_1 T^*E \longrightarrow P$ by restricting to vertical vectors. We end up with the affine bundle

$$\begin{tikzcd} J^{+}E = \Lambda^m_1 T^*E \arrow[r] & P \arrow[l, bend right, "\mathcal{H}"'] \end{tikzcd}$$

Hamiltonians are sections of this bundle.

The identification is important because $J^{+}E$ carries a tautological and multisymplectic form. The tautological form ϑ is defined as usual:

$$\begin{tikzcd} & T\Lambda^1_n T^*E \ni v_i \quad i=1\dots m \arrow[dl] \arrow[dr, "T\zeta"] & \\ \Lambda^1_n T^*E & & TE \end{tikzcd}$$

$$\vartheta(v_1, \dots, v_m) = A(T\zeta(v_1), \dots, T\zeta(v_m))$$

$$\Theta = p^\nu_a dy^a \wedge \eta_\nu + \rho\eta$$

$$\omega = d\Theta = dp^\nu_a \wedge dy^a \wedge \eta_\nu + d\rho \wedge \eta$$

This is the canonical multisymplectic form.

Hamiltonian field equations coming from the Hamiltonian $H : P \longrightarrow J^{+}E = \Lambda^{m}_{1} T^{*}E$ can be written in the following way (after Leo Ryvkin and Tillman Wurzbacher)

P is a solution of field equations $p : M \longrightarrow P$ if for all $V \in VP$

$$p^{*}(i_{V} H^{*}\omega) = 0$$

multisymplectic form is used here.

ω - $\overset{m+1}{a}$form on $J^{+}E$, $H^{*}\omega$ - a $\overset{m+1}{\cancel{\Delta}}$-form on P, $i_{V} H^{*}\omega$ - m-form on P, $p^{*}(i_{V} H^{*}\omega)$ - m form on M. This is quite complicated formula, but it works, in a sense that it gives proper field equations, namely

$$y^{a}_{,\mu} = \frac{\partial h}{\partial p^{\mu}_{a}} \qquad p_{a}{}^{\mu}{}_{,\mu} = -\frac{\partial h}{\partial y^{a}} \qquad H : P \longrightarrow \Lambda^{m}_{1} T^{*}E$$

$$p^{\mu}_{a}\, dy^{a} \wedge \eta_{\mu} + h(x, y, p)\eta$$

NON MULTISYMPLECTIC APPROACH

Now, the question is, if we can forget the multisymplectic structure, which definitely exists there and is canonical and use instead a structure native to P rather than to the values of hamiltonian. Keeping in mind the mechanical example I would like to construct a map

affine differentials of hamiltonians.

$$\beta : J^{1}P \longrightarrow PJ^{+}E$$

AFFINE DIFFERENTIAL

Our hamiltonians are not functions and also not ordinary forms. We would like to be able to differentiate it in vertical directions. To do this we can use the geometry of affine values:

$$\begin{tikzcd} J^{+}E \arrow[d] \\ P \arrow[u, bend right, "h"'] \end{tikzcd}$$
is an affine bundle modelled on the trivial vector bundle
$$\begin{tikzcd} P\times\Omega^{m} \arrow[d] \\ P \end{tikzcd}$$

If h_1, h_2 are sections, then the differen~~tiat~~ce $h_1 - h_2$ is a map $P \longrightarrow \Omega^m$. We can take a vertical differential $\delta(h_1 - h_2)$ which is a section of $V^*P \otimes \Omega^m \longrightarrow P$

We say that $dh_1(p) = dh_2(p)$ if $\delta(h_1 - h_2)(p) = 0$.
This means that both sections have the same jet in vertical directions. The set of all equivalence classes will be denoted by PJ^+E. It is an affine bundle modelled on $V^*P \otimes \Omega^m$. One can show that $PJ^+E \simeq V^*J^1E \otimes \Omega^m$ as double vector-affine bundle. This is simmilar to $T^*E^* \simeq T^*E$ as d.v.b. What I did before, I used this isomorphism and α to construct β. Now, I can do it directly using ω_P, i.e. the canonical vertical two form on P with values in Ω^{m-1}.
To do this, we need to recognize PJ^+E as certain forms on P, similarly, as we did it for J^+E itself.

PJ^+E as very special forms on P

The starting point is $\Lambda^{m+1}_2 T^*P$ i.e. forms on P with no more than two vertical component. Using coordinates, we can span these forms by $P : (x^\mu, y^a, p^\nu_b)$

$$\langle dy^a \wedge dy^b \wedge \eta_\mu ,\ dp^\nu_a \wedge dp^\lambda_b \wedge \eta_\alpha ,\ dy^a \wedge dP^\mu_b \wedge \eta_\nu ,$$
$$dy^a \wedge \eta ,\ dp^\mu_a \wedge \eta \rangle \ni B$$

Such form restricted to VP gives an element of $\Lambda^2 V^*P \otimes \Omega^{m-1}$

$$\begin{tikzcd} \Lambda^{m+1}_2 T^*P \arrow[r, "\zeta"] & \Lambda^2 V^*P \otimes \Omega^{m-1} \arrow[r] & P \arrow[l, bend right, "\omega_P"'] \end{tikzcd}$$

$\langle \partial y^a \wedge \partial y^b \otimes \eta_\mu ,\ \partial p^\nu_a \wedge \partial p^\mu_b \otimes \eta_\alpha ,$
$\partial y^a \wedge \partial p^\mu_b \otimes \eta_\nu \rangle$

In $\Lambda^2 V^*P \otimes \Omega^{m-1}$ we have a canonical submanifold being the image ~~as~~ of ω_P. Let us then define

$$\Sigma = \zeta^{-1}(\omega_P(P)) \longrightarrow P$$

↖ this projects on P and consists of forms

$$B = dp^\mu_a \wedge dy^a \wedge \eta_\mu + F_a\, dy^a \wedge \eta + G^a_\mu\, dp^\mu_a \wedge \eta$$

We can show that $\Sigma \simeq PJ^+E$

The construction of $\beta : J^1P \longrightarrow PJ^+E \simeq \Sigma$

Now we can play the same trick as identifying J^1E with $\Lambda^m_1 T^*E$. If I have a jet $u = j^1p(x)$

$$(x^\mu, y^a, p^\nu_b, y^c_\alpha, p^\beta_{d\gamma})$$

I can identify vertical differentials ∂y^a, ∂p^ν_a

With actual differentials dy^a, dp^ν_b

$$\Phi_u : \partial y^a \longmapsto dy^a - y^a_\mu \, dx^\mu$$

$$\partial p^\nu_b \longmapsto dp^\nu_b - p^\nu_{b\mu} \, dx^\mu$$

Using the jet u and $\omega_P = \partial p^\mu_a \wedge \partial y^a \otimes \eta_\mu$ I can find a particular element of Σ corresponding to u:

$$u \xmapsto{\beta} \Phi_u(\partial p^\mu_a) \wedge \Phi_u(\partial y^a) \wedge \eta_\mu =$$

$$= dp^\mu_a \wedge dy^a \wedge \eta_\mu - \underbrace{p^\mu_{a\mu}}_{F_a} dy^a \wedge \eta + \underbrace{y^a_\nu}_{G^a_\nu} dp^\nu_a \wedge \eta$$

The map β uses ω_P and jet, much as β_M uses ω_M and a vector $v \in TT^*M$. To construct this map we do not use multisymplectic form at all. The whole diagram

$$\begin{tikzcd}
PJ^\dagger E \arrow[dr, "dH"'] & & J^1P \arrow[ll, "\beta"'] \arrow[rr, "\alpha"] \arrow[dl] \arrow[dr] & & V^*J^1(\tau) \otimes \Omega^m \arrow[dl] \\
& P & & J^1(\tau) \arrow[ll, "\lambda"'] \arrow[ur, bend right, "\delta L"'] &
\end{tikzcd}$$

Is constructed based on jet bundle structure and ω_P. The relation between the jet bundle (multicontact, Cartan distribution) and ω_P is not clear to me now, but I bet it is somewhere there. Of course, in GIMMSY and other multisymplectic papers there is much more to the multisymplectic structure than just expressing the equations -

There is the whole industry of momentum maps, reductions, conservation laws .. ets. I wonder now, how long can we go not using multisymplectic structure but only what naturally lives on P.

KONIEC